

## THE CONVERGENCE THEOREMS FOR THE AP-INTEGRAL

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ABSTRACT. In this paper, we study the convergence theorem for the AP-integral based on the condition UAP and pointwise boundedness.

### 1. Introduction

The AP-integral was defined by Burkill. It was a Perron type definition using major and minor functions [2]. After an equivalent Riemann type definition of this integral was given by Henstock, and was detailed by Bullen [2], the notation AP also was used to denote any of above two types of integral without further distinction. In [5], Lin gave the weak uniform integrability theorem for the AP-integral based on the conditions UAP and UASL.

In this paper, we give the convergence theorem for the AP-integral based on the condition UAP and the pointwise boundedness.

### 2. Some Basic Definitions

Throughout this paper,  $[a, b]$  denotes a fixed finite closed interval and all functions are real-valued.

DEFINITION 2.1.

- (1) An approximate neighborhood (ap neighborhood) of  $x \in [a, b]$  is a measurable set  $D_x \subset [a, b]$  containing  $x$  and having density 1 at  $x$ .

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Received by the editors on July 12, 1999.

1991 *Mathematics Subject Classifications*: 26A39.

Key words and phrases: AP-integral, UAP[a, b], UASL[a, b].

- (2) Given a measurable set  $E \subset [a, b]$ , a set-valued function  $\Delta : E \rightarrow 2^{[a, b]}$  is an ap-neighborhood function (ANF) on  $E$ , if for every  $x \in E$ , there exists an ap-neighborhood  $D_x \subset [a, b]$  of  $x$  such that  $\Delta(x) = D_x$ .
- (3) Given an ANF  $\Delta$  on  $E \subset [a, b]$ , a finite collection of interval-point pairs  $P = \{(I, x)\}$  is a  $\Delta$ -fine partial partition on  $E$ , denoted by  $P \in P_1(E, \Delta)$ , if the intervals in  $P$  are non-overlapping and their endpoints belong to  $\Delta(x)$  when  $x \in I \cap E$ . If  $E \subset \bigcup I$ , then we call  $P$  a  $\Delta$ -fine partition on  $E$ , denoted by  $P \in P(E, \Delta)$ .

Let  $f, F$  be real-valued functions defined on  $[a, b]$ ,  $E \subset [a, b]$ ,  $\Delta$  be an ANF on  $E$ , and  $P = \{(I, x)\} \in P_1(E, \Delta)$ .

We define

$$\begin{aligned}\sigma(f, P) &= (P)\Sigma f(x)|I|, \\ \sigma(|f|, P) &= (P)\Sigma |f(x)||I|, \\ \sigma(F, P) &= (P)\Sigma F(I),\end{aligned}$$

where  $|I| = v - u$  and  $F(I) = F(v) - F(u)$  if  $I = [u, v]$ .

In addition,

$$\begin{aligned}\sigma(|F|, P) &= (P)\Sigma |F(I)|, \\ \sigma(f - F, P) &= (P)\Sigma [f(x)|I| - F(I)], \\ \sigma(|f - F|, P) &= (P)\Sigma |f(x)|I| - F(I)|, \\ \sigma(P) &= (P)\Sigma |I|.\end{aligned}$$

It is well-known that for any ANF  $\Delta$  on  $[a, b]$ ,  $P([a, b], \Delta) \neq \emptyset$  [5, Proposition 2.1].

DEFINITION 2.2. Let  $f : [a, b] \rightarrow R$ ,  $F : [a, b] \rightarrow R$ , and let  $E \subset [a, b]$  be a measurable set.

- (1)  $f$  is said to be AP-integrable on  $[a, b]$  ( $f \in AP[a, b]$ ) if for every  $\varepsilon > 0$  there exists an ANF  $\Delta$  on  $[a, b]$  such that  $|\sigma(f - F, P)| < \varepsilon$  whenever  $P \in P([a, b], \Delta)$ , where  $F$  is the primitive of  $f$ .
- (2)  $F$  is said to satisfy the approximate strong Lusin condition on  $E$  ( $F \in ASL(E)$ ) if for every  $Z \subset E$  of measure zero and for every  $\varepsilon > 0$  there exists an ANF  $\Delta$  on  $E$  such that  $\sigma(|F|, P) < \varepsilon$  whenever  $P \in P_1(Z, \Delta)$ .

DEFINITION 2.3. Let  $\{f_n\}$ ,  $\{F_n\}$  be sequences of functions defined on  $[a, b]$ , and let  $E \subset [a, b]$  be a measurable set.

- (1)  $\{f_n\}$  is said to be uniformly AP-integrable on  $[a, b]$  ( $\{f_n\} \in UAP[a, b]$ ), if for every  $\varepsilon > 0$  there exists an ANF  $\Delta$  on  $[a, b]$  such that  $|\sigma(f_n - F_n, P)| < \varepsilon$  for all  $n$  whenever  $P \in P([a, b], \Delta)$ , where  $F_n$  is the primitive of  $f_n$  for each  $n$ .
- (2)  $\{F_n\} \in UASL(E)$  if for every  $Z \subset E$  of measure zero and for every  $\varepsilon > 0$  there exists an ANF  $\Delta$  on  $E$  such that  $\sigma(|F_n|, P) < \varepsilon$  for all  $n$  whenever  $P \in P_1(Z, \Delta)$ .

### 3. The Convergence Theorem for the AP-integral

The following theorem was proved by Lin [5].

THEOREM 3.1. Let  $\{f_n\}$  be a sequence of functions defined on  $[a, b]$  satisfying the following conditions;

- (1)  $f_n(x) \rightarrow f(x)$  a.e. on  $[a, b]$  as  $n \rightarrow \infty$
- (2)  $\{F_n\} \in UASL[a, b]$ , where  $F_n$  is the primitive of  $f_n$
- (3)  $\{f_n\} \in UAP[a, b]$

Then  $f \in AP[a, b]$  and  $(AP) \int_a^b f = \lim_{n \rightarrow \infty} (AP) \int_a^b f_n$ .

To prove Theorem 3.3, we need the following lemma.

LEMMA 3.2. Let  $\{f_n\}$  be sequence of functions defined on  $[a, b]$  satisfying the following conditions;

- (1)  $\{f_n\}$  is pointwise bounded on  $[a, b]$
- (2)  $\{f_n\} \in UAP[a, b]$

Then  $\{F_n\} \in UASL[a, b]$ , where  $F_n$  is the primitive of  $f_n$ .

*Proof.* Let  $Z \subset [a, b]$  be of measure zero and let  $\varepsilon > 0$ . For each  $i$ , set  $Z_i = \{x \in Z : i - 1 \leq \sup_n |f_n(x)| < i\}$  and let  $\varepsilon_i = \frac{\varepsilon}{2^i}$ . Choose an open set  $O_i$  such that  $Z_i \subset O_i$  and  $|O_i| < \frac{\varepsilon_i}{i}$ . Since  $\{f_n\} \in UAP[a, b]$ , there exists an ANF  $\Delta'$  on  $[a, b]$  such that  $|\sigma(f_n - F_n, P)| < \varepsilon$  for all  $n$  whenever  $P \in P([a, b], \Delta')$ . Take  $\delta(x) > 0$  on  $Z_i$  so that  $(x - \delta(x), x + \delta(x)) \subset O_i$  when  $x \in Z_i$ . Now define an ANF  $\Delta$  on  $[a, b]$  by

$$\Delta(x) = \begin{cases} \Delta'(x) \cap (x - \delta(x), x + \delta(x)) & \text{if } x \in Z_i \text{ for } i = 1, 2, 3, \dots, \\ \Delta'(x) & \text{if } x \in [a, b] - \bigcup Z_i. \end{cases}$$

Suppose that  $P \in P_1(Z, \Delta)$ . For each  $i$ , let  $P_i = \{(I, x) \in P : x \in Z_i\}$ . Then  $P = \bigcup_i P_i$ , where  $P_i \in P_1(Z_i, \Delta) \subset P_1(Z_i, \Delta')$ . By the Saks-Henstock Lemma,

$$\begin{aligned} \sigma(|F_n|, P) &\leq \sigma(|f_n - F_n|, P) + \sigma(|f_n|, P) \\ &< 4\varepsilon + \sum_{i=1}^{\infty} \sigma(|f_n|, P_i) \\ &< 4\varepsilon + \sum_{i=1}^{\infty} i|O_i| \\ &< 4\varepsilon + \sum_{i=1}^{\infty} \varepsilon_i = 5\varepsilon \end{aligned}$$

for all  $n$ . Hence  $\{F_n\} \in UASL[a, b]$ . □

From Lemma 3.2, we have the following Theorem.

THEOREM 3.3. Let  $\{f_n\}$  be a sequence of functions defined on  $[a, b]$  satisfying the following conditions ;

- (1)  $f_n(x) \rightarrow f(x)$  a.e. on  $[a, b]$  as  $n \rightarrow \infty$
- (2)  $\{f_n\}$  is pointwise bounded on  $[a, b]$
- (3)  $\{f_n\} \in UAP[a, b]$ .

Then  $f \in AP[a, b]$  and  $(AP) \int_a^b f = \lim_{n \rightarrow \infty} (AP) \int_a^b f_n$ .

*Proof.* For each  $n$ , let  $F_n$  be the primitive of  $f_n$ . Since  $\{F_n\} \in UASL[a, b]$  by Lemma 3.2, the conclusion follows from Theorem 3.1.  $\square$

COROLLARY 3.4. Let  $\{f_n\}$  be a sequence of measurable functions defined on  $[a, b]$  satisfying the following conditions;

- (1)  $f_n(x) \rightarrow f(x)$  a.e. on  $[a, b]$  as  $n \rightarrow \infty$
- (2)  $\{f_n\}$  is uniformly bounded on  $[a, b]$ .

Then  $f \in AP[a, b]$  and  $(AP) \int_a^b f = \lim_{n \rightarrow \infty} (AP) \int_a^b f_n$ .

*Proof.* Suppose that  $|f_n(x)| \leq K$  for all  $n$  and all  $x \in [a, b]$  where  $K$  is a positive constant. By the condition(1),  $f$  is also measurable and bounded a.e. on  $[a, b]$ . Hence each  $f_n$  and  $f$  are Lebesgue integrable on  $[a, b]$  and hence  $AP$ -integrable on  $[a, b]$ . Since  $\{f_n\} \in UAP[a, b]$  by [5, Lemma 3.5], the conclusion follows from Theorem 3.3.  $\square$

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