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SVD(Singular Value Decomposition)을 이용한 간편한 잡음 제거법

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A Simple Noise Reduction Method using SVD(Singular Value Decomposition)

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ABSTRACT

저차 동적계(특히 카오스계)에서 측정한 시계열의 잡음을 제거하기 위해서 SVD(Singular Value Decomposition)을 이용한 새로운 간편하고 매우 효과적인 새로운 잡음 제거법이 소개되었다. 이 방법은 위상궤적(phase portrait)을 재구성하는데 중점을 두었으며, 궤적행렬(trajectory matrix)을 구성하는데 그 기본을 두었다. 이 궤적행렬에 SVD를 반복적으로 사용하여 신호와 잡음을 분리하였다. 이 방법은 Duffing 계에서 측정한 잡음이 섞인 카오스 신호에 적용되었으며, 또한 실험에의한 진폭변조된 신호에도 적용되었다.

Key Words : SVD, Method of delays, Phase portrait(위상궤적), Noise reduction(잡음제거)

1. Introduction

Over the last couple of decades, chaotic systems have been extensively studied and many remarkable results have been achieved in understanding very complex phenomena produced by simple non-linear dynamical systems. The majority of these works have been based on computer simulations. However, noise is always problematic in a practical situation. Noise-reduction for a time series may be considered as filtering of a noisy signal to extract a relatively clean signal. There are many methods of filter based noise-reduction, such as optimal filtering (Wiener filter). However, the filtered chaotic time series may be altered fundamentally, so the inherent dynamical properties (dimensions, Lyapunov exponents, etc.) of the original noise-free chaotic signal may not be obtained successfully from the filtered time series. For example, Badii *et al* [1] show that filtering processes may introduce additional spurious Lyapunov exponents and may cause an increase of the fractal dimension. Broomhead *et al* [2], however, proved that finite impulse response (FIR) filters (finite order and non-recursive filter) do not have this effect. Thus, filters for chaotic time series must be FIR filters and applicable to non-stationary time series because chaotic time series is generally strongly non-stationary. Recently many different noise-reduction methods for chaotic time series have been developed [3 - 13], many of them based on the considerations of geometrical properties by using the embedding methods such as the 'method of delays' [14] and SVD (Singular Value Decomposition) [15]. Some of them are claimed to produce very good noise reduction. However, these methods are usually very complicated and difficult to implement, and also require many aspects

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to be considered carefully. In this paper, a very simple and effective noise reduction method is presented. The term 'simple' means that there are only two parameters, which are the sampling rate and the embedding dimension, are required to successfully apply this method. This method is based on the algorithm of reconstruction of phase portrait in [15] which is very useful for the reconstruction of phase portrait from a noisy signal.

In this paper, we assume that the noise is additive and white. This apart we assume that we have no prior knowledge of the noise-free signal. The signal used in this paper is from Duffing equation

$$\ddot{x} + c\dot{x} - kx(1 - x^2) = A\cos(\omega t) \qquad (1)$$

with parameters for $k = 1$, $c = 0.4$, $A = 0.4$ and $\omega = 1$. A sampled displacement signal 'x(k)' is obtained using 4-th order Runge-Kutta method with fixed integration step size 0.1 second which gives the sampling frequency f_s = 10Hz, and this signal is assumed to be clean. The Gaussian white noise n(k) is added to the clean signal, so the noise contaminated signal s(k) is given by

$$s(k) = x(k) + n(k) \qquad (2)$$

The standard deviation of noise is 50% of standard deviation of the clean signal. The corresponding Signal-to-Noise Ratio ($SNR = 10\log_{10}(\sigma_x^2/\sigma_n^2)$) is 6dB, where σ_x and σ_n are the standard deviation of the clean signal and the noise signal respectively. These signals are shown in Fig. 1.

(a) (b)

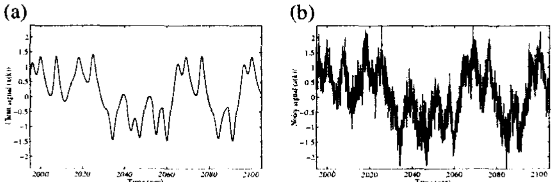

Fig. 1(a) clean signal
(b) noisy signal: 50 % white noise added (SNR = 6dB)

2. Reconstruction of Phase Portrait and SVD

A useful method of reconstruction of phase portraits based on SVD (Singular Value Decomposition) was introduced by Broomhead *et al.* [15]. This method provides phase portrait reconstruction with a little of noise reduction. From the measured discrete time series $\{v_i \mid i = 1, \cdots, N_T\}$, where N_T is the number of data points, a sequence of vectors $\{\mathbf{x}_i \in \Re^n \mid i = 1, \cdots, N\}$ can be generated and the trajectory matrix $\mathbf{X}$ can be constructed in n-embedding dimension

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_N^T \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ v_2 & v_3 & & v_{n+1} \\ \vdots & & \ddots & \vdots \\ v_N & v_{N+1} & \cdots & v_{N+n-1} \end{bmatrix} \qquad (3)$$

where $N = N_T-(n-1)$. Note that the matrix $\mathbf{X}$ is the pseudo phase portrait (by the 'method of delays') in n-dimensional pseudo phase space with a delay time of 'one unit'. The SVD of the trajectory matrix gives

$$\mathbf{X} = \mathbf{S}\mathbf{\Sigma}\mathbf{C}^T \qquad (4)$$

where, $\mathbf{S}$ is the N×n matrix of eigenvectors of $\mathbf{XX}^T$, $\mathbf{C}$ is the n×n matrix of eigenvectors of $\mathbf{X}^T\mathbf{X}$ and Σ is the $n \times n$ diagonal matrix consisting of singular values, i.e., $\mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$. Rearranging (4),

$$\mathbf{XC} = \mathbf{S\Sigma} \qquad (5)$$

The matrix $\mathbf{XC}$ is the trajectory matrix projected onto basis $\{\mathbf{c}_i\}$, where '$\mathbf{c}_i$' is the i-th column of $\mathbf{C}$. One can think of the trajectory as exploring on average, an n-dimensional ellipsoid, where the $\{\mathbf{c}_i\}$ represent directions and the $\{\sigma_i\}$ represent the lengths of the principal axes of the ellipsoid [15]. The main concept of this method is to extract the dimensionality n' (minimum embedding dimension) of the subspace containing the embedded manifold, where, $n' \le n$. The dimensionality n' is the rank of the eigenvector matrices (Rank($\mathbf{S}$) = Rank($\mathbf{C}$)), where the rank is the number of non-zero singular values. At this point, one can intuitively think of the physical meaning of the dimensionality n' as an effective embedding dimension. In other words, the matrix $\mathbf{XC}$ with embedding dimension n' has no less information than the matrix with embedding dimension 'n'. Also, note that the SVD ensures that each column of the matrix $\mathbf{XC}$ is linearly independent. In the presence of noise, the noise causes all the singular values of the trajectory matrix to be non-zero. However, assuming the noise is white, the noise will cause all the singular values of $\mathbf{X}$ to be shifted uniformly, i.e., they can be written as

$$\begin{aligned} \sigma^2_i &= \overline{\sigma}^2_i + \sigma^2_{noise} \qquad i = 1, 2, \cdots, k \\ \sigma^2_{k+1} &=, \cdots, = \sigma^2_n = \sigma^2_{noise} \end{aligned} \tag{6}$$

where σ_{noise} are the singular values of the noise floor, and the trajectory matrix can also be written as

$$\begin{aligned} \mathbf{X} &= \overline{\mathbf{X}} + \mathbf{N} \\ &= [\mathbf{S}_1 \quad \mathbf{S}_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{C}_1^T \\ \mathbf{C}_2^T \end{bmatrix} \end{aligned} \tag{7}$$

where $\overline{\mathbf{X}}$ is the deterministic part of the trajectory matrix, $\mathbf{N}$ is the noise dominated part, $\mathbf{S}_1 \in \Re^{N \times k}$, $\Sigma_1 \in \Re^{k \times k}$, and $\mathbf{C}_1 \in \Re^{n \times k}$. In order to separate the noise dominated part from the trajectory matrix, one can estimate the deterministic part $\overline{\mathbf{X}}$ by either least-squares or minimum variance estimate. The least squares estimate of $\overline{\mathbf{X}}$ is given by [16 -18]

$$\overline{\mathbf{X}}_e = \mathbf{S}_1 \Sigma_1 \mathbf{C}_1^T \tag{8}$$

and the minimum variance estimate is given by [16, 17]

$$\overline{\mathbf{X}}_e = \mathbf{S}_1 \Sigma_1^{-1} (\Sigma_1^2 - \sigma^2_{noise} \mathbf{I}_k) \mathbf{C}_1^T \tag{9}$$

where, $\mathbf{I}_k$ is the 'k × k' identity matrix. From (8) or (9), we can see that the deterministic part of the trajectory matrix can be estimated by using the SVD of $\mathbf{X}$. The matrix $\mathbf{XC}$ now becomes $\mathbf{X}_e\mathbf{C}_1$ which is less noisy. However, if the smallest singular value of the deterministic part is not significantly greater than the noise level, the above method is a little problematic especially for reconstruction of phase portraits.

Equation (6) has a special meaning in that the ratio of singular values '$\sigma_i > \sigma_{noise}$' (above the noise floor) and σ_{noise} represents the signal-to-noise ratios which are associated with each singular vector $\mathbf{s}_i$ (each column of the matrix $\mathbf{S}$), and so the signal-to-noise ratio of the i-th left singular vector can be written as

$$SNR_i = 10 \log \frac{\sigma_i^2 - \sigma_{noise}^2}{\sigma_{noise}^2} \tag{10}$$

Thus, '$\sigma^2_1/\sigma^2_{noise}$' represents the SNR of the first column of the matrix $\mathbf{X}_e\mathbf{C}_1$, and '$\sigma^2_2/\sigma^2_{noise}$' represents the SNR of the second column of the matrix $\mathbf{X}_e\mathbf{C}_1$, and so on. Thus, the reconstructed phase portraits using the above method may be degraded by the part associated with the singular values which are not significantly greater than σ_{noise}. The singular values of the trajectory matrix constructed by the clean signal 'x(k)' and the pseudo phase portrait by SVD are shown in Fig. 2. The above problem is shown in Fig. 3. The singular values of the trajectory matrix constructed by the noisy signal 's(k) = x(k) + n(k)' and the corresponding pseudo phase portrait by the first and second column of the matrix $\mathbf{X}_e\mathbf{C}_1$ are shown in Fig. 3(a), (b). The first two columns of the matrix $\mathbf{X}_e\mathbf{C}_1$ of the noisy signals are shown in Fig. 3(c), (d). From these Figures, it can be shown that the reconstructed phase portraits are greatly degraded by the second column of the matrix $\mathbf{X}_e\mathbf{C}_1$ which has far lower SNR compared to the first column of the matrix $\mathbf{X}_e\mathbf{C}_1$. Also the dimensionality (number of non-zero singular values) n′ is estimated '2' for noisy signals (Fig. 3(a)) rather than '3' for the clean signal (Fig. 2(a)).

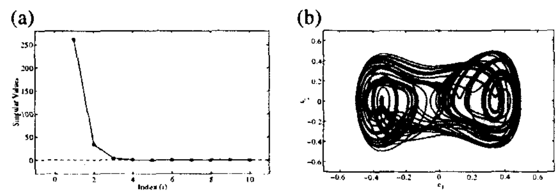

Fig. 2(a) Singular values of the trajectory matrix constructed by the clean signal x(k)
(b) Pseudo Phase Portraits reconstructed by SVD (normalised version of the matrix $\overline{\mathbf{X}}_e\mathbf{C}_1$)

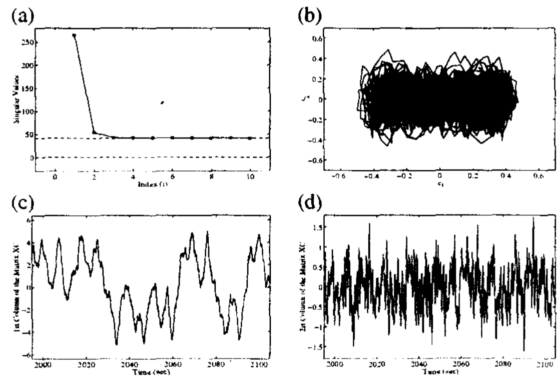

Fig. 3 Reconstructed phase portraits by using the SVD for noisy signal
(a) Singular values
(b) Pseudo Phase Portraits reconstructed by SVD (normalised version of the matrix $\overline{\mathbf{X}}_e\mathbf{C}_1$)
(c) 1st column of the matrix $\overline{\mathbf{X}}_e\mathbf{C}_1$
(d) 2nd column of the matrix $\overline{\mathbf{X}}_e\mathbf{C}_1$

3. The Iterative SVD Method and Applications

The method used to overcome the problem, described in the previous section, is named as the 'Iterative SVD method'. For the purpose of noise reduction (not reconstruction of phase portrait), if we can use the first singular value only in (8) or (9), then we can maximise the signal-to noise ratio. In order to do this, the first singular value must contain most of the energy of the deterministic signal. This will happen when dealing with low dimensional systems (Lorenz equation, Duffing equation, etc.). First, an example using a sinusoid is considered since in this case the first singular value carries most of the energy of the signal. If the trajectory matrix is constructed from a sinusoidal signal, we can write

$$\mathbf{X}=\begin{bmatrix} \sin(\omega t) & \sin(\omega t+\phi) & \cdots & \sin(\omega t+(n-1)\phi) \\ \sin(\omega t+\phi) & \sin(\omega t+2\phi) & \cdots & \sin(\omega t+n\phi) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(\omega t+(N-1)\phi) & \sin(\omega t+N\phi) & \cdots & \sin(\omega t+(N+n-2)\phi) \end{bmatrix} \tag{11}$$

where, the phase delay '$n\phi$' corresponds to the time delay 'ωnT_s', and 'T_s' is the sampling time. Assuming each column of the matrix contains exact periods of the signal, the autocovariance matrix '$\mathbf{X}^T\mathbf{X}$' becomes

$$\mathbf{X}^T\mathbf{X}=\sigma_s^2\begin{bmatrix} 1 & \cos(\phi) & \cos(2\phi) & \cdots & \cos((n-1)\phi) \\ \cos(\phi) & 1 & \cos(\phi) & \cdots & \vdots \\ \cos(2\phi) & \cos(\phi) & 1 & \cdots & \cos(2\phi) \\ \vdots & \vdots & \vdots & \vdots & \cos(\phi) \\ \cos((n-1)\phi) & \cos((n-2)\phi) & \cdots & \cos(\phi) & 1 \end{bmatrix} \tag{12}$$

where, σ_s^2 is the variance of the signal. It is shown that the rank of the matrices (11) and (12) is two, and the two non-zero eigenvalues of (12) are given by [19]

$$\lambda_1,\lambda_2=\frac{\sigma_s^2}{2}\left[n\pm\frac{\sin(n\phi)}{\sin(\phi)}\right] \tag{13}$$

Since the square roots of the eigenvalues of (12) are the singular values of (11), the two non-zero singular values can be written as

$$\sigma_1,\sigma_2=\sqrt{\frac{\sigma_s^2}{2}\left[n\pm\left|\frac{\sin(n\phi)}{\sin(\phi)}\right|\right]} \tag{14}$$

Equation (14) permits us to determine when the first singular value carries the most energy of the signal. Note that it is related to both embedding dimension (n) and sampling time ($\phi=\omega T_s$). For a given N×n trajectory matrix, we may express the energy carried by the first singular value by

$$E_1=\frac{\sigma_1^2}{\sum_{i=1}^{n}\sigma_i^2} \tag{15}$$

and when the signal contains white noise this becomes

$$E_1=\frac{\sigma_1^2-\sigma_{noise}^2}{\sum_{i=1}^{k}\left(\sigma_i^2-\sigma_{noise}^2\right)} \tag{16}$$

Given the sampling rate, assuming that each column of the matrix has exactly one period, the ratio of the energy carried by the first singular value of (11) is shown in Fig. 4(a). From this Figure, it is shown that if we make the dimension of the trajectory matrix '(n / N)<0.1, then the first singular value carries more than 97% of the energy. In other words, each row of the matrix contains one tenth of the period. This also tells us the necessary sampling rate. For example, if the embedding dimension (or column dimension) 'n' is estimated as 5, then more than 50 samples per period is required. The row dimension 'N' of the trajectory matrix does not significantly affect the nature of the signal compression as long as the sampling rate and the column dimension are fixed. This is shown in Fig. 4(b).

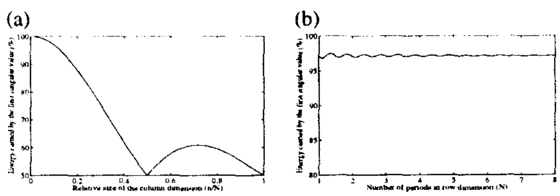

Fig. 4(a) Energy carried by the first singular value (one sinusoid) with different embedding dimension 'n'. In this case, the row dimension and the sampling rate is fixed.

(b) Energy carried by the first singular value (one sinusoid) with different row dimension 'N'. In this case, the embedding dimension and the sampling rate is fixed.

Similar results are obtained for the multiple sinusoid case. In this case the results depend on the variance and frequency of each sinusoid. Details of these can be found in reference [20]. The above results require very high sampling rates especially when the embedding dimension is estimated to be large, i.e., the sampling rate becomes more than '10n' times the highest frequency component. However, for low dimensional dynamical systems, such as the Lorenz and Duffing equations, carried out in this study, the sampling rate of approximately more than 10 times the cut-off frequency is shown to be satisfactory. The case of the Duffing equation with the same parameters as in section 2 is shown in Fig. 5(a). Each column, in this case, has approximately one forcing period. For the Duffing equation, if the sampling rate is roughly 10 times the cut-off frequency and the embedding dimension is set to '$n = T_s / T_c$' where T_s is the sampling time, $T_c = 1/f_c$ and f_c is the cut-off frequency, then the size of row dimension 'N' is not very important. This is shown in Fig. 5(b).

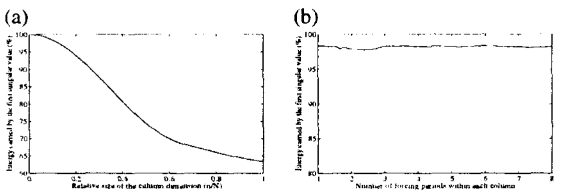

Fig. 5(a) Energy carried by the first singular value (Duffing equation) with different embedding dimension 'n'. The row dimension and the sampling rate is fixed.
(b) Energy carried by the first singular value (Duffing equation) with different row dimension 'N'. In this case, the embedding dimension and the sampling rate is fixed.

Once we make sure that the energy of the signal is compressed toward the first singular value, then we can use only the first singular value to estimate $\overline{\mathbf{X}}_e$ in equation (8) or (9), and this will maximise the signal-to-noise ratio of the recovered signal, i.e., equations become

$$\overline{\mathbf{X}}_{e1} = \sigma_1 \mathbf{s}_1 \mathbf{c}_1^T \tag{17}$$

$$\overline{\mathbf{X}}_{e1} = \left(\frac{\sigma_1^2 - \sigma_{noise}^2}{\sigma_1}\right) \mathbf{s}_1 \mathbf{c}_1^T \tag{18}$$

where, $\mathbf{s}_1$ and $\mathbf{c}_1$ are the first columns of the corresponding singular vectors in (8) or (9). This procedure can be considered as optimal filtering. Consider a linear FIR filter which can be expressed as

$$y_i = \mathbf{w}^T \mathbf{x}_i \tag{19}$$

where $\mathbf{x}_i$ is the measured sequence with length 'n' ($\mathbf{x}_i = [x_i\ x_{i-1}\ \dots\ x_{i-n+1}]^T$), and $\mathbf{w}$ is the filter with length 'n' ($\mathbf{w} = [w_1\ w_2\ \dots\ w_n]^T$). One can find the FIR filter which maximises the output variance subject to the constraint $\sum_{i=1}^{n} w_i^2 = \mathbf{w}^T\mathbf{w} = 1$, i.e.,

$$\text{maximise: } E[y_i^2] = E[\mathbf{w}^T\mathbf{x}\mathbf{x}^T\mathbf{w}] = \mathbf{w}^T\mathbf{R}\mathbf{w}$$
$$\text{subject to } \mathbf{w}^T\mathbf{w}=1 \tag{20}$$

To solve this optimisation problem the method of Laglange multiplier is used.

$$\frac{\partial}{\partial \mathbf{w}}\left\{\mathbf{w}^T\mathbf{R}\mathbf{w} + \lambda(\mathbf{w}^T\mathbf{w} - 1)\right\} = \mathbf{R}\mathbf{w} + \lambda\mathbf{w} = 0 \tag{21}$$

$$\mathbf{R}\mathbf{w} = -\lambda\mathbf{w} \tag{22}$$

where $\mathbf{R}$ is autocovariance matrix of $\mathbf{x}$. Thus, $\mathbf{w}$ are the eigenvectors of the autocovariance matrix $\mathbf{R}$, and λ are the eigenvalues of the matrix $\mathbf{R}$. Since $\mathbf{w}^T\mathbf{R}\mathbf{w} = E[y_i^2] = -\lambda$, the filter $\mathbf{w}$ (eigenvector) associated with the largest eigenvalue gives maximum output variance. Also the matrix '$\mathbf{X}^T\mathbf{X}$' from the trajectory matrix is the autocovariance matrix. Hence the singular vector associated with the largest singular value is also the FIR filter which maximise the output power. Thus equation (17) or (18) can be considered as the optimal FIR filter.

From equation (17) or (18), it is easily noticed that $\overline{\mathbf{X}}_{e1}$ is an N×n matrix, where N is the length of each column vector of the trajectory matrix and n is the embedding dimension. Each column of the matrix $\overline{\mathbf{X}}_{e1}$ can be considered as a candidate for the noise reduced signal which is delayed by (n−1) sampling unit. To maximise the SNR, we can average each column of the matrix $\overline{\mathbf{X}}_{e1}$ by compensating for the delays. Then we obtain the new noise reduced signal $x_{e1}(k)$. Because $\overline{\mathbf{X}}_{e1}$ is only an estimate of the true deterministic part $\overline{\mathbf{X}}$, the recovered signal is not noise free. Thus it may need several iterations. From the noise reduced signal $x_{e1}(k)$, we can construct a new trajectory matrix, and then do the SVD and construct $\overline{\mathbf{X}}_{e1}$ again to obtain the further noise reduced signal. This procedure is iterated until '$\sigma_{noise} \approx 0$' or '$\|\sigma^k_{noise} - \sigma^{k-1}_{noise}\| < \varepsilon$', where σ^k_{noise} is the k-th iterated noise floor, and ε is the tolerance which determines that the noise floor does not change significantly further. The procedure of this iteration method is summarised in below

1. Construct the trajectory matrix $\mathbf{X}$ from the noisy signal s(k)
2. Singular Value Decomposition, $\mathbf{X} = \mathbf{S}\Sigma\mathbf{C}^T$

3. Construct the matrix $\overline{\mathbf{X}}_{e1}$ using (17) or (18)
4. Obtain the noise reduced signal $x_{e1}(k)$ by averaging each column of $\overline{\mathbf{X}}_{e1}$
5. Repeat the above procedure (1 to 4) until '$\sigma_{noise} \approx 0$' or '$\| \sigma^{k}_{noise} - \sigma^{k-1}_{noise} \| < \varepsilon$'

For white noise considered in this paper, very few iterations are required to recover the noise reduced signal, just 2 iterations are required. The noise reduced signals, using equation (17), are shown in Fig. 6(a). It is observed that equations (17) and (18) do not differ much for this example. From this noise reduced signal, we can reconstruct the pseudo phase portrait by SVD described in section 2. The singular values of the trajectory matrix constructed from the recovered signals are shown in Fig. 6(b). From this, we can see the dimensionality n' is well recovered and estimated as '3'. The reconstructed pseudo phase portraits are shown in Fig. 6(c). This Figure shows remarkable recovery compared to the noisy pseudo phase portrait in Fig. 3(b). The above method (Iterative SVD Method) is also applied to a real experimental system [20] to which the Force-state mapping method is applied. The noise reduction results in [20] are briefly given below. The measured noisy acceleration signal is shown in Fig. 7(a), and the corresponding Force-state map is shown in Fig. 7(b). After applying the 'Iterative SVD Method' appropriately, the acceleration signal and the Force-state map are shown in Fig. 7(c) and 7(d) respectively.

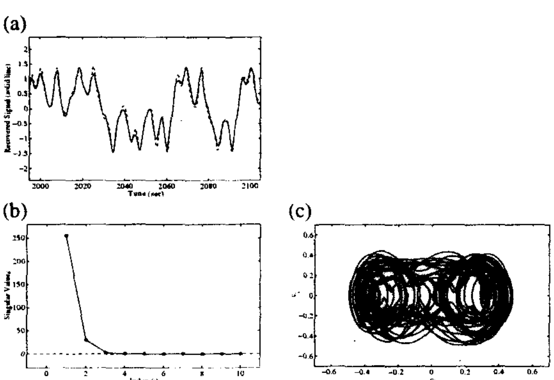

Fig. 6 Reconstructed phase portraits and time series by using the 'Iterative SVD method'
(a) Recovered time series (solid line), dashed line is the original clean signal (the recovered signal is obtained by 2 iterations of the 'Iterative SVD method')
(b) Singular values of the trajectory matrix constructed from the recovered signal
(c) Pseudo Phase Portraits reconstructed from the recovered signal (normalised version of the matrix $\overline{\mathbf{X}}_{e}\mathbf{C}_{1}$)

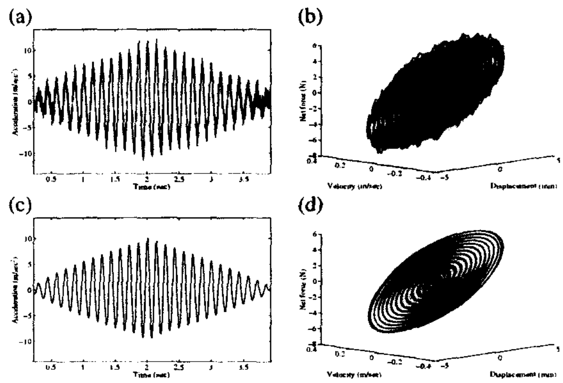

Fig. 7 Results from an experimental signal (Force-state mapping method)
(a), (b) Measured noisy acceleration signal, and corresponding Force-state map
(c), (d) Recovered acceleration signal using the 'Iterative SVD Method', and corresponding Force-state map

Conclusion

It can be seen, from the results given in this paper, that the 'Iterative SVD method' is a very useful and simple method to suppress white noise. Also, this method is not only applicable to chaotic time series but to ordinary deterministic signals as shown in section 3. Chaotic time series is essentially a broad band signal, thus the above method can also be applied to other types of broad band signal. However, the signal must be a deterministic, i.e., directly measured signal from a physical system. This method also has the additional advantage, especially for the reconstruction of phase portrait, that information about the minimum embedding dimension of the system can be obtained. This is a very important feature since estimation of the embedding dimension is one of the most important aspects for reconstruction of the phase portrait. This technique is only valid for filtering out white noise due to the limitation of the algorithm used in this paper. However, one may attempt to apply to other types of noise (e.g., pink noise, brown noise, etc.) by incorporating other modern signal processing methods. This deserves further investigations.

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