〈Original Paper〉

On the Study of Nonlinear Normal Mode Vibration via Poincare Map and Integral of Motion

푸앙카레 사상과 운동적분를 이용한 비선형 정규모드 진동의 연구

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ABSTRACT

The existence, bifurcation, and the orbital stability of periodic motions, which is called nonlinear normal mode, in a nonlinear dual mass Hamiltonian system, which has 6th order homogeneous polynomial as a nonlinear term, are studied in this paper. By direct integration of the equations of motion, Poincare Map, which is a mapping of a phase trajectory onto 2 dimensional surface in 4 dimensional phase space, is obtained. And via the Birkhoff-Gustavson canonical transformation, the analytic expression of the invariant curves in the Poincare Map is derived for small value of energy. It is found that the nonlinear system, which is considered in this paper, has 2 or 4 nonlinear normal modes depending on the value of nonlinear parameter. The Poincare Map clearly shows that the bifurcation modes are stable while the mode from which they bifurcated out changes from stable to unstable.

요 약

6승의 비선형 항을 가지는 두개의 질량으로 구성된 비선형 해밀톤계에 대해서, 비선형 정규모드인 주기운동의 존재성, 분기현상 및 궤도 안정성을 연구하였다. 운동방정식의 직접적분을 통해 4차원 위상공간에서의 운동궤적을 2차원 면으로 투영하는 푸앙카레 사상을 구하였고, 또한 버크호프-구스타프슨 표준 변환을 통해 구한 운동적분을 이용하여 에너지가 작을때 푸앙카레 사상에 나타나는 불변 곡선들의 해석적인 표현을 유도하였다. 본 논문에서 연구한 진동계는 비선형 계수의 값에 따라 2개 또는 4개의 비선형 정규모드를 가짐이 밝혀졌다. 푸앙카레 사상은, 분기된 모드는 안정하고, 원래의 모드는 안정한 상태에서 불안정한 상태로 변한다는 것을 분명하게 보여주었다.

1. Introduction

R. M. Rosenburg extended the notion of normal modes occuring in linear systems to nonlinear systems[1]. A vibration in a Nonlinear Normal Mode (NNM) is one that satisfies the following properties, especially in two degree of freedom (DOF) systems,

(a) The motion of each mass has the same

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period.

(b) During any time interval of one period, both masses simultaneously pass through their equilibrium configuration exactly twice (zero potential energy) and the velocities of both masses simultaneously vanish (zero kinetic energy) precisely twice.

The existence[2], the bifurcation[3], and the stabilities[4] of NNMs have been studied in numerous papers.

In this paper we will consider a nonlinear 2 DOF Hamiltonian system $\mathcal{R}$ in Fig. 1. It is assumed that the system $\mathcal{R}$ has unit masses and the restoring force F for the anchor springs is given by $F=d+kd^5$, while for the coupling spring $F=d^5$. Therefore, the Hamiltonian of the system $\mathcal{R}$ consists of $H(2)$ and $H(6)$ ($H(s)$ is a homogeneous polynomial of degree s.) as follows :

$$H=\frac{1}{2}(x_1^2+y_1^2+x_2^2+y_2^2) + \frac{1}{6}(x_1-x_2)^6+\frac{k}{6}(x_1^5+x_2^6) \quad (1)$$

where $y_i=\dot{x}_i$.

In the previous works[3,5], some Hamiltonian systems, which are composed of $H(2)$ and $H(4)$, were investigated, and it was revealed that these systems have 2 or 4 NNMs and each NNM may be either stable or unstable. Now, we will investigate the effect of the $H(6)$ on the NNMs by adopting the system $\mathcal{R}$. For the study on the existence of normal modes and their stability in our nonlinear system $\mathcal{R}$, the technique of Poincare Map[6~8] is utilized in this paper. We provide two procedures to obtain Poincare Map for the system $\mathcal{R}$. The first one is the direct integration of the equations of motion. The second procedure is based on using the Birkhoff-Gustavson canonical transformation[11,12] to obtain the approximation for the Poincare Map.

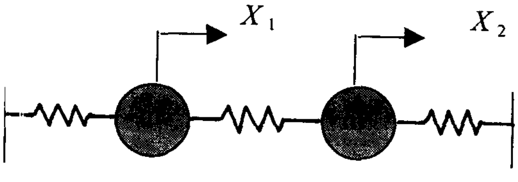


Fig. 1 The System $\mathcal{R}$

These method offer some advantages[5] over the Floquet theory approach. Specifically, these methods does not require linearization in the neighborhood of any particular motion and hence they yield a global picture of the motion flow. Thus, they not only permit conclusions to be drawn about the stability of NNMs but they provide insight into the dynamical structure of the system. Moreover this method is successful in determining the stability for the systems for which the Floquet theory approach fails to predicts the stability for a given NNM[5].

2. Poincare Map

In the case of 2 DOF Hamiltonian systems the phase space is 4 dimensional. Then it is very difficult to picture trajectories of the system since it is impossible to picture 4 dimensions. The idea proposed by H. Poincare provides us with a method of peering into 4 dimensional space. This idea is well-known as the Poincare Map. Let us start with $R^4(x_1, x_2, y_1, y_2)$. The motion is restricted to a 3 dimensional space by Hamiltonian $H=h$, i.e., each energy level set is 3 dimensional. If another integral (called an integral of motion or a first integral in this paper) exists, then the 3 dimensional manifold $H=h$ is fibered by 2 dimensional tori and then can be represented in ordinary 3 dimensional space as a family of concentric tori. We now construct a 2 dimensional plane in this 3 dimensional surface by slicing the surface with the $x_1=0$ hyperplane. A phase trajectory beginning in such a plane returns to it after making a circuit around the torus. In this way we can make a mapping onto itself, which is known as Poincare Map[6~8]. The 2 dimensional (x_2, y_2) surface $\{x_1=0\}\cap\{H=h\}$ is the surface of section. We look at successive intersections of trajectories with this surface of section. It should be noted that periodic orbits correspond to fixed points of this map. It is also noted that if a periodic motion is orbitally stable then the (x_2, y_2) plane will contain a fixed point surrounded by concentric circles, i.e., center. If a periodic motion is unstable it look like a saddle point in the (x_2, y_2) plane.

To insure the trajectories actually pierce the surface of section it is required that when we slice the $H=h$ surface we make a tranversal (not tangential) cut. The trajectory (x_1, x_2, y_1, y_2) will fail to intersect the surface of section tranversally whenever the normal to the surface is perpendicular to the tangent of the trajectory. Thus whenever

$$(x_1, x_2, y_1, y_2)(1, 0, 0, 0) = 0$$

or $x_1 = y_1 = 0$, the tranversality condition is violated. Thus it is further noted that we require either $y_1 > 0$ always when $x_1 > 0$ or $y_1 < 0$ always when $x_1 = 0$. This restriction is to insure that the trajectory always pierces the surface of section from the same side. If we want to numerically obtain a picture of the Poincare Map for fixed h, it is required to proceed direct integration scheme (forward integration) in the following manner :

(a) Let $x_1 = 0$.

(b) Use $H=h$ to obtain $y_1 = y_1(x_2, y_2, h)$.

(c) Select initial point (x_2, y_2) and integrate equations of motion until $x_1 = 0$ and $y_1 > 0$: $x_2 = x_2^*$, $y_2 = y_2^*$.

(d) Using x_2^*, y_2^*, repeat above steps to get next points.

(e) In order to get different trajectories it is required to start with different initial points.

By following the above procedure we obtain results shown in Figs. 2~6. All the figures are bounded by $y_1 = y_1(x_2, y_2, h) = 0$. It is important

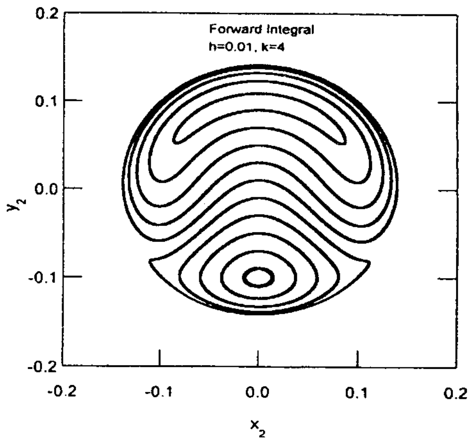


Fig. 2 Poincare map by forward integration for $h = 0.01$, $k = 4$

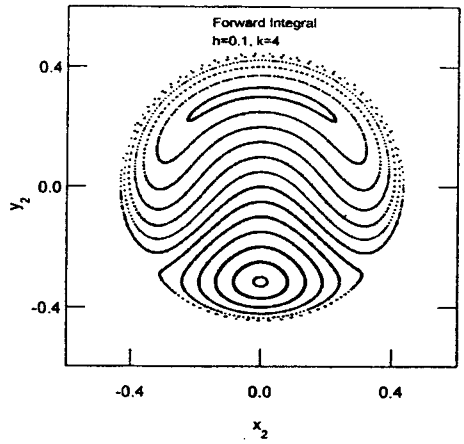


Fig. 3 Poincare map by forward integration for $h = 0.1$, $k = 4$

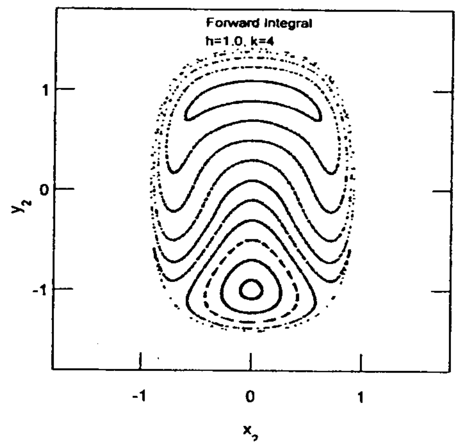


Fig. 4 Poincare map by forward integration for $h = 1$, $k = 4$

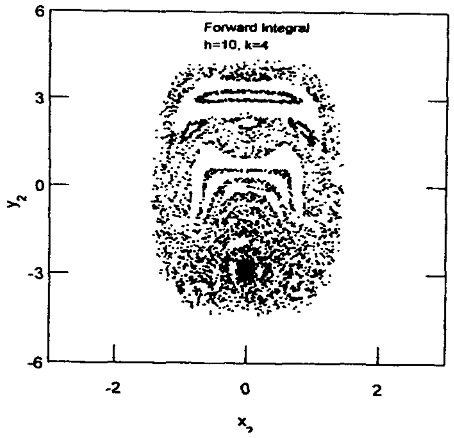


Fig. 5 Poincare map by forward integration for h =10, k =4

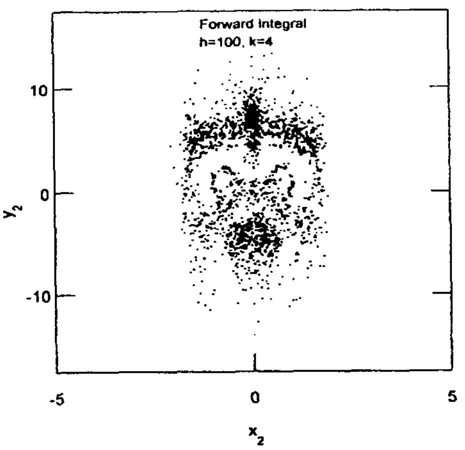


Fig. 6 Poincare map by forward integration for h =100, k=4

to notice that for small values of energy the picture is divided into invariant curves. This means that the totality of trajectories forms a one parameter family of curves which fills the region inside the bounding curve $y_1 = 0$. These invariant curves represent invariant regions in space. The fact that the space is divided into invariant regions suggests the existence of a first integral, which is independent of energy conservation $H=h$, for small values of energy. Notice from Figs. 2 ~ 6 that as the energy h increases the invariant curves seem to disintegrate. This is related to a series of bifurcation which occurs as the energy increases[9,10]. This represents a type of motion known as ergodic or chaotic motion.

In this paper we would like to find an analytic expression for the invariant curves in the Poincare Map for small values of energy. In order to do this, we look for an approximate first integral $f(x_1, x_2, y_1, y_2)$ =const. independent of Hamiltonian $H=h$. With the analytic forms of the invariant curves in the Poincare Map, we will discuss the existence, stability and bifurcation of the NNMs depending on the value of nonlinear parameter k for our nonlinear system $\mathcal{R}$. In the next section we obtain the approximate first integral independent of Hamiltonian.

3. Integral of Motion

In this section we will discuss a method that has been developed for constructing first integrals of a Hamiltonian system formally. The basic idea is to try to transform the Hamiltonian by using canonical transformation until the Hamiltonian has the form of uncoupled linear oscillators plus higher order terms.

The method is divided into two cases. The first case is the so-called nonresonance case, which means that the linearized natural frequencies are not commensurable. Birkhoff[11] considered this problem. Gustavson[12] extended this work by considering Hamiltonians with internal resonance, which means that the linearized natural frequencies are commensurable, i.e., rationally dependent. Our system $\mathcal{R}$ is in the latter case (1 to 1 resonance case).

Let us start with the system of differential equations

$$\dot{x} = \frac{\partial H}{\partial y}(x, y),$$

$$\dot{y} = \frac{\partial H}{\partial x}(x, y) \qquad (2)$$

where x and y are the generalized coordinate and momentum vectors, respectively. These equations are generated by a Hamiltonian,

$$H(x, y) = H(2)(x, y) + H(3)(x, y) + \cdots, \qquad (3)$$

a power series in x and y which is convergent in the neighborhood of the origin. The assumption that the Hamiltonian begins with quadratic terms implies that the origin $x=0$, $y=0$ is an equlibrium point of the system (2). In addition it is restricted that $H(2)(x,y)$ is a positive definite quadratic form. In general we can write $H(2)(x,y)$ in the form

$$H(2)(x,y)=\sum_{\nu=1}^{n}\frac{\alpha_\nu}{2}(x^2+y^2) \tag{4}$$

where α_ν's (positive quantities) represent the natural frequencies and n is the number of DOF. The higher order terms of H plays the role of coupling the oscillators.

For the purpose of explanation we will consider a Hamiltonian of the form

$$H=H(2)(x,y)+H(6)(x,y) \tag{5}$$

since the system $\mathcal{R}$ is in this category.

The first step is a canonical tranformation $(x,y)\rightarrow(\xi,\eta)$ generated by :

$$S(x,\eta)=x\eta+W(6)(x,\eta):$$
$$\xi=x+\frac{\partial W(6)}{\partial\eta}(x,\eta),$$
$$y=\eta+\frac{\partial W(6)}{\partial x}(x,\eta), \tag{6}$$

with $W(6)(x,\eta)$ a homogeneous polynomial of degree 6.

As a first approximation we neglect terms higher than O(6). Therefore we have

$$W(6)(x,\eta)=W(6)(\xi,\eta), \tag{7}$$

$$\frac{\partial W(6)}{\partial x}(x,\eta)=\frac{\partial W(6)}{\partial\eta}(\xi,\eta), \tag{8}$$

$$H(6)(x,y)=H(\xi,\eta), \tag{9}$$

and

$$H(2)(x,y)=\sum_{\nu=1}^{n}\frac{\alpha_\nu}{2}(\xi_\nu^2+\eta_\nu^2)\sum_{\nu=1}^{n}\alpha_\nu \times\left(\eta_\nu\frac{\partial W(6)}{\partial\xi_\nu}-\xi_\nu\frac{\partial W(6)}{\partial\eta_\nu}\right) \tag{10}$$

Thus we have a new Hamiltonian,

$$H(\xi,\eta)=H(2)(\xi,\eta)+DW(6)(\xi,\eta)+H(6)(\xi,\eta), \tag{11}$$

where

$$D=\sum_{\nu=1}^{n}\alpha_\nu(\eta_\nu\frac{\partial}{\partial\xi_\nu})-\xi_\nu(\eta_\nu\frac{\partial}{\partial\eta_\nu}). \tag{12}$$

So far, we have not specified the function $W(6)$. We now wish to choose $W(6)(\xi,\eta)$ so that

$$DW(6)(\xi,\eta)+H(6)(\xi,\eta)=0. \tag{13}$$

Then we will have from $H(\xi,\eta)=H(2)(\xi,\eta)$, n independent first integrals of the form

$$\frac{\alpha_\nu}{2}(\xi_\nu^2+\eta_\nu^2),\ \nu=1,2,3,\cdots,n. \tag{14}$$

Since both $W(s)$ and $H(s)$ are homogeneous polynomials of degree s in $2n$ variables, they can be considered as elements of a function space under the usual addition and scalar multiplication. The dimension m of the space is

$$m=\frac{(2n+s-1)!}{(2n-1)!s!}. \tag{15}$$

The monomial terms $\xi^k\eta^l$, $|k|+|l|=s$, $|i|=\sum_{\nu=1}^{n}i_\nu$ where form a basis for this vector space. Therefore, for the system $\mathcal{R}$, $H(6)$ and $W(6)$ are homogeneous polynomials consisting of at most 84 terms ($n=2$, $s=6$).

There is an isomorphism between the vector space of homogeneous polynomials of degree 6 in four variables and R^{84}. The components of the vectors in R^{84} correspond to the coefficients $C_1, C_2, C_3,\cdots, C_{84}$ of the basis vectors in the function space. Refer to Table 1. In Table 1 the notation $(\beta,\gamma,\sigma,\tau)$ represents the vector ξ_1^β, ξ_2^γ, η_1^σ, η_2^τ.

In order to solve Eq. (13) we define the second canonical transformation :

$$(\xi,\eta)\rightarrow(p,q):$$
$$\xi_\nu=\frac{1}{\sqrt{2}}(q+ip),$$
$$\eta_\nu=\frac{1}{\sqrt{2}}(q-ip), \tag{16}$$

Table 1 Isomorphism between R^{84} and homogeneous polynomials of degree 6 ($Cx_1^{\beta} x_2^{\gamma} y_1^{\sigma} y_2^{\tau}$)

C_1(0 0 0 6)	C_2(0 0 1 5)	C_3(0 0 2 4)	C_4(0 0 3 3)	C_5(0 0 4 2)	C_6(0 0 5 1)
C_7(0 0 6 0)	C_8(0 1 0 5)	C_9(0 1 1 4)	C_{10}(0 1 2 3)	C_{11}(0 1 3 2)	C_{12}(0 1 4 1)
C_{13}(0 1 5 0)	C_{14}(0 2 0 4)	C_{15}(0 2 1 3)	C_{16}(0 2 2 2)	C_{17}(0 2 3 1)	C_{18}(0 2 4 0)
C_{19}(0 3 0 3)	C_{20}(0 3 1 2)	C_{21}(0 3 2 1)	C_{22}(0 3 3 0)	C_{23}(0 4 0 2)	C_{24}(0 4 1 1)
C_{25}(0 4 2 0)	C_{26}(0 5 0 1)	C_{27}(0 5 1 0)	C_{28}(0 6 0 0)	C_{29}(1 0 0 5)	C_{30}(1 0 1 4)
C_{31}(1 0 2 3)	C_{32}(1 0 3 2)	C_{33}(1 0 4 1)	C_{34}(1 0 5 0)	C_{35}(1 1 0 4)	C_{36}(1 1 1 3)
C_{37}(1 1 2 2)	C_{38}(1 1 3 1)	C_{39}(1 1 4 0)	C_{40}(1 2 0 3)	C_{41}(1 2 1 2)	C_{42}(1 2 2 1)
C_{43}(1 2 3 0)	C_{44}(1 3 0 2)	C_{45}(1 3 1 1)	C_{46}(1 3 2 0)	C_{47}(1 4 0 1)	C_{48}(1 4 1 0)
C_{49}(1 5 0 0)	C_{50}(2 0 0 4)	C_{51}(2 0 1 3)	C_{52}(2 0 2 2)	C_{53}(2 0 3 1)	C_{54}(2 0 4 0)
C_{55}(2 1 0 3)	C_{56}(2 1 1 2)	C_{57}(2 1 2 1)	C_{58}(2 1 3 0)	C_{59}(2 2 0 2)	C_{60}(2 2 1 1)
C_{61}(2 2 2 0)	C_{62}(2 3 0 1)	C_{63}(2 3 1 0)	C_{64}(2 4 0 0)	C_{65}(3 0 0 3)	C_{66}(3 0 1 2)
C_{67}(3 0 2 1)	C_{68}(3 0 3 0)	C_{69}(3 1 0 2)	C_{70}(3 1 1 1)	C_{71}(3 1 2 0)	C_{72}(3 2 0 1)
C_{73}(3 2 1 0)	C_{74}(3 3 0 0)	C_{75}(4 0 0 2)	C_{76}(4 0 1 1)	C_{77}(4 0 2 0)	C_{78}(4 1 0 1)
C_{79}(4 1 1 0)	C_{80}(4 2 0 0)	C_{81}(5 0 0 1)	C_{82}(5 0 1 0)	C_{83}(5 1 0 0)	C_{84}(6 0 0 0)

which diagonalizes the matrix D.
Then D of Eq. (13) becomes

$$E=\sum_{\nu=1}^{n} i\alpha_\nu \left(q_\nu \frac{\partial}{\partial q_\nu} - p_\nu \frac{\partial}{\partial p_\nu}\right). \tag{17}$$

And $H(2)(\xi,\eta)$ becomes

$$K(2)(q,p)=\sum_{\nu=1}^{n} i\alpha_\nu p_\nu q_\nu, \tag{18}$$

giving us the new Hamiltonian

$$K(q,p)=K(2)(q,p) + EW(6)(q,p)+K(6)(q,p), \tag{19}$$

where $K(6)(q,p)$ is $K(6)(\xi,\eta)$ written in terms of q and p.

Let us now assume we have 2 DOF. It is shown that E is a diagonal matrix with diagonal element,

$$i\alpha_1(\beta-\sigma)+i\alpha_2(\gamma-\tau). \tag{20}$$

We now try to choose $K(6)(q,p)$ so that

$$EW(6)(q,p)+K(6)(q,p)=0. \tag{21}$$

It should be noted that E is a singular matrix. Therefore Eq. (21) will not have a solution unless $K(6)(q,p)$ lies in the range of E. However, we can write that

$$EW(6)(q,p)=-PK(6)(q,p). \tag{22}$$

where P is the projection operator onto the range of E. $PK(6)(q,p)$ may be thought as

$$PK(6)(q,p)= K(6)(q,p)-N_1(6)(q,p) - N_2(6)(q,p) \tag{23}$$

where $N_1(6)[N_2(6)]$ is the projection of $K(6)$ on the null space of E due to nonresonance [resonance]. It has been used that since E is a singular matrix, the range and null space of E are complementary. For the system $\mathcal{R}$, the bases for the $N_1(6)$ and $N_2(6)$ consist of the following vectors :

$N_1(6)$: $C_{19}\, q_2^3 p_2^3$, $C_{41}\, q_1 q_2^2 p_1 p_2^2$, $C_{57}\, q_1^2 q_2 p_1^2 p_2$, $C_{68}\, q_1^3 p_1^3$

$N_2(6)$: $C_{20}\, q_2^3 p_1 p_2^2$, $C_{21}\, q_2^3 p_1^2 p_2$, $C_{22}\, q_2^3 p_1^3$, $C_{42}\, q_1 q_2^2 p_1^2 p$, $C_{43}\, q_1 q_2^2 p_1^3$, $C_{55}\, q_1^2 q_2 p_2^3$, $C_{56}\, q_1^2 q_2 p_1 p_2^2$, $C_{58}\, q_1^2 q_2 q_1^3$, $C_{65}\, q_1^3 p_2^3$, $C_{66}\, q_1^3 p_1 p_2^2$, $C_{67}\, q_1^3 p_1^2 p_2$ (24)

Thus our new Hamiltonian is of the form

$$K(q,p)=K(2)(q,p)+N_1(6)(q,p)+N_2(q,p), \quad (25)$$

where $K(2)(q,p)$ and $N_1(6)(q,p)$ is a function of only the product terms q_ν, p_ν, but $N_2(6)(q,p)$ is not. Therefore, if in the nonresonance case, that is, $N_2(6)(q,p)=0$, then q_ν, p_ν can be chosen as the first integrals.

We can determine an approximate first integral as follows. First it is noted that $EN_1(6)(q,p)=0$ and $EN_2(6)(q,p)=0$. Also we see by inspection that $EK(2)(q,p)=0$. Therefore,

$$\begin{aligned} EK(q,p)= & \ EK(2)(q,p)+EN_1(6)(q,p) \\ & +EN_2(6)(q,p)=0 \end{aligned} \quad . \quad (26)$$

Next it can be shown that $K(2)(q,p)$ is a first integral as follows :

$$\begin{aligned} \dot{K}(2)(q,p) &= \sum_{\nu=1}^{n} i\alpha_\nu(\dot{p}_\nu q_\nu+p_\nu\dot{q}_\nu) \\ &= \sum_{\nu=1}^{n} i\alpha_\nu\left(-\frac{\partial K}{\partial q_\nu}q_\nu+\frac{\partial K}{\partial p_\nu}p_\nu\right) \quad (27) \\ &= -EK=0 \end{aligned}$$

Since $K(2)$ and K are first integrals, we also have that

$$K(q,p)-K(2)(q,p)=N_1(q,p)+N_2(q,p) \quad (28)$$

is a first integral. Since we neglect terms of degree higher than 6, we need only transform from $(q,p)\rightarrow(\xi,\eta)$, which means that we do not need to calculate $W(6)(q,p)$ to find the approximate integral.

By the above method we can obtain an approximate first integral for the system $\mathcal{R}$. The integral is

$$\begin{aligned} &\frac{15}{32}\{(x_1^2+y_1^2)^2(x_2^2+y_2^2)+(x_1^2+y_1^2)(x_2^2+y_2^2)^2\} \\ &+\frac{5}{96}(k+1)\{(x_1^2+y_1^2)^3+(x_2^2+y_2^2)^3\} \\ &+\frac{5}{16}\{(x_1x_2+y_1y_2)^2-(x_1x_2+y_2y_1)^2\} \\ &\quad (x_1^2+x_2^2+y_1^2+y_2^2) \\ &-\frac{5}{48}\{(x_1^3-3x_1y_1^2)^2(x_2^3-3x_2y_2^2)^2 \\ &\quad +(3x_1^2y_1-y_1^3)(3x_2^2y_2-y_2^3)\} \\ &-\frac{15}{16}(x_1x_2+y_1y_2)(x_1^2+y_1^2)(x_2^2+y_2^2) \\ &-\frac{5}{16}(x_1x_2+y_1y_2)\{(x_1^2+y_1^2)^2+(x_2^2+y_2^2)^2\}=G \end{aligned} \quad (29)$$

To check that Eq. (29) is indeed a first integral, we need to differentiate Eq. (29) with respect to time. We can find $\dot{G}=0$ if we neglect terms of $O(8)$ and use the fact that

$$\begin{aligned} &\dot{x}_1=y_1, && \dot{x}_2=y_2, \\ &\dot{y}_1=-x_1+O(5), && \dot{y}_2=-x_2+O(5). \end{aligned} \quad (30)$$

Therefore the expression given by Eq. (29) is indeed a first integral up to terms of O(6).

4. Invariant Curves in the Poincare Map Surface of Section

The Birkhoff-Gustavson approximate integral of Eq. (29) will be used to find the invariant curves of the Poincare Map.

Let us start with the Hamiltonian of Eq. (1) and the first integral of Eq. (29). We form the surface of section from $\{x_1=0\}\cap\{H=h\}$. Successive intersections of trajectories with this two dimensional surface of section (x_2, y_2) form invariant curves. We can analytically determine these invariant curves as follows :

$$\begin{aligned} &\frac{15}{32}\{y_1^4(x_2^2+y_2^2)+y_1^2(x_2^2+y_2^2)^2\} \\ &+\frac{5}{96}(k+1)\{y_1^6+(x_2^2+y_2^2)^3\} \\ &-\frac{5}{16}(y_1^2y_2^2-x_2^2y_1^2)(y_1^2+x_2^2+y_2^2)+\frac{5}{48}y_1^3(3x_2^2y_2-y_2^3) \\ &-\frac{15}{16}y_1^3y_2(x_2^2+y_2^2)+\frac{5}{16}y_1y_2\{y_1^4+(x_2^2+y_2^2)^2\}=G, \end{aligned} \quad (31)$$

where

$$y_1\sqrt{2h-\frac{k+1}{3}x_2^6-x_2^2-y_2^2}.$$

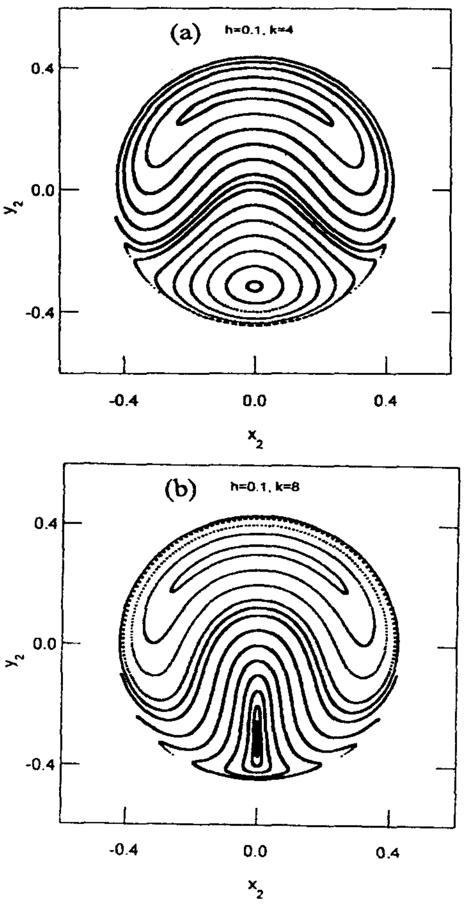
It should be noted that all NNMs appear as fixed points of the Poincare Map which lies on the y_2 axis. This follows from the fact that a NNM is a periodic motion in which both masses simultaneously pass through their equilibrium point. Thus for a NNM, it is required that

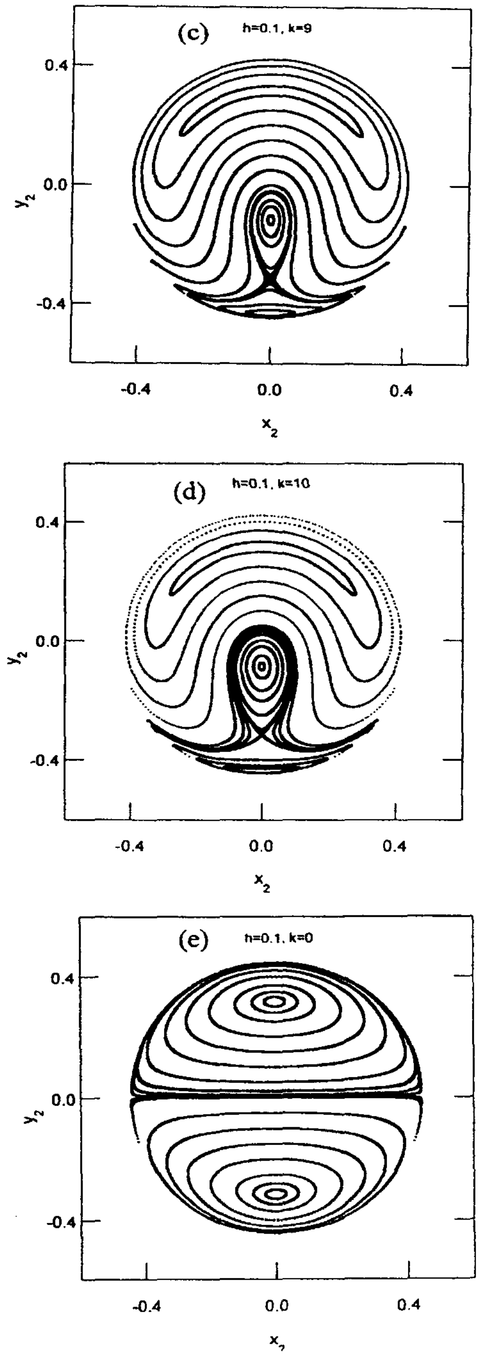
$$\begin{aligned}\frac{\partial G}{\partial y_2}(0,y_2)&=0\\&=\frac{5}{8}(y_2^2-h)\Big[2(k-4)hy_2+5y_2^2(2h-y_2^2)^{\frac{1}{2}}\\&\quad+(2h-y_2^2)^{-\frac{1}{2}}(y_2^4-2hy_2^2+4h^2)\Big] \qquad (32)\end{aligned}$$

From Eq. (32) we find that there are at least 2 NNMs, $y_2=\pm\sqrt{h}$, for any given k and h, and that in addition [ ] $=0$ in Eq. (32) may have another roots. The roots independent of k correspond to the two Similar Normal Modes (SNMs), $x_2=cx_1$, $c=\pm 1$, which means in-phase and out-of-phase modes, respectively. Next we are supposed to solve [ ] $=0$ in Eq. (32). It is assumed that the system $\mathcal{R}$ has only SNMs. For the equations of motion

$$\ddot{x}_1=-\frac{\partial V}{\partial x_1} \quad \text{and} \quad \ddot{x}_2=-\frac{\partial V}{\partial x_2}, \qquad (33)$$

where $V=\frac{1}{2}(x_1^2+x_2^2)+\frac{k}{6}(x_1^6+x_2^6)+\frac{1}{6}(x_1-x_2)^6$.





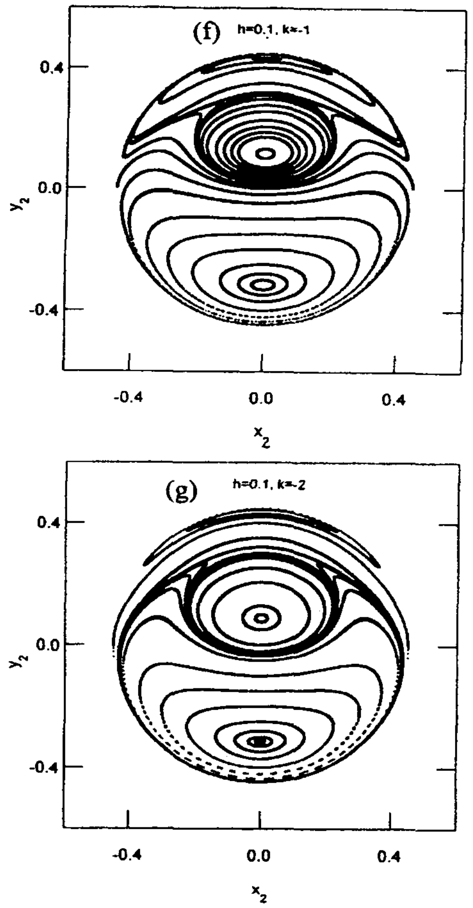


Fig. 7 Poincare map by the integral of motion for $h=0.1$: (a) $k=4$ (b) $k=8$ (c) $k=9$ (d) $k=10$ (e) $k=0$ (f) $k=-1$ (g) $k=-2$.

to have a solution of the form $x_2=cx_1$, it is necessary and sufficient[3] that

$$(c^2-1)\{c^4+(k-4)c^3+6c^2+(k-4)c+1\}=0. \quad (34)$$

From Eq. (34), when $0 \le k \le 8$ there are only 2 SNMs $x_2=\pm x_1$. An additional pair of SNMs bifurcate out of the $x_2=-x_1$, when $k>8$. Another pair of SNMs bifurcate out of the $x_2=x_1$ mode when $k<0$. We can easily see that the roots of Eqs. (32) and (34) are identical with the help of computer. Therefore it is clear that the system $\mathcal{R}$ has only SNMs, which are at most 4.

To get the Poincare Map we can plot the invariant curves on the (x_2, y_2) plane as shown in Fig. 7. By comparing Fig. 7(a) with Fig 3 for $k=4$, it is remarkable to note that they are matching very well. In Fig. 7 the nonlinear parameter k is varied. As expected from the Eq. (34), when $0 \le k \le 8$ there exist 2 NNMs, in-phase and out-of-phase. We can clearly see that the two modes are stable because they are center points in the Poincare Map. When $k<0$, as shown in Figs. 7, two additional modes bifurcate out of the in-phase mode. The bifurcation modes are stable while the original in-phase mode changes from stable to unstable. When $k>8$ two additional modes bifurcate out of the out-of-phase mode. The bifurcation modes are stable while the original out-of-phase mode changes from stable to unstable.

5. Conclusions

The dynamical structure of a nonlinear dual mass coupled oscillator, of which Hamiltonian consists of 2nd and 6th order homogeneous polinomials, was investigated by picturing the Poincare Map by direct integration of the equations of motion, and also by generating an approximation for the Poincare Map via Birkhoff-Gustavson canonical transformation for small values of energy. In particular the existence and the stability of Nonlinear Normal Mode have been studied. It is found that the system considered in this paper has 2 or 4 Similar Nonlinear Normal Modes depending on the valus of the nonlinear parameter k. The bifurcating modes enter as stable while the mode from which they bifurcated changes from stable to unstable.

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