Optimal Decomposition of Convex Structuring Elements on a Hexagonal Grid

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Abstract

In this paper, we present a new technique for the optimal local decomposition of convex structuring elements on a hexagonal grid, which are used as templates for morphological image processing. Each basis structuring element in a local decomposition is a local convex structuring element, which can be contained in hexagonal window centered at the origin. Generally, local decomposition of a structuring element results in great savings in the processing time for computing morphological operations. First, we define a convex structuring element on a hexagonal grid and formulate the necessary and sufficient conditions to decompose a convex structuring element into the set of basis convex structuring elements. Further, a cost function was defined to represent the amount of computation or execution time required for performing dilations on different computing environments and by different implementation methods. Then the decomposition condition and the cost function are applied to find the optimal local decomposition of convex structuring elements, which guarantees the minimal amount of computation for morphological operation. Simulation shows that optimal local decomposition results in great reduction in the amount of computation for morphological operations. Our technique is general and flexible since different cost functions could be used to achieve optimal local decomposition for different computing environments and implementation methods.

I. Introduction

Mathematical morphology is a powerful tool for image processing and computer vision [1], [2], [3], [4], [5]. Morphological image processing is based on the following two basic operations called dilation and erosion. In the below, A and B are subsets of E^N, where E^N is the N-dimensional Euclidean or digital space.

Dilation:

$$A \oplus B = \{ a+b \mid a \in A, b \in B \} \quad (1)$$

Erosion:

$$A \ominus B = \{ c \mid c+b \in A \text{ for every } b \in B \} \quad (2)$$

In the above, A generally represents an image and B is called a structuring element. Different image processing operations could be achieved by choosing structuring elements of appropriate sizes and shapes, and putting the dilation and erosion operations in chained sequences.

Dilation and erosion operation can be implemented by simple algorithms. But, it's often inefficient to use a large structuring element when an input image has a large amount of data. Also, some parallel architectures can only compute with structuring elements that fit inside a 3×3 window centered on the origin. Therefore, it is desirable to decompose a large structuring element into a sequence of dilation of smaller structuring elements. By the chain rule for dilations [2], if structuring element B is decomposed into B_1, B_2, ..., B_m, i.e

$$B = B_1 \oplus B_2 \oplus \ldots \oplus B_m \quad (3)$$

then the dilation of A by B can be computed by the sequence of dilations as

$$(((A \oplus B_1) \oplus B_2) \oplus \ldots) \oplus B_n \quad (4)$$

instead of a single dilation by the original structuring element. Generally, the amount of computation for the sequence of dilations as in (4) is less than that for a single dilation operation as $A \oplus B$. Similarly, the erosion of A by B can be computed by

Also, the erosion of A by B can be computed by

$$(((A \ominus B_1) \ominus B_2) \ominus \ldots) \ominus B_n\,. \quad (5)$$

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Since an erosion operation is defined as the dual operation of a dilation operation, the decomposition for erosion operations can be similarly obtained like the decomposition for dilation operations, and we will omit the discussion on erosion operation.

Haralick and Zhuang first proposed methods for decomposition of structuring elements [6]. They developed algorithms for two-point decomposition of structuring elements for Image Flow machine. Since then, many researchers have developed algorithms and techniques for the local decomposition of convex shaped structuring elements for parallel image processing architectures. The local decomposition consists of the set of local structuring elements, which can be contained in 3×3 local window centered on the origin. Xu [7] reported an algorithm to get an optimal decomposition for Cytocomputer [8]. Park [9] reported an algorithm to find an optimal decomposition for four-connected MPP type machines. Kanungo [10] proved that there exists a linear transformation between 13 primal basis digital convex polygons and the 8 edges of an input digital convex polygon. In [11] and [12], we proposed general methods for finding optimal decompositions of convex structuring elements on square grid for different types of pipelined or parallel processing architectures. In this paper, we apply the method to the decomposition convex structuring elements on a hexagonal grid. Much research effort was done to the decomposition methods for square grid. However, the decomposition on a hexagonal grid is undeveloped and untouched. The hexagonal grid has good geometrical properties [13] since a point on the hexagonal grid has the same connectivity to all of its 6 neighbor points, and the hexagonal grid is preferred by many morphologists. Our method is based on the Shephard's theorem [14], [15] for the decomposition of Euclidean convex polytopes. We derived a set of linear constraints on the lengths of edges of convex polygons on hexagonal grid, which is also defined to be a convex structuring element. This set of linear constraints will serve as the necessary and sufficient conditions to decompose a convex structuring element into a set of basis convex structuring element. We define a cost function that represents the total processing time or the cost to execute morphological operations with a set of basis structuring elements. The decomposition problem is formulated as an integer programming problem where we seek to minimize the cost (or objective) function, given the set of linear constraints. The decomposition method is applied to local decomposition of a convex structuring element, in which each basis is a local convex structuring element or a local basis in short. A local convex structuring element is a small convex structuring element contained in the hexagonal window centered at the origin. Our method results in the optimal local decomposition with respect to the cost function, which in turn, represents the execution time. For different implementation methods for dilation operations, different cost functions could be defined.

This paper is organized as follows. In Section 2, we define convex structuring element on hexagonal grid and discuss the decomposition conditions for convex structuring elements. In Section 3, we present a technique to find an optimal local decomposition of a convex structuring element. Finally, Section 4 is our conclusion.

II. Decomposition of Convex Structuring Elements on Hexagonal Grid

2.1 Hexagonal Grid

First, we define basis vectors α and β for hexagonal grid as follows.

$$\alpha = (0, 1) \tag{6}$$

$$\beta = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \tag{7}$$

The angle between α and β is $\frac{\pi}{3}$ as shown in Figure 1. The basis vectors α and β generate hexagonal grid. Any point (x, y) such that $(x, y) = x'\alpha + y'\beta$, where x' and y' are integers, is a hexagonal grid point. See Figure 2 for hexagonal grid. We will use x' and y' to denote two coordinates of hexagonal grid.

Each point on hexagonal grid has six neighbor point. The hexagonal grid has the advantage that all neighbor points of a point have the same form of contact with the point, and the distances from a point to its all neighbors are same. In a square grid, we must distinguish between strong neighbors (horizontal or vertical neighbors) and weak neighbors (diagonal neighbors), and the distances to two types of neighbors are different. [13]

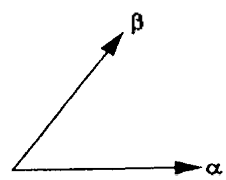

Figure 1. Basis vectors of hexagonal grid.

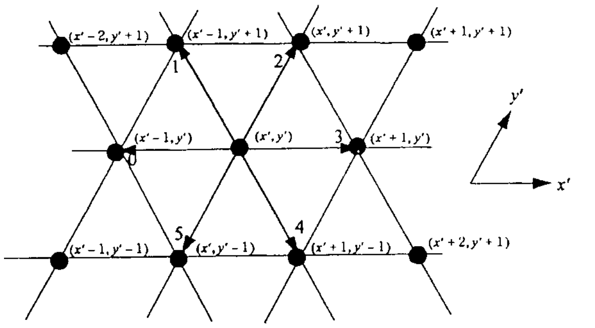

Figure 2. Hexagonal grid and chain code directions to six neighbor points.

2.2 Convex Structuring Elements on Hexagonal Grid

In our method, a convex structuring element on hexagonal grid is represented using its boundary chain codes [20]. See Figure 2 for the different chain code directions.

Definition 1: The set of hexagonal grid points A is a called a hexagonal convex structuring element, HCSE in short, if the boundary of the image can be represented as a chain code in the form of $0^{c0}1^{c1}2^{c2}3^{c3}4^{c4}5^{c5}$ and it has no holes inside.

See Figure 3 for an example HCSE P. The chain code representation of the boundary of P is $0^5 1^3 2^4 3^3 4^5 5^2$, starting with points and the lengths of edges of P are $e_h(P,0) = 5$, $e_h(P,1) = 3$, $e_h(P,2) = 4$, $e_h(P,3) = 3$, $e_h(P,4) = 5$, $e_h(P,5) = 2$.

Definition 2: Suppose the boundary of HCSE A is represented as chain code sequence $0^{c0}1^{c1}2^{c2}3^{c3}4^{c4}5^{c5}$. The length of the i th edge of HCSE A, denoted as $e(A,i)$, can be defined as the chain code length c_i in the chain code sequence.

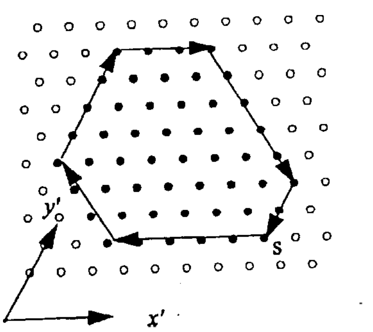

Figure 3. Example HCSE P. The chain code of the boundary of P is 051324334552 starting with points.

2.3 Decomposition of HCSEs

Definition 3: Two sets of points A and B are said to be shape-equivalent and denoted as $A \cong B$ if $A = B_t$ for a proper vector t, where B_t represents the translation of B by vector t.

Performing digital morphological operations on HCSEs is similar to performing Euclidean morphological operations on convex polygons[23]. We have the following Lemma on the relationships between the edges of original HCSE and decomposed HCSEs. We avoid rigorous proof of propositions in this paper and show illustrative examples instead. Refer [23] for proofs of Lemma and Propositions.

Lemma 1: Suppose P, Q, and R are HCSEs.

$$P \cong Q \oplus R \tag{8}$$

if and only if

$$e(P,i) = e(Q,i) + e(R,i) \ , \tag{9}$$

for $i = 0, ..., 5$.

Figure 4 shows an illustra $e(Q,i) + e(R,i) = e(P,i)$ tive example of Lemma 1. In Figure 4, for $i = 0, ..., 5$ and $P \cong Q + R$. Lemma 1 can be extended to n -terms:

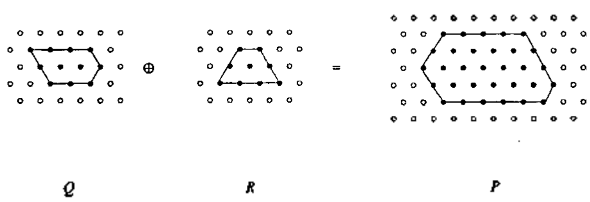

Figure 4. Illustration of Lemma 1.

Proposition 1: Suppose that P and Q_k, where $k = 1, ..., n$, are HCSEs.

$$P \cong Q_1 \oplus \ldots \oplus Q_n \tag{10}$$

if and only if for $i = 0, ..., 5$,

$$e(P,i) = \sum_{k=1}^{n} e(Q_k, i) \ . \tag{11}$$

Proposition 1 can be extended to a linear combination form as in Proposition 2. In the following, a_kQ_k represents a_k-fold dilation of Q_k.

Proposition 2 : Suppose P and Q_k, where $k = 1, ..., n$, are HCSEs.

$$P \cong a_1Q_1 \oplus \ldots \oplus a_nQ_n \tag{12}$$

if and only if, for $i = 0, ..., 5$,

$$e(P,i) = \sum_{k=0}^{n} a_k e(Q_k, i) \ . \tag{13}$$

Propositions 2 provides the necessary and sufficient condition for a HCSE P to be decomposed into a_1 Q_1s, ..., a_n Q_ks by considering only the shapes of HCSEs.

So far we haven't considered the positions of the HCSEs. In this section, we discuss the relationship between the positions of the HCSEs. In the following, $\min_{x'}(P)$ denotes the minimum x' coordinate of the region occupied by the set of points P. Similarly, $\min_{y'}(P)$ denotes the minimum y' coordinate.

Since P consists of the points which is the vector addition of every points in Q and R, we have Proposition 3.

Propositions 3: Suppose P, Q, and R are HCSEs. If

$$P = Q \oplus R \ , \tag{14}$$

then

$$\begin{aligned} \min_{x'}(P) &= \min_{x'}(Q) + \min_{x'}(R) \\ \min_{y'}(P) &= \min_{y'}(Q) + \min_{y'}(R) \end{aligned} \ . \tag{15}$$

Extending Proposition 3 to a linear combination form, we obtain Proposition 4.

Proposition 4: Suppose P and Q_ks, where $k = 1, ..., n$, are HCSEs. If

$$P = a_1Q_1 \oplus \ldots \oplus a_nQ_n \ , \tag{16}$$

then

$$\begin{aligned} \min_{x'}(P) &= \sum_{k=1}^{n} a_k \min_{x'}(Q_k) \\ \min_{y'}(P) &= \sum_{k=1}^{n} a_k \min_{y'}(Q_k) \end{aligned} \ . \tag{17}$$

From Propositions 2 and 4, the necessary and sufficient conditions for a HCSE P to be decomposed into a_1 Q_1s, ..., a_n Q_ks are as follows:

Condition 1. For $i = 0, ..., 5$,

$$e(P,i) = \sum_{k=1}^{n} a_k e(Q_k, i) \ . \tag{18}$$

Condition 2.

$$\begin{aligned} \min_{x'}(P) &= \sum_{k=1}^{n} a_k \min_{x'}(Q_k) \\ \min_{y'}(P) &= \sum_{k=1}^{n} a_k \min_{y'}(Q_k) \end{aligned} \tag{19}$$

Condition 1 considers shapes only. If Condition 1 is satisfied, P and $a_1Q_1 \oplus \ldots \oplus a_nQ_n$ are shape-equivalent. If Condition 2 is satisfied in addition to Condition 1, P and $a_1Q_1 \oplus \ldots \oplus a_nQ_n$ are located at the same position. Since Condition 1 and 2 are the necessary and sufficient conditions, the space of solution n-tuples of $(a_1, \ldots, a_n)$ for Condition 1 and 2 contains all and only n-tuples $(a_1, \ldots, a_n)$ such that a_1 Q_1s, ..., a_n Q_ns is the decomposition of P.

III. Optimal Local Decomposition of HCSEs

In this section, we develop a technique to find the optimal local decomposition of a convex structuring element into a set of local convex structuring elements using the decomposition conditions presented in Section 2. If a HCSE consists of a subset of origin point and its 6 neighbors, then it is called a local HCSE or a local basis, in short.

3.1 Local Decomposition of HCSEs

The set of all local basis contains 41 elements, and the elements are denoted as $H_0, \ldots, H_{40}$. Figure 6 shows some example local basis. Given HCSE P, the solution space of n-tuples $(a_0, \ldots, a_{40})$ satisfying Condition 1 and 2 involving the set of all the local basis represents all the n-tuples such that

$$P = a_0H_0 \oplus \ldots \oplus a_{40}H_{40} \ . \tag{20}$$

The solution space thus obtained represents only and all the feasible local decompositions of P.

3.2 Cost Function

Suppose c_k, where $k = 1, ..., n$, is the cost or time to compute dilation of an image by structuring element Q_k. Then the total cost or time to compute the sequence of dilation operations by $a_1 Q_1 s, \ldots, a_n Q_n s$ can be represented by the cost function

$$\sum_{k=1}^{n} a_k c_k \tag{21}$$

A dilation operation of input image A by structuring element B can be executed by the union of the translation of the input image as

$$\bigcup_{b \in B} A_b \tag{22}$$

where A_b represents the translation of input image by vector b[3]. On a plain von Neumann type computer which has a CPU and a main memory, a dilation operation can be implemented by first translating input image by each elements in B and then ORing the translated images. The computational complexity of the implementation method is proportional to the number of the translations of input image, which in turn, is the number of elements in structuring element B. Thus the cost function for the sequence of dilation operations by structuring elements $a_1 Q_1 s, \ldots, a_n Q_n s$ is

$$\sum_{k=1}^{n} a_k p_k \tag{23}$$

where p_k is the number of the elements in structuring element Q_k.

The implementation method presented in the above is one of the many implementation methods. Different cost functions can be obtained for different implementation methods. If the cost of the dilation operations by a implementation method is known, the cost function for the implementation method can be easily derived.

3.3 Finding the optimal solution n-tuple

Given a structuring element P, the optimal local decomposition of P is determined as

$$a_0 H_0 \oplus \ldots \oplus a_{40} H_{40}\ , \tag{24}$$

where the n-tuple $(a_0, \ldots, a_{40})$ minimizes a cost function while satisfying Condition 1 and 2 involving all the local basis. We use linear integer programming technique [21], [22] to find the optimal solution n-tuple. The set of constraints used in linear integer programming is the set of linear integer equations generated from Condition 1 and 2 involving the set of all local bases. The objective function to be minimized is the cost function obtained by the set of all local bases. By giving different cost functions, optimal decompositions for different implementation algorithms can be obtained.

3.4 Decomposition Examples

Table 1 shows the optimal local decompositions of example HCSEs in Figure 5. Figure 6 graphically depicts the local bases that appear in the local optimal decomposition in Table 1. In Table 2, we compared the cost for performing dilation by the original example HCSEs and the cost for performing dilation by the sequences of decomposed structuring elements shown in Table 1. The cost for performing dilation represents the number of translation operations of images required for executing dilation, which in turn, is the number of elements in the structuring elements involved in dilation. In case of dilation by an original structuring element, the number of translation operations is the number of the elements in the original structuring element. In case of dilation by the decomposed sequence resulted from optimal local decomposition, the number of translation operations is the number of the elements in all the bases contained in the decomposed sequence. The simulation shows more reduction in the amount of computation when using optimal local decomposition for larger structuring elements.

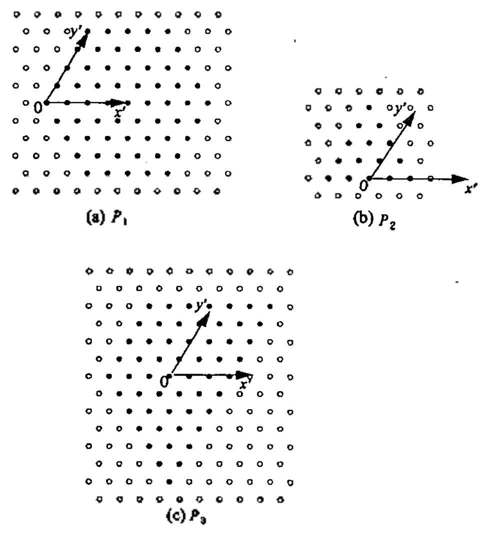

Figure 5. Example HCSEs.

IV. Conclusion

In this paper, we formulated a general framework to find an optimal local decomposition of convex structuring elements on hexagonal grid. Optimal criteria for decomposition vary widely depending on computing environments and implementation methods. By choosing different cost functions as the object function for the integer programming, our method can easily adopt a variety of optimality criteria for different computing environments and the implementation algorithms. Our method gives the optimal local decomposition which guarantees the minimal computation for a morphological operation. The optimal local decomposition results can be used for the implementation of time-critical applications such as real time video processing or military target recognition.

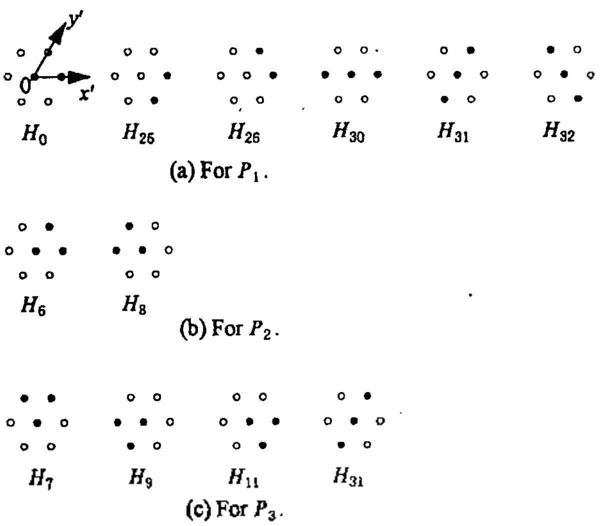

Figure 6. Local basis in the decomposition of example HCSEs.

Table 1 Decomposition results

Example HCSE	Optimal Local Decomposition
P_1	$2H_0 \oplus 2H_{25} \oplus 2H_{26} \oplus H_{30} \oplus H_{31} \oplus H_{32}$
P_2	$2H_6 \oplus 2H_8$
P_3	$3H_7 \oplus H_9 \oplus 2H_{11} \oplus 2H_{31}$

Table 2. Comparison of the numbers of the translation operation of images required for dilation by original structuring elements and by decomposed sequences of optimal local decomposition.

Example HCSE	Original Structuring Element	Decomposed Sequence by Optimal Local Decomposition
P_1	61	21
P_2	15	12
P_3	66	24

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