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Fuzzy r-derived Sets in Fuzzy Topological Spaces

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ABSTRACT

In this paper, we introduce the notions of fuzzy r-adherent points, fuzzy r-accumulation points and fuzzy r-derived sets in fuzzy topological spaces and investigate some of their properties.

1. Introduction and preliminaries

A.P. Sostak [10] introduced the fuzzy topology as an extension of Chang's fuzzy topology [1]. It has been developed in many directions [2-6,9]. In [8,11], the concepts of fuzzy adherent points, fuzzy accumulation points and fuzzy derived sets were introduced in L-fuzzy topological spaces in a different viewpoint.

In this paper, we introduce the concept of fuzzy r-adherent points, fuzzy r-accumulation points and fuzzy r-derived sets in a smooth fuzzy topological space in a view of [8]. We investigate some of their properties. We study relationship between fuzzy r-derived sets and fuzzy closure operators. We give an example of it.

In this paper, let X be a nonempty set, $I=[0,1]$ and $I_0=(0,1]$. A *fuzzy point* x_t for $t\in I_0$ is an element of I^X such that, for $y\in X$,

$$x_t(y)=\begin{cases} t \text{ if } y=x,\\ 0 \text{ if } y\neq x.\end{cases}$$

The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t\in\lambda$ iff $t\leq\lambda(x)$. A fuzzy set λ is *quasi-coincident* with μ, denoted by $\lambda\, q\, \mu$, if there exists $x\in X$ such that $\lambda(x)+\mu(x)>1$. If λ is not quasi-coincident with μ, we denote $\lambda\, \bar{q}\, \mu$. Let $f:X\to Y$ be a function, $\mu\in I^X$ and $\nu\in I^Y$. We define

$$f(\mu)(y)=\begin{cases}\sup\{\mu(x)\mid x\in f^{-1}(\{y\})\}, & \text{if } f^{-1}(\{y\})\neq\varnothing,\\ 0, & \text{if } f^{-1}(\{y\})=\varnothing,\end{cases}$$

and $f^{-1}(\nu)(x)=\nu(f(x))$ for all $x\in X$.

All the other notations and the other definitions are standard in fuzzy set theory.

Definition 1.1.[10] A function $\tau: I^X\to I$ is called a *fuzzy topology* on X if it satisfies the following conditions:

(O1) $\tau(\tilde{0})=\tau(\tilde{1})=1$, where $\tilde{0}(x)=0$ and $\tilde{1}(x)=1$ for all $x\in X$.
(O2) $\tau(\mu_1\wedge\mu_2)\geq\tau(\mu_1)\wedge\tau(\mu_2)$, for any $\mu_1,\mu_2\in I^X$.
(O3) $\tau(\bigvee_{i\in\Gamma}\mu_i)\geq\bigwedge_{i\in\Gamma}\tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma}\subset I^X$.

The pair (X,τ) is called a *fuzzy topological space*.

Let τ_1 and τ_2 be fuzzy topologies on X. We say τ_2 is finer than τ_1 (τ_1 is *coarser* than τ_2), denoted by $\tau_1\leq\tau_2$, if $\tau_1(\lambda)\leq\tau_2(\lambda)$ for all $\lambda\in I^X$.

Theorem 1.2.[2] Let (X,τ) be a fuzzy topological space. For each $r\in I_0$ and $\lambda\in I^X$, we define a fuzzy closure operator $C_\tau: I^X\times I_0\to I^X$ as follows:

$$C_\tau(\lambda,r)=\wedge\{\rho\mid\lambda\leq\rho,\ \tau(\tilde{1}-\rho)\geq r\}.$$

For $\lambda,\mu\in I^X$ and $r,s\in I_0$, it satisfies the following properties:

(C1) $C_\tau(\tilde{0},r)=\tilde{0}$.
(C2) $\lambda\leq C_\tau(\lambda,r)$.
(C3) $C_\tau(\lambda,r)\vee C_\tau(\mu,r)=C_\tau(\lambda\vee\mu,r)$.
(C4) $C_\tau(\lambda,r)\leq C_\tau(\lambda,s)$, if $r\leq s$.
(C5) $C_\tau(C_\tau(\lambda,r),r)=C_\tau(\lambda,r)$.

Lemma 1.3.[8] Let $f:X\to Y$ be a function. For $\lambda,\mu,\rho\in I^X$ and $\nu\in I^Y$, the following properties hold:

(1) $\lambda\,\bar{q}\,\mu$ iff $\lambda\leq\tilde{1}-\mu$.
(2) If $\lambda\leq\mu$ and $\rho\, q\,\lambda$, then $\rho\, q\,\mu$.
(3) If $\lambda\, q\,\mu$, then $f(\lambda)\, q\, f(\mu)$.
(4) $\nu\, q\, f(\mu)$ iff $f^{-1}(\nu)\, q\,\mu$.
(5) $\nu\geq f(f^{-1}(\nu))$ with equality if f is surjective.
(6) $\mu\leq f^{-1}(f(\mu))$ with equality if f is injective.
(7) $f^{-1}(\tilde{1}-\nu)=\tilde{1}-f^{-1}(\nu)$.

2. Derived sets in fuzzy topological spaces

In [8], from the definition of the quasi-difference of

L-fuzzy sets, we can redefine it as taking $L = I$.

Definition 2.1. Let λ, $\mu \in I^X$. Define the *fuzzy quasi-difference* of λ and μ, denoted by $\lambda \backslash \mu$, as

$$(\lambda \backslash \mu)(x) = \begin{cases} \lambda(x), & \text{if } \mu(x) = 0, \\ 0, & \text{if } \lambda(x) \geq \mu(x) > 0, \\ \lambda(x), & \text{if } \lambda(x) < \mu(x). \end{cases}$$

Lemma 2.2. For λ, $\mu \in I^X$ and $x_t \in Pt(X)$, the following properties hold:

(1) $\lambda \backslash \mu \leq \lambda$.

(2) $\lambda \backslash \tilde{0} = \lambda$.

(3) If $x_t \notin \lambda$, then $\lambda \backslash x_t = \lambda$ If $x_t \in \lambda$, then, for each $y \in X$,

$$(\lambda \backslash x_t)(y) = \begin{cases} \lambda(y), & \text{if } y \neq x, \\ 0, & \text{if } y = x. \end{cases}$$

(4) $\chi_A \backslash \chi_B = \chi_{A \cap B^c}$ where χ is a characteristic function.

(5) $(\lambda \vee \mu) \backslash x_t \leq (\lambda \backslash x_t) \vee (\mu \backslash x_t)$.

(6) If $f: X \to Y$ is injective, then $f(\lambda \backslash \mu) = f(\lambda) \backslash f(\mu)$.

Proof. We prove (5) and (6). Others are easily proved.

(5) If $y \neq x$, by (3), it is trivial. Let $y = x$. If $t > (\lambda \vee \mu)(x)$, then $t > \lambda(x)$ and $t > \mu(x)$. Hence

$((\lambda \vee \mu) \backslash x_t)(x) = (\lambda \vee \mu)(x) = (\lambda \backslash x_t)(x) \vee (\mu \backslash x_t)(x)$.

If $t \leq (\lambda \vee \mu)(x)$, then $t \leq \lambda(x)$ or $t \leq \mu(x)$. Hence

$0 = ((\lambda \vee \mu) \backslash x_t)(x) \leq (\lambda \backslash x_t)(x) \vee (\mu \backslash x_t)(x)$.

(6) We show that $f(\lambda \backslash \mu)(y) = (f(\lambda) \backslash f(\mu))(y)$ for each $y \in Y$.

If $f^{-1}(\{y\}) = \emptyset$, it is trivial.

If $f^{-1}(\{y\}) \neq \emptyset$, there exists a unique $x \in f^{-1}(\{y\})$ because f is injective. Since $f(\lambda)(y) = \lambda(x)$ and $f(\mu)(y) = \mu(x)$, we have

$$(f(\lambda) \backslash f(\mu))(y) = \begin{cases} \lambda(x), & \text{if } \mu(x) = 0, \\ 0, & \text{if } \lambda(x) \geq \mu(x) > 0, \\ \lambda(x), & \text{if } \lambda(x) < \mu(x). \end{cases}$$

It follows $(f(\lambda) \backslash f(\mu))(y) = (\lambda \backslash \mu)(x) = f(\lambda \backslash \mu)(y)$. □

Remark 2.3. In Lemma 2.2 (4), if we interpret χ_A as A in an ordinary set, then $\chi_A \backslash \chi_B$ just becomes $A \backslash B$ in an ordinary set.

Remark 2.4. Let $X = \{a, b\}$ be a set. Define μ, $\rho \in I^X$ as follows:

$\mu(a) = 0.3$, $\mu(b) = 0.4$, $\rho(a) = 0.6$, $\rho(b) = 0.2$.

Since

$((\mu \vee \rho) \backslash a_{0.5}) = b_{0.4}$,

$(\mu \backslash a_{0.5}) = \mu$ and $(\rho \backslash a_{0.5}) = b_{0.2}$,

we have $((\mu \vee \rho) \backslash a_{0.5}) \neq (\mu \backslash a_{0.5}) \vee (\rho \backslash a_{0.5})$. □

Notation. Let (X, τ) be a fuzzy topological space. For each $x_t \in Pt(X)$ and $r \in I_0$, we denote $N(x_t, r) = \{\mu \in I^X \mid x_t\, q\, \mu$ and $\tau(\mu) \geq r\}$.

Definition 2.5. Let (X, τ) be a fuzzy topological space, $\lambda \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. x_t is called an *fuzzy r-adherent point* of λ if for every $\mu \in N(x_t, r)$ we have $\lambda\, q\, \mu$. x_t is called a *fuzzy r-accumulation point* of λ if for every $\mu \in N(x_t, r)$, we have $\mu\, q(\lambda \backslash x_t)$. Define the *fuzzy r-derived set* of λ, denote by $D_\tau(\lambda, r)$, as

$D_\tau(\lambda, r) = \vee \{x_t \in Pt(X) \mid x_t$ is a fuzzy r-accumulation point of $\lambda\}$.

Theorem 2.6. Let (X, τ) be a fuzzy topological space. For each $\lambda \in I^X$ and $r \in I_0$, we have

$C_\tau(\lambda, r) = \vee \{x_t \in Pt(X) \mid x_t$ is a fuzzy r-adherent point of $\lambda\}$.

Proof. Put $\rho = \vee \{x_t \in Pt(X) \mid x_t$ is a fuzzy r-adherent point of $\lambda\}$.

Suppose $C_\tau(\lambda, r) \not\leq \rho$. Then there exist $x \in X$ and $t \in I_0$ such that

$C_\tau(\lambda, r)(x) \geq t > \rho(x)$.

Since $\rho(x) < t$, x_t is not a fuzzy r-adherent point of λ. Hence there exists $\mu \in N(x_t, r)$ such that $\lambda\, \bar{q}\, \mu$, that is, $\lambda \leq \tilde{1} - \mu$. Then $\lambda \leq C_\tau(\lambda, r) \leq \tilde{1} - \mu$. Since $x_t\, q\, \mu$ and $\mu \leq \tilde{1} - C_\tau(\lambda, r)$, we have $x_t\, q\, \tilde{1} - C_\tau(\lambda, r)$. It implies $t > C_\tau(\lambda, r)(x)$. It is a contradiction. Hence $C_\tau(\lambda, r) \leq \rho$.

Suppose $C_\tau(\lambda, r) \ngeq \rho$. Then there exists a fuzzy r-adherent point $x_t \in Pt(X)$ of λ such that

$C_\tau(\lambda, r)(x) < t \leq \rho(x)$.

Since $C_\tau(\lambda, r)(x) < t$, then $x_t\, q\, \tilde{1} - C_\tau(\lambda, r)$ and $\tau(\tilde{1} - C_\tau(\lambda, r)) \geq r$ from the definition of C_τ and (O3) of Definition 1.1. Moreover, since $\lambda \leq C_\tau(\lambda, r)$, by Lemma 1.3(1),

$\lambda\, \bar{q}\, \tilde{1} - C_\tau(\lambda, r)$.

So x_t is not a fuzzy r-adherent point of λ. It is a contradiction. Hence

$C_\tau(\lambda, r) \geq \rho$. □

Theorem 2.7. Let (X, τ) be a fuzzy topological space. For λ, $\mu \in I^X$ and $r, s \in I_0$, the following properties hold:

(1) $D_\tau(\lambda, r) \leq C_\tau(\lambda, r)$.
(2) $C_\tau(\lambda, r) = \lambda \vee D_\tau(\lambda, r)$.
(3) $C_\tau(\lambda, r) = \lambda$ iff $D_\tau(\lambda, r) \leq \lambda$.
(4) If $r \leq s$, then $D_\tau(\lambda, r) \leq D_\tau(\lambda, s)$.
(5) $D_\tau(\lambda \vee \mu, r) \leq D_\tau(\lambda, r) \vee D_\tau(\mu, r)$.

Proof. (1) It is clear because every fuzzy r-accumulation point of λ is a fuzzy r-adherent point of λ.
(2) Since $\lambda \leq C_\tau(\lambda, r)$ and $D_\tau(\lambda, r) \leq C_\tau(\lambda, r)$, we have $\lambda \vee D_\tau(\lambda, r) \leq C_\tau(\lambda, r)$.
Conversely, suppose $C_\tau(\lambda, r) \not\leq \lambda \vee D_\tau(\lambda, r)$. Then there exist $x \in X$ and $t \in I_0$ such that
$C_\tau(\lambda, r)(x) > t > \lambda(x) \vee D_\tau(\lambda, r)(x)$.

Since $\lambda(x) \vee D_\tau(\lambda, r)(x) < t$, then $x_t \notin \lambda$ and x_t is not a fuzzy r-accumulation point of λ. Hence there exists $\mu \in N(x_t, r)$ such that
$\mu \bar{q} (\lambda \setminus x_t)$.

Since $x_t \notin \lambda$, by Lemma 2.2(3), we have $(\lambda \setminus x_t) = \lambda$. Thus $\mu \bar{q} \lambda$. It implies $\lambda \leq C_\tau(\lambda, r) \leq \tilde{1} - \mu$. Since $x_t\, q\, \mu$, that is, $(\tilde{1} - \mu)(x) < t$,
$C_\tau(\lambda, r)(x) \leq (\tilde{1} - \mu)(x) < t$.
It is a contradiction. Hence $C_\tau(\lambda, r)) \leq \lambda \vee D_\tau(\lambda, r)$.
(3) It follows immediately from (2).
(4) Suppose $D_\tau(\lambda, r) \not\leq D_\tau(\lambda, s)$. Then there exists a fuzzy r-accumulation point $x_t \in Pt(X)$ of λ such that
$D_\tau(\lambda, r)(x) \geq t > D_\tau(\lambda, s)(x)$.
Since $D_\tau(\lambda, s)(x) < t$, then x_t is not a fuzzy s-accumulation point of λ. Hence there exists $\rho \in N(x_t, s)$ such that $\rho \bar{q} (\lambda \setminus x_t)$. Since $\tau(\rho) \geq s \geq r$. Then x_t is not a fuzzy r-accumulation point of λ. It is a contradiction.
(5) Suppose $D_\tau(\lambda \vee \mu, r) \not\leq D_\tau(\lambda, r) \vee D_\tau(\mu, r)$. Then there exists a fuzzy r-accumulation point $x_t \in Pt(X)$ of $\lambda \vee \mu$ such that
$D_\tau(\lambda \vee \mu, r)(x) \geq t > D_\tau(\lambda, r)(x) \vee D_\tau(\mu, r)(x)$.

Since $D_\tau(\lambda, r)(x) < t$ and $D_\tau(\mu, r)(x) < t$, then x_t is not a fuzzy r-accumulation point of either λ or μ. Hence there exist $\rho_1, \rho_2 \in N(x_t, r)$ such that
$\rho_1 \bar{q} (\lambda \setminus x_t)$ and $\rho_2 \bar{q} (\mu \setminus x_t)$.
Take $\rho = \rho_1 \wedge \rho_2$. Then $x_t\, q\, \rho_1 \wedge \rho_2$ and $\tau(\rho_1 \wedge \rho_2) \geq r$, that is, $\rho_1 \wedge \rho_2 \in N(x_t, r)$. Moreover,

$$\begin{aligned}(\lambda \vee \mu) \setminus x_t &\leq (\lambda \setminus x_t) \vee (\mu \setminus x_t) \quad \text{(by Lemma 2.2 (5))}\\ &\leq (\tilde{1} - \rho_1) \vee (\tilde{1} - \rho_2) \quad \text{(by Lemma 1.3 (1))}\\ &= \tilde{1} - (\rho_1 \wedge \rho_2)\\ &= \tilde{1} - \rho.\end{aligned}$$

Hence $\rho \bar{q} ((\lambda \vee \mu) \setminus x_t)$. Thus x_t is not a fuzzy r-accumulation point of $\lambda \vee \mu$. It is a contradiction. Therefore $D_\tau(\lambda \vee \mu, r) \leq D_\tau(\lambda, r) \vee D_\tau(\lambda, r)$. □

The following example is that $\lambda \leq \mu$ but $D_\tau(\lambda, r) \not\leq D_\tau(\mu, r)$.

Example 2.8. Let $X = \{a, b\}$ be set. Define $\mu, \rho \in I^X$ as follows:
$\mu(a) = 0.3$, $\mu(b) = 0.4$, $\rho(a) = 0.6$, $\rho(b) = 0.2$.

We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \tilde{0} \text{ or } \tilde{1},\\ \frac{1}{2}, & \text{if } \lambda = \mu,\\ \frac{2}{3}, & \text{if } \lambda = \rho,\\ \frac{3}{4}, & \text{if } \lambda = \mu \wedge \rho,\\ \frac{1}{2}, & \text{if } \lambda = \mu \vee \rho,\\ 0, & \text{otherwise.} \end{cases}$$

For $a_1 \in Pt(X)$, let a_t with $0 < t \leq 1$ and $r = 1/2$. Since $(a_1 \setminus a_t) = \tilde{0}$, for all $\lambda \in N(a_t, 1/2)$, we have
$\lambda\, \bar{q}\, ((a_1 \setminus a_t) = \tilde{0})$.

Hence a_t is not a fuzzy 1/2-accumulation point of a_1.
Let b_s with $0 < s \leq 1$ and $r = 1/2$. Since $(a_1 \setminus b_s) = a_1$, for all $\lambda \in N(b_s, 1/2)$, we have
$\lambda\, q ((a_1 \setminus b_s) = a_1)$.

Hence b_s is a fuzzy 1/2-accumulation point of a_1. Therefore
$D_\tau(a_1, 1/2) = b_1$.

For $a_{0.8} \in Pt(X)$. let a_t with $0.8 < t \leq 1$ and $r = 1/2$. Since $(a_{0.8} \setminus a_t) = a_{0.8}$, for all $\lambda \in N(a_t, 1/2)$, we have
$\lambda\, q\, ((a_{0.8} \setminus a_t) = a_{0.8})$.
Hence a_t is a fuzzy 1/2-accumulation point of $a_{0.8}$.
Let b_s with $0 < s \leq 1$ and $r = 1/2$. Since $(a_{0.8} \setminus b_s) = a_{0.8}$, for all $\lambda \in N(b_s, 1/2)$, we have
$\lambda\, q\, ((a_{0.8} \setminus b_s) = a_{0.8})$.

Thus b_s is a fuzzy 1/2-accumulation point of $a_{0.8}$. Therefore
$D_\tau(a_{0.8}, 1/2) = \tilde{1}$.

Hence $a_{0.8} \leq a_1$ but $D_\tau(a_{0.8}, 1/2) \not\leq D_\tau(a_1, 1/2)$. Furthermore,
$D_\tau(a_{0.8} \vee a_1, 1/2) \neq D_\tau(a_{0.8}, 1/2) \vee D_\tau(a_1, 1/2)$. □

Example 2.9. In Example 2.8, let $\nu \in I^X$ as follows:
$\nu(a) = 0.7$, $\nu(b) = 0.3$.
From Theorem 1.2, we obtain:

$$C_\tau(\nu,r)=\begin{cases}\tilde{1}-\mu, & \text{if } 0<r\le\frac{1}{2},\\ \tilde{1}-(\mu\wedge\rho), & \text{if } \frac{1}{2}<r\le\frac{3}{4},\\ \tilde{1}, & \text{if } \frac{3}{4}<r\le 1.\end{cases}$$

We can show whether it is a fuzzy accumulation point or not from the following statements (1)-(11):

(1) If a_t with $0.7<t\le 1$ and $0<r\le 3/4$, there exists $\mu\wedge\rho$ with $a_t\, q(\mu\wedge\rho)$ and $\tau(\mu\wedge\rho)=3/4$ such that
$(\mu\wedge\rho)\bar{q}((\nu\setminus a_t)=\nu)$.
Hence a_t is not a fuzzy r-accumulation point of ν.

(2) If a_t with $0.7<t\le 1$ and $3/4<r\le 1$, there only exists $\tilde{1}$ with $a_t\ q\ \tilde{1}$ and $\tau(\tilde{1})=1$ such that
$\tilde{1}\ q((\nu\setminus a_t)=\nu)$.
Hence a_t is a fuzzy r-accumulation point of ν.

(3) If a_t with $0.4<t\le 0.7$ and $0<r\le 2/3$, there exists ρ with $a_t q\,\rho$ and $\tau(\rho)=2/3$ such that
$\rho\ \bar{q}((\nu\setminus a_t)=b_{0.3})$.
Hence a_t is not a fuzzy r-accumulation point of ν.

(4) If a_t with $0.4<t\le 0.7$ and $2/3<r\le 1$, there only exists $\tilde{1}$ with $a_t\ q\ \tilde{1}$ and $\tau(\tilde{1})=1$ such that
$\tilde{1}\, q((\nu\setminus a_t)=b_{0.3})$.
Hence a_t is a fuzzy r-accumulation point of ν.

(5) If a_t with $0<t\le 0.4$ and $0<r\le 1$, there only exists $\tilde{1}$ with $a_t\ q\ \tilde{1}$ and $\tau(\tilde{1})=1$ such that
$\tilde{1}\, q((\nu\setminus a_t)=b_{0.3})$.
Hence a_t is a fuzzy r-accumulation point of ν.

(6) If b_s with $0.8<t\le 1$ and $0<r\le 3/4$, there exists $\mu\wedge\rho$ with $b_s q\,\mu\wedge\rho$ and $\tau(\mu\wedge\rho)=3/4$ such that
$(\mu\wedge\rho)\bar{q}((\nu\setminus b_s)=\nu)$.
Hence b_s is not a fuzzy r-accumulation point of ν.

(7) If b_s with $0.8<s\le 1$ and $3/4<r\le 1$, there only exists $\tilde{1}$ such that $b_s q\ \tilde{1}$ and $\tau(\tilde{1})=1$
$\tilde{1}q((\nu\setminus b_s)=\nu)$.
Hence b_s is a fuzzy r-accumulation point of ν.

(8) If b_s with $0.6<s\le 0.8$ and $0<r\le 1/2$, there exists μ with $b_s q\,\mu$ and $\tau(\mu)=1/2$ such that
$\mu\ \bar{q}((\nu\setminus b_s)=\nu)$.
Hence b_s is not a fuzzy r-accumulation point of ν.

(9) If b_s with $0.6<s\le 0.8$ and $1/2<r\le 1$, there only exists $\tilde{1}$ with $b_s q\,\tilde{1}$ and $\tau(\tilde{1})=1$ such that
$\tilde{1}q((\nu\setminus b_s)=\nu)$.
Hence b_s is a fuzzy r-accumulation point of ν.

(10) If b_s with $0.3<s\le 0.6$ and $0<r\le 1$, there only exists $\tilde{1}$ with $a_t q\,\tilde{1}$ and $\tau(\tilde{1})=1$ such that
$\tilde{1}q((\nu\setminus b_s)=\nu)$.
Hence b_s is a fuzzy r-accumulation point of ν.

(11) If b_s with $0<s\le 0.3$ and $0<r\le 1$, there only exists $\tilde{1}$ with $a_t q\,\tilde{1}$ and $\tau(\tilde{1})=1$ such that
$\tilde{1}q((\nu\setminus b_s)=a_{0.7})$.
Hence b_s is a fuzzy r-accumulation point of ν.

From the above statements (1)-(11), we find fuzzy r-derived sets:

$$D_\tau(\nu,r)=\begin{cases}D_\tau(\nu,r)(a)=0.4, D_\tau(\nu,r)(b)=0.6, & \text{if } 0<r\le\frac{1}{2},\\ D_\tau(\nu,r)(a)=0.4, D_\tau(\nu,r)(b)=0.8, & \text{if } \frac{1}{2}<r\le\frac{2}{3},\\ \tilde{1}-(\mu\wedge\rho), & \text{if } \frac{2}{3}<r\le\frac{3}{4},\\ 1, & \text{if } \frac{3}{4}<r\le 1.\end{cases}$$

We easily show that $C_\tau(\nu, r)=\nu\vee D_\tau(\nu, r)$.

In general, for $x_t\in D_\tau(\lambda, r)$, x_t need not be a fuzzy r-accumulation point of λ, that is, it does not satisfy Multiple-Choice Principle (see [9]). Consider the fuzzy 2/3-derived set of $a_{0.5}$, that is, $D_\tau(a_{0.5}, 2/3)$. By a similar method, we obtain

$D_\tau(a_{0.5}, 2/3)(a)=0.7,\ D_\tau(a_{0.5}, 2/3)(b)=0.8.$

We know that $a_{0.4}\in D_\tau(a_{0.5}, 2/3)$ but $a_{0.4}$ is not a fuzzy 2/3-accumulation point of $a_{0.5}$. Because there only exists $\tilde{1}$ with $a_{0.4}q\,\tilde{1}$ and $\tau(\tilde{1})=1$ but
$\tilde{1}\ \bar{q}((a_{0.5}\setminus a_{0.4})=\tilde{0})$. □

Theorem 2.10. Let τ_1 and τ_2 be fuzzy topologies on X. Let τ_1 is coarser than τ_2. Then, for each $\lambda\in I^X$ and $r\in I_0$, we have:

(1) $C_{\tau_2}(\lambda, r)\le C_{\tau_1}(\lambda, r)$.

(2) $D_{\tau_2}(\lambda, r)\le D_{\tau_1}(\lambda, r)$.

Proof. (1) For each $\lambda\in I^X$, $r\in I_0$, we have the following:

$$\begin{aligned}C_{\tau_2}(\lambda, r)&=\wedge\{\mu\mid\mu\ge\lambda,\ \tau_2(\tilde{1}-\mu)\ge r\}\\ &\le\wedge\{\mu\mid\mu\ge\lambda,\ \tau_1(\tilde{1}-\mu)\ge r\}(\tau_1\le\tau_2)\\ &=C_{\tau_1}(\lambda, r).\end{aligned}$$

(2) Suppose $D_{\tau_2}(\lambda, r)\not\le D_{\tau_1}(\lambda, r)$. Then there exists a fuzzy r-accumulation point $x_t\in Pt(X)$ of λ on (X, τ_2) such that

$D_{\tau_2}(\lambda, r)(x)\ge t> D_{\tau_1}(\lambda, r)(x)$.

Since $D_{\tau_1}(\lambda, r)(x)<t$, then x_t is not a fuzzy r-accumulation point of λ on (X, τ_1). Hence there exists $\rho\in I^X$ with $x_t q\,\rho$ and $\tau_1(\rho)\ge r$ such that
$\rho\,\bar{q}(\lambda\setminus x_t)$.

Since $\tau_2(\rho)\ge\tau_1(\rho)\ge r$. Then x_t is not a fuzzy r-accumulation point of λ on (X, τ_2). It is a contradiction. □

Definition 2.11. Let (X, τ_1) and (Y, τ_2) be fuzzy topological spaces. A function $f: (X, \tau_1) \to (Y, \tau_2)$ is called *fuzzy continuous* if $\tau_2(v) \leq \tau_1(f^{-1}(v))$ for all $v \in I^Y$.

Theorem 2.11. Let (X, τ_1) and (Y, τ_1) be fuzzy topological spaces. Let $f: X \to Y$ be an injective function. The following statements are equivalent.

(1) $f: (X, \tau_1) \to (Y, \tau_2)$ is fuzzy continuous.

(2) $f(D_{\tau_1}(\lambda, r)) \leq D_{\tau_2}(f(\lambda), r)$ for each $\lambda \in I^X$ and $r \in I_0$.

(3) $f(C_{\tau_1}(\lambda, r)) \leq C_{\tau_2}(f(\lambda), r)$ for each $\lambda \in I^X$ and $r \in I_0$.

Proof. (1) $\Rightarrow$ (2) Suppose there exist λ and $r \in I_0$ such that

$f(D_{\tau_1}(\lambda, r)) \not\leq D_{\tau_2}(f(\lambda), r)$.

Then there exists a $y \in Y$ such that

$f(D_{\tau_1}(\lambda, r))(y) > f(D_{\tau_2}(f(\lambda), r)(y)$.

Since f is injective, there exists a unique $x \in f^{-1}(y)$ such that

$f(D_{\tau_1}(\lambda, r))(y) \geq D_{\tau_1}(\lambda, r)(x) > D_{\tau_2}(f(\lambda), r)(y)$.

There exists a fuzzy r-accumulation point x_t of λ on (X, τ_1) such that

$D_{\tau_1}(\lambda, r)(x) \geq t > D_{\tau_2}(f(\lambda), r)(f(x))$.

Therefore $f(x)_t = f(x_t)$ is not a fuzzy r-accumulation point of $f(\lambda)$. Hence there exist $\rho \in I^X$ with $f(x_t) q\, \rho$ and $\tau_2(\rho) \geq r$ such that

$\rho\ \bar{q}(f(\lambda) \setminus f(x_t))$.

Since f is injective, by Lemma 2.1(6),

$f(\lambda) \setminus f(x_t) = f(\lambda \setminus x_t)$.

Since $f: (X, \tau_1) \to (Y, \tau_2)\}$ is fuzzy continuous, then

$\tau_1(f^{-1}(\rho)) \geq \tau_2(\rho) \geq r$.

From Lemma 1.3(4), we have

$f(x_t) q\, \rho \Rightarrow x_t q f^{-1}(\rho)$,

$\rho\, \bar{q} f(\lambda \setminus x_t) \Rightarrow f^{-1}(\rho)\, \bar{q}(\lambda \setminus x_t)$.

Hence x_t is not a fuzzy r-accumulation point of λ. It is a contradiction. Hence $f(D_{\tau_1}(\lambda, r)) \leq D_{\tau_2}(f(\lambda), r)$ for each $\lambda \in I^X$ and $r \in I_0$.

(2) $\Rightarrow$ (3) It is easily proved from the following: for each $\lambda \in I^X$ and $r \in I_0$,

$$\begin{aligned} f(C_{\tau_1}(\lambda, r)) &= f(\lambda \vee D_{\tau_1}(\lambda, r)) \quad \text{(By Theorem 2.7(2))} \\ &= f(\lambda) \vee f(D_{\tau_1}(\lambda, r)) \\ &\leq f(\lambda) \vee D_{\tau_2}(f(\lambda), r) \quad \text{(by (2))} \\ &= C_{\tau_2}(f(\lambda), r). \end{aligned}$$

(3) $\Rightarrow$ (1) Suppose that there exists $v \in I^Y$ such that

$\tau_2(v) > \tau_1(f^{-1}(v))$

Then there exists $r \in I_0$ such that $\tau_2(v) \geq r > \tau_1(f^{-1}(v))$ and hence $C_{\tau_2}(\tilde{1} - v, r) = \tilde{1} - v$.

On the other hand, we have

$$\begin{aligned} f(C_{\tau_1}(\tilde{1} - f^{-1}(v), r)) &= f(C_{\tau_1}(f^{-1}(\tilde{1} - v), r)) \\ &\qquad \text{(by Lemma 1.3(7))} \\ &\leq C_{\tau_2}(f(f^{-1}(\tilde{1} - v)), r) \quad \text{(by (3))} \end{aligned}$$

(by Lemma 1.3(5) and theorem 1.2(3))

$\leq C_{\tau_2}(\tilde{1} - v, r) = \tilde{1} - v$.

Using Lemma 1.3(6), we have

$$\begin{aligned} C_{\tau_1}(\tilde{1} - f^{-1}(v), r) &\leq f^{-1}(f(C_{\tau_1}(\tilde{1} - f^{-1}(v) r))) \\ &\leq \tilde{1} - f^{-1}(v). \end{aligned}$$

Hence, by (C2) of Theorem 1.2,

$C_{\tau_1}(\tilde{1} - f^{-1}(v), r) = \tilde{1} - f^{-1}(v)$.

From the definition of C_{τ_1} and (O3) of Definition 1.1, we have $\tau_1(f^{-1}(v)) \geq r$. It is a contradiction. □

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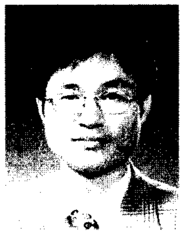

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