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Some Properties of Product Smooth Fuzzy Topological Spaces

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ABSTRACT

We will investigate some properties of product smooth fuzzy topological spaces. We will show that a projection map in product smooth fuzzy topological spaces need not be a fuzzy open map. Furthermore, a slice need not be homeomorphic to the coordinate space which is parallel to it.

1. Introduction

A.P. Sostak [10,11] introduced the smooth fuzzy topology as an extension of Chang's fuzzy topology [1]. It has been developed in many directions [2-9]. In [5,8], it was introduced the existence of initial smooth fuzzy topological spaces as a generalization of subspaces and products of smooth fuzzy topological spaces.

In this paper, we investigate some properties of product smooth fuzzy topological spaces. In general topology, every projection map is a open map and every slice is homeomorphic to the coordinate space which is parallel to it. We show that the above facts are not satisfied in smooth fuzzy topological spaces.

But they are satisfied in stratified smooth fuzzy topological spaces as an extension of Lowen's fuzzy topology [7].

In this paper, let X be a nonempty set, $I=[0,1]$ and $I_0=(0,1]$. For $\alpha\in I$, $\bar{\alpha}(x)=\alpha$ for all $x\in X$. All the other notations and the other definitions are standard in fuzzy set theory.

2. Preliminaries

Definition 2.1[8,10] A function $\tau: I^X\to I$ is called a *smooth fuzzy topology* on X if it satisfies the following conditions:

(O1) $\tau(\bar{0})=\tau(\bar{1})=1$.

(O2) $\tau(\mu_1\wedge\mu_2)\geq\tau(\mu_1)\wedge\tau(\mu_2)$ for each $\mu_1,\ \mu_2\in I^X$.

(O3) $\tau(\vee_{i\in\Gamma}\mu_i)\geq\wedge_{i\in\Gamma}\tau(\mu_i)$ for any $\{\mu_i\}_{i\in\Gamma}\subset I^X$.

The pair (X,τ) is called a *smooth fuzzy topological space.*

A smooth fuzzy topological space (X,τ) is called *stratified* if

(S) $\tau(\bar{\alpha})=1$ for each $\alpha\in I$.

Let τ_1 and τ_2 be smooth fuzzy topologies on X. We say τ_1 is *finer* than τ_2(τ_2 is *coarser* than τ_1) if $\tau_2(\mu)\leq\tau_1(\mu)$ for all $\mu\in I^X$.

Definition 2.2 [5] Let $\bar{0}\notin\Theta_X$ be a subset of I^X. A function $\beta:\Theta_X\to I$ is called a *smooth fuzzy topological base* on X if it satisfies the following conditions:

(B1) $\beta(\bar{1})=1$.

(B2) $\beta(\mu_1\wedge\mu_2)\geq\beta(\mu_1)\wedge\beta(\mu_2)$, for all $\mu_1,\ \mu_2\in\Theta_X$.

A smooth fuzzy topological base β always *generates* a smooth fuzzy topology τ_β on X in the following sense:

Theorem 2.3[5] Let β be a smooth fuzzy topological base on X. Define the function $\tau_\beta: I^X\to I$ as follows: for each $\mu\in I^X$,

$$\tau_\beta(\mu)=\begin{cases}\vee\{\wedge_{i\in J}\beta(\mu_i)\} & \text{if } \mu=\vee_{i\in J},\ \mu_j\in\Theta_X,\\ 1 & \text{if } \mu=\bar{0},\\ 0 & \text{otherwise}\end{cases}$$

where the first $\vee$ is taken over all families $\{\mu_i\in\Theta_X \mid \mu=\vee_{i\in J}\mu_i\}$.

Then (X,τ_β) is a smooth fuzzy topological space.

Definition 2.4[5] If β is a smooth fuzzy topological base on X, then τ_β is called the smooth fuzzy topology generated by β. The pair (X,τ_β) is called a smooth fuzzy topological space *generated by a base* β on X.

Definition 2.5[2] Let (X,τ) be a smooth fuzzy topological space. *A fuzzy closure operator* is a function $C_\tau: I^X\times I_0\to I^X$ defined by, for each $\lambda\in I^X$ and $r\in I_0$, $C_\tau(\lambda, r)=\wedge\{\rho\mid\lambda\leq\rho,\ \tau(\bar{1}-\rho)\geq r\}$.

Definition 2.6 Let (X,τ_1) and (Y,τ_2) be smooth fuzzy topological spaces and $f: X \to Y$ a function.

(1) f is called *fuzzy continuous* if $\tau_2(\mu) \le \tau_1(f^{-1}(\mu))$ for all $\mu \in I^Y$.

(2) f is called *fuzzy open* if $\tau_1(\lambda) \le \tau_2(f(\lambda))$ for all $\lambda \in I^X$.

(3) f is called a *fuzzy homeomorphism* if f is bijective fuzzy continuous and f^{-1} is fuzzy continuous.

Definition 2.7 Let $(X_i, \tau_i)_{i \in \Gamma}$ be smooth fuzzy topological spaces and X a set and $f_i: X \to X_i$ a function, for each $i \in \Gamma$. *The initial structure* τ is the coarsest smooth fuzzy topology on X for which each f_i is fuzzy continuous.

Theorem 2.8[5,8] (Existence of initial structures) Let $(X_i, \tau_i)_{i \in \Gamma}$ be smooth fuzzy topological spaces and X a set and $f_i: X \to X_i$ a function, for each $i \in \Gamma$. Let $\Theta_X = \{\overline{0} \ne \mu = \wedge_{i \in F} f_i^{-1}(\nu_i) \mid \tau_i(\nu_i) > 0, i \in F\}$ be given, for every finite index set $F \subset \Gamma$.

Define a function $\beta: \Theta_X \to I$ on X by

$$\beta(\mu) = \vee \{\bigwedge_{i \in F} \tau_i(\nu_i) \mid \mu = \bigwedge_{i \in F} f_i^{-1}(\nu_i)\}$$

where the first $\vee$ is taken over all finite index subset F of Γ. Then:

(1) β is a smooth fuzzy topological base on X.

(2) The smooth fuzzy topology τ_β generated by β is the initial smooth fuzzy topology on X for which each $i \in \Gamma$, f_i is fuzzy continuous.

(3) A map $f: (Z, \tau_Z) \to (X, \tau_\beta)$ is fuzzy continuou iff for each $i \in \Gamma$, $f_i \cdot f$ is fuzzy continuous.

Let (X, τ) be a smooth fuzzy topological space and A be a subset of X. The pair $(A, \tau|_A)$ is said to be a *subspace* of (X, τ) if $\tau|_A$ is endowed with the initial smooth fuzzy topology on A for which the inclusion map i is fuzzy continuous.

Let X be the product $\Pi_{i \in \Gamma} X_i$ of the family $\{(X_i, \tau_i) \mid i \in \Gamma\}$ of smooth fuzzy topological spaces. The initial smooth fuzzy topology $\tau = \otimes \tau_i$ on X for which each the projections $\pi_i: X \to X_i$ is fuzzy continuous is called the *product smooth fuzzy topology* of $\{\tau_i \mid i \in \Gamma\}$, and (X, τ) is called the *product smooth fuzzy topology space*.

3. Some properties of product smooth fuzzy topological spaces

Lemma 3.1 Let X be a product of the family $\{X_i \mid i \in \Gamma\}$ of sets and, for each $i \in \Gamma$, $\pi_i: X \to X_i$ a projection map. For each $\lambda \in I^X$, $i, j \in \Gamma$ and $\lambda_i \in I^{X_i}$, we have the following properties.

(1) $\pi_i(\pi_i^{-1}(\lambda_i) \wedge \lambda) = \lambda_i \wedge \pi_i(\lambda)$.

(2) If $\vee_{x^i \in X_i} \lambda_i(x^i) = \alpha_i$ for $i \in F$ with each finite index subset F of $\Gamma \setminus \{j\}$ and put $\alpha = \wedge_{i \in F} \alpha_i$, then:

(a) $\vee_{x \in X}(\wedge_{i \in F} \pi_i^{-1}(\lambda_i))(x) = \alpha$.

(b) $\pi_j(\wedge_{i \in F} \pi_i^{-1}(\lambda_i)) = \overline{\alpha}$.

Proof. (1) Since $I = [0, 1]$ is infinitely distributive, we have for all $x^i \in X_i$,

$$\begin{aligned}\pi_i(\pi_i^{-1}(\lambda_i) \wedge \lambda)(x^i) &= \vee \{(\pi_i^{-1}(\lambda_i) \wedge \lambda)(x) \mid \pi_i(x) = x^i\} \\ &= \vee \{\lambda_i(\pi_i(x)) \wedge \lambda(x) \mid \pi_i(x) = x^i\} \\ &= \vee \{\lambda_i(x^i) \wedge \lambda(x) \mid \pi_i(x) = x^i\} \\ &= \lambda_i(x^i) \wedge (\vee \{\lambda(x) \mid \pi_i(x) = x^i\}) \\ &= \lambda_i(x^i) \wedge \pi_i(\lambda)(x^i) \\ &= (\lambda_i \wedge \pi_i(\lambda))(x^i).\end{aligned}$$

(2) (a) Since $I = [0, 1]$ is infinitely distributive, we have

$$\begin{aligned}\vee_{x \in X}(\wedge_{i \in F} \pi_i^{-1}(\lambda_i))(x) &= \wedge_{i \in F}(\vee_{x \in X} \lambda_i(\pi_i(x))) \\ &= \wedge_{i \in F}(\vee_{x^i \in X^i} \lambda_i(x^i)) \ (\pi_i(x) = x^i) \\ &= \wedge_{i \in F} \alpha_i \\ &= \alpha.\end{aligned}$$

(b) For each $x^j \in X_j$, we have

$$\begin{aligned}\pi_j(\wedge_{i \in F} \pi_i^{-1}(\lambda_i))(x^j) &= \vee (\wedge_{i \in F} \pi_i^{-1}(\lambda_i))(x) \mid \pi_j(x) = x^j\} \\ &= \vee_{x \in X}(\wedge_{i \in F} \pi_i^{-1}(\lambda_i))(x) \ (F \subset \Gamma \setminus \{j\}) \\ &= \alpha. \text{ (by (a))}\end{aligned}$$

Hence $\pi_j(\wedge_{i \in F} \pi_i^{-1}(\lambda_i)) = \overline{\alpha}$.

□

Example 3.2 Let $X = \{x^1, x^2, x^3\}$, $Y = \{y^1, y^2\}$ and $Z = \{z^1, z^2\}$ be sets and $W = X \times Y \times Z$ a product set. Let $\pi_1: W \to X$, $\pi_2: W \to Y$ and $\pi_3: W \to Z$ be the projection maps. Define $\lambda_1 \in I^X$, $\lambda_2 \in I^Y$ and $\lambda_3 \in I^Z$ as follows:

$\lambda_1(x^1) = 0.5$, $\lambda_1(x^2) = 0.2$, $\lambda_1(x^3) = 0.3$,

$\lambda_2(y^1) = 0.4$, $\lambda_2(y^2) = 0.7$ and $\lambda_3(z^1) = 0.6$, $\lambda_3(z^2) = 0.1$.

Then

$\vee_{i \in \{1,2,3\}} \lambda_1(x^i) = 0.5$, $\vee_{i \in \{1,2\}} \lambda_2(y^i) = 0.7$, $\vee_{i \in \{1,2\}} \lambda_3(z^i) = 0.6$.

From Lemma 3.1 (2)(b), we have the followings:

$\pi_3(\pi_1^{-1}(\lambda_1)) = \overline{0.5}$, $\pi_3(\pi_2^{-1}(\lambda_2)) = \overline{0.7}$,

$\pi_3(\pi_1^{-1}((\lambda_1) \wedge \pi_2^{-1}(\lambda_2))) = \overline{0.5 \wedge 0.7} = \overline{0.5}$,

$\pi_2(\pi_1^{-1}(\lambda_1)) = \overline{0.5}$, $\pi_2(\pi_3^{-1}(\lambda_3)) = \overline{0.6}$,

$\pi_2(\pi_1^{-1}((\lambda_1) \wedge \pi_3^{-1}(\lambda_3))) = \overline{0.5 \wedge 0.6} = \overline{0.5}$,

$\pi_1(\pi_2^{-1}(\lambda_2)) = \overline{0.7}$, $\pi_1(\pi_3^{-1}(\lambda_3)) = \overline{0.6}$,

$\pi_1(\pi_2^{-1}((\lambda_2)\wedge\pi_3^{-1}(\lambda_3)))=\overline{0.7}\wedge\overline{0.6}=\overline{0.6}.$

Lemma 3.3 Let (Y, τ) be smooth fuzzy topological space and β a smooth fuzzy topological base on X. If $f: (X, \beta)\to(Y, \tau)$ is a function such that $\beta(\lambda)\leq\tau(f(\lambda))$ for all $\lambda\in\Theta_X$, then $f: (X, \tau_\beta)\to(Y, \tau)$ is fuzzy open.

Proof. Suppose that there exists $\mu\in I^X$ such that $\tau_\beta(\mu)>\tau(f(\mu))$. Then there exists a family $\{\lambda_i\in\Theta_X \mid \mu=\vee_{i\in\Gamma}\lambda_i\}$ such that

$$\tau_\beta(\mu)\geq\wedge_{i\in\Gamma}\beta(\lambda_i)>\tau(f(\mu)).$$

On the other hand, since $\beta(\lambda)\leq\tau(f(\lambda))$ or all $\lambda\in\Theta_X$, we have

$$\begin{aligned}\wedge_{i\in\Gamma}\beta(\lambda_i)&\leq\wedge_{i\in\Gamma}\tau(f(\lambda_i))\\&\leq\tau(\vee_{i\in\Gamma}(f(\lambda_i)))\text{ (by (O3) of Definition 2.1)}\\&=\tau(f(\vee_{i\in\Gamma}\lambda_i))\\&=\tau(f(\mu)).\end{aligned}$$

It is a contradiction. Hence f is fuzzy open.

Theorem 3.4 Let (X, τ_β) be a product space of a family $\{(X_i, \tau_i)\mid i\in\Gamma\}$ of smooth fuzzy topological spaces. Then the following statements are equivalent:

(1) A projection $\pi_j: (X, \tau_\beta)\to(X_j, \tau_j)$ is fuzzy open.

(2) For every $\mu=\wedge_{i\in\Gamma}\pi_i^{-1}(\lambda_i)$ such that $\vee_{x\in X_i}\lambda_i(x)=\alpha_i$ for each $\alpha_i\in I$ and $i\in F$ such that a finite index subset F of $\Gamma\setminus\{j\}$ and $\tau_i(\lambda_i)>0$, we have $\wedge_{i\in F}\tau_i(\lambda_i)\leq\tau_j(\overline{\alpha})$ where $\alpha=\wedge_{i\in F}\alpha_i$.

Proof. (1)⇒(2) For every $\mu=\wedge_{i\in F}\pi_i^{-1}(\lambda_i)$ such that $\vee_{x\in X_i}\lambda_i(x)=\alpha_i$ for each finite index subset F of $\Gamma\setminus\{j\}$, by Lemma 3.1 (2)(b), we have, for $\alpha=\wedge_{i\in F}\alpha_i$,

$$\pi_j(\mu)=\pi_j(\wedge_{i\in F}\pi_i^{-1}(\lambda_i))=\overline{\alpha}.$$

Since $\mu\in\Theta_X$, by Theorem 2.8, we have

$$\wedge_{i\in F}\tau_i(\lambda_i)\leq\beta(\mu)\leq\tau_\beta(\mu).$$

Furthermore, since π_j is fuzzy open, we have

$$\tau_\beta(\mu)\leq\tau_j(\pi_j(\mu))=\tau_j(\overline{\alpha}).$$

Hence

$$\wedge_{i\in F}\tau_i(\lambda_i)\leq\tau_j(\overline{\alpha}).$$

(2)⇒(1) From Lemma 3.3, we only show that $\beta(\lambda)\leq\tau_j(\pi_j(\lambda))$ for all $\lambda\in\Theta_X$.

Suppose that there exists $\nu\in\Theta_X$ such that $\beta(\nu)>\tau_j(\pi_j(\nu))$. Then there exists a finite index subset F of $\Gamma\setminus j$ with $\nu=\pi_j^{-1}(\lambda_j)\wedge(\wedge_{i\in F}\pi_i^{-1}(\lambda_i))$ (if necessary, we can take $\lambda_j=\overline{1}$) such that

$$\beta(\nu)\geq\tau_j(\lambda_j)\wedge(\wedge_{i\in F}\tau_i(\lambda_i))>\tau_j(\pi_j(\nu)).$$

On the other hand, by Lemma 3.1(2), we have

$$\begin{aligned}\pi_j(\nu)&=\pi_j(\pi_j^{-1}(\lambda_j)\wedge(\wedge_{i\in F}\pi_i^{-1}(\lambda_i)))\\&=\lambda_j\wedge\pi_j(\wedge_{i\in F}\pi_i^{-1}(\lambda_i))\\&=\lambda_j\wedge\overline{\alpha}\end{aligned}$$

where $\vee_{x\in X_i}\lambda_i(x)=\alpha_i$ and $\alpha=\wedge_{i\in F}\alpha_i$. Since $\wedge_{i\in F}\tau_i(\lambda_i)\leq\tau_j(\overline{\alpha})$, we have

$$\begin{aligned}\tau_j(\pi_j(\nu))&=\tau_j(\lambda_j\wedge\overline{\alpha})\\&\geq\tau_j(\lambda_j)\wedge\tau_j(\overline{\alpha})\\&\geq\tau_j(\lambda_j)\wedge(\wedge_{i\in F}\tau_i(\lambda_i)).\end{aligned}$$

It is a contradiction.

□

Example 3.5 Example 3.2, we define smooth fuzzy topologies as follows: for each $\lambda\in I^X$, $\mu\in I^Y$ and $\rho\in I^Z$,

$$\tau_1(\lambda)=\begin{cases}1 & \text{if } \nu=\overline{0}\text{ or }\overline{1},\\ \frac{1}{2} & \text{if } \lambda=\lambda_1,\\ 0 & \text{otherwise,}\end{cases}\qquad \tau_2(\mu)=\begin{cases}1 & \text{if } \mu=\overline{0}\text{ or }\overline{1},\\ \frac{1}{4} & \text{if } \mu=\lambda_2,\\ 0 & \text{otherwise,}\end{cases}$$

$$\tau_3(\rho)=\begin{cases}1 & \text{if } \rho=\overline{0}\text{ or }\overline{1},\\ \frac{2}{3} & \text{if } \rho=\lambda_3,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\Theta_W=\{\overline{0}\neq\nu=\wedge_{j\in F}\pi_j^{-1}(\lambda_j)\mid\tau_j(\lambda_j)>0\}$ for every finite index set $F\subset\{1, 2, 3\}$. From Theorem 2.8, we can obtain a smooth fuzzy topological base $\beta: \Theta_W\to I$. We have

$$\beta(\pi_2^{-1}(\lambda_2)\wedge\pi_3^{-1}(\lambda_3))\geq\tau_2(\lambda_2)\wedge\tau_3(\lambda_3)=\frac{1}{4}.$$

Since $\pi_1(\pi_2^{-1}(\lambda_2)\wedge\pi_3^{-1}(\lambda_3))=\overline{0.6}$, we have $\tau_1(\overline{0.6})=0$. Thus

$$\beta(\pi_2^{-1}(\lambda_2)\wedge\pi_3^{-1}(\lambda_3))\geq\frac{1}{4}>\tau_1(\pi_1(\pi_2^{-1}(\lambda_2)\wedge\pi_3^{-1}(\lambda_3)))=0.$$

Hence the projection $\pi_1: W\to X$ is not fuzzy open.

On the other hand, from Theorem 3.4 (2), we can find the coarsest smooth fuzzy topology τ^* on X for which π_1 is fuzzy open Since

$$\pi_1(\pi_2^{-1}(\lambda_2))=\overline{0.7},\quad \pi_1(\pi_3^{-1}(\lambda_3))=\overline{0.6},$$
$$\pi_1(\pi_2^{-1}(\lambda_2)\wedge\pi_3^{-1}(\lambda_3))=\overline{0.7}\wedge\overline{0.6}=\overline{0.6},$$

we have

$$\tau_2(\lambda_2)\leq\tau^*(\overline{0.7}),\ \tau_3(\lambda_3)\leq\tau^*(\overline{0.6}),\ \tau_2(\lambda_2)\wedge\tau_3(\lambda_3)\leq\tau^*(\overline{0.6})$$

for which π_1 is fuzzy open. Then

$$\tau^*(\lambda)=\begin{cases}1 \text{ if } \lambda=\overline{0} \text{ or } \overline{1},\\ \frac{1}{2} \text{ if } \lambda=\lambda_1,\\ \frac{1}{4} \text{ if } \lambda=\overline{0.7},\\ \frac{2}{3} \text{ if } \lambda=\overline{0.6},\\ 0 \text{ otherwise.}\end{cases}$$

Theorem 3.6 Let (X, τ) be a product space of a family $\{(X_i, \tau_i) \mid i\in\Gamma\}$ of smooth fuzzy topological spaces and (X_j, τ_j) be stratified. Then we have the following properties:

(1) The product smooth fuzzy topological space (X, τ) is stratified.

(2) A projection $\pi_j: X\to X_j$ is fuzzy open.

Proof (1) It is clear from the following: for each $\alpha \in I$,

$$\begin{aligned}\tau(\overline{\alpha})&\geq\beta(\overline{\alpha})\\ &=\vee\{\wedge_{i\in F}\tau_i(\lambda_i) \mid \overline{\alpha}=\wedge_{i\in F}\pi_i^{-1}(\lambda_i)\}\\ &\geq\tau_j(\overline{\alpha})=1.\ (\overline{\alpha}=\pi_j^{-1}(\overline{\alpha}))\end{aligned}$$

(2) Since $\tau_j(\overline{\alpha})=1$ for each $\alpha\in I$, it satisfies the condition of Theorem 3.4(2).

Theorem 3.7 Let (X, τ) be a product space of a family $\{(X_i, \tau_i) \mid i\in\Gamma\}$ of smooth fuzzy topological spaces and (X_j, τ_j) be stratified. Then for every slice $\widetilde{X}_j$ in X parallel to X_j, $\pi_j|_{\widetilde{X}_j}\ \widetilde{X}_j\to X_j$ is a fuzzy homeomorphism.

Proof. (1) Let $\widetilde{X}_j=X_j\times\Pi\{y^i \mid i\neq j]$ be a slice. Since $i: \widetilde{X}_j\to X_j$ and $\pi_j: X\to X_j$ are fuzzy continuous, $\pi_j \circ i=\pi_j|_{\widetilde{X}_j}$ is fuzzy continuous. Moreover, $\pi_j|_{\widetilde{X}_j}$ is bijective.

We only show that $\pi_j|_{\widetilde{X}_j}$ is fuzzy open. Suppose that there exists $\mu\in I^{\widetilde{X}_j}$ such that

$$\tau|_{\widetilde{X}_j}(\mu)>\tau_j(\pi_j|_{\widetilde{X}_j}(\mu)).$$

Then there exists $\nu\in I^X$ with $\mu=i^{-1}(\nu)$ such that

$$\tau|_{\widetilde{X}_j}(\mu)\geq\tau(\nu)>\tau_j(\pi_j|_{\widetilde{X}_j}(\mu)).$$

From the definition of τ, there exists a family $\{\nu_k\in\Theta_X \mid \nu=\vee_{k\in K}\nu_k\}$ such that

$$\tau(\nu)\geq\wedge_{k\in K}\beta(\nu_k)>\tau_j(\pi_j|_{\widetilde{X}_j}(\mu)). \quad \text{(A)}$$

On the other hand, since each $\nu_k\in\Theta_X$, there exists a finite index subset F_k of $\Gamma\setminus\{j\}$ with

$$\nu_k=\pi_j^{-1}(\lambda_{kj})\wedge(\wedge_{i\in F_k}\pi_i^{-1}(\lambda_i)).$$

Since $\pi_i(x)=y^i$ for $i\neq j$, then, for each $x\in\widetilde{X}_j$,

$$\wedge_{i\in F_k}\pi_i^{-1}(\lambda_i)(x)=(\wedge_{i\in F_k}\lambda_i)(y^i).$$

Put $\alpha_k=(\wedge_{i\in F_k}\lambda_i)(y^i)$. Let $\mu_k=i^{-1}(\nu_k)$ for each $k\in K$. Then

$$\begin{aligned}\pi_j|_{\widetilde{X}_j}(\mu_k)(x^j)&=\vee\{\mu_k(x) \mid x\in\widetilde{X}_j,\ \pi_j|_{\widetilde{X}_j}(x)=x^j\}\\ &=\vee\{(i^{-1}(\nu_k)(x) \mid x\in\widetilde{X}_j,\ \pi_j(x)=x^j((\mu_k=i^{-1}(\nu_k))\\ &=\vee\{\nu_k(x) \mid x\in\widetilde{X}_j,\ \pi_j(x)=x^j\}\\ &=\vee\{\pi_j^{-1}(\lambda_{kj}))(x)\wedge(\wedge_{i\in F_k}\pi_i^{-1}(\lambda_i)(x) \mid x\in\widetilde{X}_j,\ \pi_j(x)=x^j\}\\ &=\vee\{\lambda_{kj}(\pi_j(x))\wedge(\wedge_{i\in F_k}\lambda_i)(\pi_i(x)) \mid x\in\widetilde{X}_j, \pi_j(x)=x_j\}\\ &=\lambda_{kj}(x^j)\wedge(\wedge_{i\in F_k}\lambda_i)(y^i)\\ &=\lambda_{kj}(x^j)\wedge\alpha_k\\ &=(\lambda_{kj}\wedge\overline{\alpha_k})(x^j).\end{aligned}$$

Hence $\pi_j|_{\widetilde{X}_j}(\mu_k)=\lambda_{kj}\wedge\overline{\alpha_k}$. Thus

$$\begin{aligned}\tau_j(\pi_j|_{\widetilde{X}_j}(\mu_k))&=\tau_j(\lambda_{kj}\wedge\overline{\alpha_k})\\ &\geq\tau_j(\lambda_{kj})\wedge\tau_j(\overline{\alpha_k})\\ &=\tau_j(\lambda_{kj})\ (\tau_j(\overline{\alpha_k})=1)\\ &\geq\tau_j(\lambda_{kj})\wedge(\wedge_{i\in F_k}\lambda_i).\end{aligned}$$

From the definition of β, it implies $\tau_j(\pi_j|_{\widetilde{X}_j}(\mu_k))\geq\beta(\nu_k)$. Thus

$$\tau_j(\pi_j|_{\widetilde{X}_j}(\mu))\geq\wedge_{k\in K}\tau_j(\pi_j|_{\widetilde{X}_j}(\mu_k))\geq\wedge_{k\in K}\beta(\nu_k).$$

It is a contradiction for (A).

□

In smooth fuzzy topological spaces, a slice $\widetilde{X}$ need not be homeormorphic to X from the following example.

Example 3.8 In Example 3.5, let $\widetilde{X}=\{(x, y^2, z^2) \mid x\in X\}$ be a slice of X.

$$\tau|_{\widetilde{X}}(\mu)=\begin{cases}1 \text{ if } \mu=\overline{0} \text{ or } \overline{1},\\ \frac{1}{2} \text{ if } \mu=\mu_1,\\ \frac{2}{3} \text{ if } \mu=\overline{0.1},\\ \frac{1}{4} \text{ if } \mu=\overline{0.7},\\ 0 \text{ otherwise.}\end{cases}$$

where

$$\mu_1(x^1, y^2, z^2)=0.5,\ \mu_1(x^2, y^2, z^2)=0.2,\ \mu_1(x^3, y^2, z^2)=0.3.$$

Then the projection $\pi_j|_{\widetilde{X}}: \widetilde{X}\to X$ is bijective fuzzy continuous. Since

$$\frac{2}{3}=\tau|_{\tilde{X}}(\overline{0.1})\not\leq \tau_1\left(\pi_1|_{\tilde{X}}(\overline{0.1})\right)=0,$$

then $\pi_j|_{\tilde{X}}$ is not fuzzy open. Hence $\widetilde{X}$ and X are not homeomorphic.

Let $\{\lambda_i \in I^{X_i} \mid i \in \Gamma\}$ be a family of fuzzy sets. Define the product of $\{\lambda_i \in I^{X_i} \mid i \in \Gamma\}$ as

$$\prod_{i\in\Gamma} \lambda_i = \bigwedge_{i\in\Gamma} \pi_i^{-1}(\lambda_i).$$

Theorem 3.9 Let (X, τ) be a product space of a family $\{(X_i, \tau_i) \mid i \in \Gamma\}$ of smooth fuzzy topological spaces. Then we have the following properties:

(1) $C_\tau(\Pi_{i\in\Gamma} \lambda_i, r) \leq \Pi_{i\in\Gamma} C_{\tau_i}(\lambda_i, r)$, for all $\lambda_i \in I^{X_i}$ and $r \in I_0$.

(2) If $C_{\tau_i}(\lambda_i, r) = \lambda_i$ for all $\lambda_i \in I^{X_i}$ and $r \in I_0$, then $C_\tau(\Pi_{i\in\Gamma} \lambda_i, r) = \Pi_{i\in\Gamma} \lambda_i$.

Proof. (1) Suppose $C_\tau(\Pi_{i\in\Gamma} \lambda_i, r) \not\leq \Pi_{i\in\Gamma} C_{\tau_i}(\lambda_i, r)$. Then there exist $x \in X$ and $t \in I_0$ such that

$$C_\tau\left(\prod_{i\in\Gamma} \lambda_i, r\right)(x) \geq t > \prod_{i\in\Gamma} C_{\tau_i}(\lambda_i, r)(x). \qquad \text{(B)}$$

Since $\Pi_{i\in\Gamma} C_{\tau_i}(\lambda_i, r)(x) < t$, there exists $j \in \Gamma$ such that

$$\prod_{i\in\Gamma} C_{\tau_i}(\lambda_i, r)(x) \leq \pi_j^{-1}(C_{\tau_j}(\lambda_j, r))(x) < t.$$

Put $\pi_j(x) = x^j$. It implies

$$C_{\tau_j}(\lambda_j, r)(x^j) < t.$$

From the definition of C_{τ_j}, there exists $\mu_j \in I^{X_j}$ with $\lambda_j \leq \mu_j$ and $\tau_j(\overline{1} - \mu_j) \geq r$ such that

$$C_{\tau_j}(\lambda_j, r)(x^j) \leq \mu_j(x^j) < t.$$

On the other hand, we have

$$\begin{aligned} \lambda_j \leq \mu_j &\Rightarrow \pi_j^{-1}(\lambda_j) \leq \pi_j^{-1}(\mu_j) \\ &\Rightarrow \Pi_{i\in\Gamma}\lambda_i = \wedge_{i\in\Gamma}\pi_i^{-1}(\lambda_i) \leq \pi_j^{-1}(\mu_j) \\ &\Rightarrow C_\tau(\Pi_{i\in\Gamma}\lambda_i, r) \leq \pi_j^{-1}(\mu_j) \end{aligned}$$

because

$$\begin{aligned} \tau(\overline{1} - \pi_j^{-1}(\mu_j)) &= \tau(\pi_j^{-1}(\overline{1} - \mu_j)) \\ &\geq \tau_j(\overline{1} - \mu_j) \geq r. \end{aligned}$$

Hence

$$\begin{aligned} C_\tau(\Pi_{i\in\Gamma}\lambda_i, r)(x) &\leq \pi_j^{-1}(\mu_j)(x) \\ &= \mu_j(x^j) < t. \end{aligned}$$

It is a contradiction for (B). Hence $C_\tau(\Pi_{i\in\Gamma}\lambda_i, r) \leq \Pi_{i\in\Gamma} C_{\tau_i}(\lambda_i, r)$.

(2) It is clear from the following:

$$\begin{aligned} \Pi_{i\in\Gamma}\lambda_i &\leq C_\tau(\Pi_{i\in\Gamma}\lambda_i, r) \\ &\leq \Pi_{i\in\Gamma}C_{\tau_i}(\lambda_i, r) \\ &= \Pi_{i\in\Gamma}\lambda_i. \end{aligned}$$

Remark 3.10 In Theorem 3.9, if each the range of τ_i is $\{0, 1\}$, then the product of every fuzzy closed sets is a fuzzy closed set in Chang's fuzzy topology [1].

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