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Some properties of fuzzy closure spaces

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ABSTRACT

We will prove the existence of initial fuzzy closure structures. From this fact, we can define subspaces and products of fuzzy closure spaces. Furthermore, the family $\Delta(X)$ of all fuzzy closure operators on X is a complete lattice. In particular, an initial structure of fuzzy topological spaces can be obtained by the initial structure of fuzzy closure spaces induced by those. We suggest some examples of it.

1. Introduction

The theory of fuzzy topological spaces introduced by C.L. Chang [2] has been developed in many directions. But the category of fuzzy topological spaces is not good one to work in for many problems. So, many attempts have been made to find suitable categories. One of them, fuzzy closure spaces were introduced by A.S. Mashhour and M.H. Ghanim [6] as a generalization of closure spaces.

We will prove the existence of initial fuzzy closure structures. From this fact, we can define subspaces and products of fuzzy closure spaces. Furthermore, the family $\Delta(X)$ of all fuzzy closure operators on X is a complete lattice.

In particular, an initial structure of fuzzy topological spaces can be obtained by the initial structure of fuzzy closure spaces induced by those. We provide some examples of it.

In this paper, all the notations and the definitions are standard in fuzzy set theory.

2. Preliminaries

Lemma 2.1. [4] If $f: X \to Y$, then we have the following properties for direct and inverse image of fuzzy sets under mappings: for $\mu, \mu_i \in I^X$ and $\nu, \nu_i \in I^Y$.

(1) $\nu \geq f(f^{-1}(\nu))$ with equality if f is surjective.
(2) $\mu \leq f^{-1}(f(\mu))$ with equality if f is injective.
(3) $f^{-1}(\tilde{1} - \nu) = \tilde{1} - f^{-1}(\nu)$.
(4) $f(\tilde{1} - \mu) = \tilde{1} - f(\mu)$ if f is bijective.
(5) $f^{-1}(\vee_{i\in\Lambda} \nu_i) = \vee_{i\in\Lambda} f^{-1}(\nu_i)$.
(6) $f^{-1}(\wedge_{i\in\Lambda} \nu_i) = \wedge_{i\in\Lambda} f^{-1}(\nu_i)$.
(7) If $\lambda \leq \mu$, then $f(\lambda) \leq f(\mu)$.
(8) $f(\vee_{i\in\Lambda} \nu_i) = \vee_{i\in\Lambda} f(\nu_i)$.
(9) $f(\wedge_{i\in\Lambda} \mu_i) \leq \wedge_{i\in\Lambda} f(\mu_i)$ with equality if f is injective.

Definition 2.2. [2] A subset τ of I^X is called a *fuzzy topology* on X if it satisfies the following conditions:

(O1) $\tilde{0}, \tilde{1} \in \tau$, where $\tilde{1}(x)=1$, $\tilde{0}(x)=0$, $\forall x \in X$.
(O2) If $\mu_1, \mu_2 \in \tau$, then $\mu_1 \wedge \mu_2 \in \tau$.
(O3) If $\mu_i \in \tau$ for each $i \in \Lambda$, then $\vee_{i\in\Lambda} \mu_i \in \tau$.

The pair (X, τ) is called a *fuzzy topological space.*

Let τ_1 and τ_2 be fuzzy topologies on X. We say that τ_1 is *finer* than τ_2(τ_2 is coarser than τ_1) if $\tau_2 \subset \tau_1$.

Definition 2.3. [4,6] A function $C: I^X \to I^X$ is called a *fuzzy closure operator* on X if it satisfies the following conditions:

(C1) $C(\tilde{0}) = \tilde{0}$.
(C2) $C(\lambda) \geq \lambda$, for all $\lambda \in I^X$.
(C3) $C(\lambda \vee \mu) = C(\lambda) \vee C(\mu)$, for all $\lambda, \mu \in I^X$.

The pair (X, C) is called a *fuzzy closure space.*

A fuzzy closure space (X, C) is called *topological* provided that

(C4) $C(C(\lambda)) = C(\lambda)$, for all $\lambda \in I^X$.

Let C_1 and C_2 be fuzzy closure operators on X. We say that C_1 is *finer* than C_2(C_2 is *coarser* than C_1) if $C_1(\lambda) \leq C_2(\lambda)$, for each $\lambda \in I^X$.

Theorem 2.4. [4] Let (X, τ) be a fuzzy topological space. We define an operator $C_\tau: I^X \to I^X$ as follows:

$C_\tau(\lambda) = \wedge\{\mu \mid \mu \geq \lambda, \ \tilde{1} - \mu \in \tau\}$.

Then (X, C_τ) be a topological fuzzy closure space.

Theorem 2.5. [4] Let (X, C) be a fuzzy closure space. Define $\tau_C \subset I^X$ on X by

$\tau_C=\{\ \tilde{1}-\lambda\in I^X \mid C(\lambda)=\lambda\}$.

Then:

(1) τ_C is a fuzzy topology on X.

(2) We have $C=C_{\tau_C}$ iff (X, C) is topological.

Let (X, τ_1), (Y, τ_2) be fuzzy topological spaces. A map $f:(X, \tau_1)\to(Y, \tau_2)$ is called *fuzzy continuous* if $f^{-1}(\mu)\in\tau_1$ for all $\mu\in\tau_2$.

Let (X, C_1), (Y, C_2) be fuzzy closure spaces. A map $f:(X, C_1)\to(Y, C_2)$ is called a *C-map* if $f(C_1(\lambda))\leq C_2(f(\lambda))$, for all $\lambda\in I^X$.

Definition 2.6. [4,5] Let $\tilde{0}\notin\beta$ be a subset of I^X. A structure β is called a *basis* on X if it satisfies the following conditions:

(B1) $\tilde{1}\in\beta$.

(B2) $\mu_1\wedge\mu_2\in\beta$, for all $\mu_1, \mu_2\in\beta$.

A basis $\beta=\{\mu_i \mid i\in\Gamma\}$ *generates* a fuzzy topology τ_β on X in the following sense:

$\tau_\beta=\{\ \tilde{0},\ \mu \mid \mu=\vee_{j\in\Lambda}\mu_j,\ \mu_j\in\beta,\ \Lambda\subset\Gamma\}$.

Theorem 2.7. [4,5] (Existence of initial fuzzy topological structures) Let $(X_i, \tau_i)_{i\in\Gamma}$ be a family of fuzzy topological spaces, X a set and, for each $i\in\Gamma$, $f_i:X\to X_i$ a function. Let

$$\beta=\{\ \tilde{0}\neq\mu=\bigwedge_{j=1}^{n} f_{kj}^{-1}(v_{kj}) \mid v_{kj}\in\tau_{kj} \text{ for all } k_j\in K\}$$

for every finite index set $K=\{k_1,\dots,k_n\}\subset\Gamma$.

Then:

(1) β is a basis on X.

(2) The fuzzy topology τ_β generated by β is the coarsest fuzzy topology on X which for each $i\in\Gamma$, f_i is a fuzzy continuous map.

(3) A map $f:(Y, \tau')\to(X, \tau_\beta)$ is a fuzzy continuous map iff for each $i\in\Gamma$, $f_i\circ f:(Y, \tau')\to(X_i, \tau_i)$ is a fuzzy continuous map.

Theorem 2.8. [4] Let (X, τ_1) and (Y, τ_2) be fuzzy topological spaces. Then the following statements are equivalent:

(a) A function $f:(X, \tau_1)\to(Y, \tau_2)$ is a fuzzy continuous map.

(b) $f:(X, C_{\tau_1})\to(Y, C_{\tau_2})$ is a C- map.

(c) $C_{\tau_1}(f^{-1}(\mu))\leq f^{-1}(C_{\tau_2}(\mu))$, for each $\mu\in I^Y$.

3. Initial fuzzy closure spaces

Now, we will prove the existence of an initial fuzzy closure space.

Definition 3.1. Let $\{(X_i, C_i)\}_{i\in\Gamma}$ be a family of fuzzy closure spaces. Let X be a set and, for each $i\in\Gamma$, $f_i:X\to X_i$ a function. The *initial structure* C is the coarsest fuzzy closure operator on X for which each f_i is a C-map.

Theorem 3.2. (Existence of initial fuzzy closure structures) Let $\{(X_i, C_i)\}_{i\in\Gamma}$ be a family of fuzzy closure spaces, X a set and $f_i:X\to X_i$ a function, for each $i\in\Gamma$. Define a function $C:I^X\to I^X$ on X by, for each $\lambda\in I^X$,

$$C(\lambda)=\inf\{\bigvee_{j=1}^{p}(\inf_{i\in\Gamma}f_i^{-1}(C_i(f_i(\lambda_j))))\}$$

where the infimum is taken over all finite families $\{\lambda_j \mid j=1,\dots, p\}$ such that $\lambda=\vee_{j=1}^{p}\lambda_j$. Then:

(1) C is the coarsest fuzzy closure operator on X which for each $i\in\Gamma$, f_i is a C-map.

(2) If $\{(X_i, C_i)\}_{i\in\Gamma}$ is a family of topological fuzzy closure spaces, then (X, C) is a topological fuzzy closure space.

(3) A map $f:(Y, C')\to(X, C)$ is a C-map iff for each $i\in\Gamma$, $f_i\circ f:(Y, C')\to(X_i, C_i)$ is a C-map.

Proof. (1) First, we will show that C is a fuzzy closure operator on X.

(C1) It is easily proved from Definition 2.3.

(C2) For all finite families $\{\lambda_j \mid j=1,\dots, p\}$ such that $\lambda=\vee_{j=1}^{p}\lambda_j$, we have for each $i\in\Gamma$,

$\lambda_j\leq f_i^{-1}(f_i(\lambda_j))\leq f_i^{-1}(C_i(f_i(\lambda_j)))$.

Hence $C(\lambda)\geq\lambda$, for all $\lambda\in I^X$.

(C3) We will show that $C(\lambda)\leq C(\mu)$ for $\lambda\leq\mu$.

Suppose $C(\lambda)\not\leq C(\mu)$ for $\lambda\leq\mu$. Then there exists $x\in X$ such that

$C(\lambda)(x)>C(\mu)(x)$.

There exists a finite family $\{\mu_k \mid k=1,\dots, q\}$ such that $\mu=\vee_{k=1}^{q}\mu_k$ with

$$C(\lambda)(x)>\bigvee_{k=1}^{q}\ (\inf_{i\in\Gamma}f_i^{-1}(C_i(f_i(\mu_k))))(x).$$

On the other hand, since $\lambda\leq\mu$, there exists a finite family $\{\lambda\wedge\mu_k \mid k=1,\dots, q\}$ such that $\lambda=\vee_{k=1}^{q}(\lambda\wedge\mu_k)$ with

$$C(\lambda)\leq\bigvee_{k=1}^{q}(\inf_{i\in\Gamma}f_i^{-1}(C_i(f_i(\lambda\wedge\mu_k))))$$

$$\leq\bigvee_{k=1}^{q}(\inf_{i\in\Gamma}f_i^{-1}(C_i(f_i(\mu_k)))).$$

It is a contradiction. Hence we have $C(\lambda)\vee C(\mu)\leq C(\lambda\vee\mu)$.

For any $\lambda, \mu\in I^X$, we will show that

$C(\lambda\vee\mu)\leq C(\lambda)\vee C(\mu)$.

Suppose that there exist $\lambda, \mu\in I^X$ and $x\in X$ such that

$t = C(\lambda \vee \mu)(x) > C(\lambda)(x) \vee C(\mu)(x)$.

There exist finite families $\{\lambda_j \mid j=1,\dots,p\}$ and $\{\mu_k \mid k=1,\dots,q\}$ such that $\lambda = \vee_{j=1}^{p} \lambda_j$ and $\mu = \vee_{k=1}^{q} \mu_k$ with

$$t > \bigvee_{j=1}^{p} inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda_j)))(x),$$

$$t > \bigvee_{k=1}^{q} (inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(\mu_k))))(x).$$

There exists a finite family $\{\lambda_j, \mu_k \mid j=1,\dots,p,\ k=1,\dots,q\}$ such that $\lambda \vee \mu = (\vee_{j=1}^{p} \lambda_j) \vee (\vee_{k=1}^{q} \mu_k)$ with

$$t > \bigvee_{j=1}^{p} (inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda_j))))(x)$$
$$\vee (\bigvee_{k=1}^{q} (inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(\mu_k))))(x)$$
$$\geq C(\lambda \vee \mu)(x) = t.$$

It is a contradiction. Therefore, C is the fuzzy closure operator on X.

Next, we will show that C is the coarsest fuzzy closure operator on X. We observe the follows from the definition of C:

$$C(\lambda) \leq inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda)))$$
$$\leq f_i^{-1}(C_i(f_i(\lambda))).$$

It implies that

$$f_i(C(\lambda)) \leq f_i(f_i^{-1}(C_i(f_i(\lambda))))$$
$$\leq C_i(f_i(\lambda)). \qquad \text{(by Lemma 2.1(1))}$$

Hence for each $i \in \Gamma$, $f_i:(X, C) \to (X_i, C_i)$ is a C-map.

If $f_i: (X, C^*) \to (X_i, C_i)$ is a C-map for every $i \in \Gamma$, then we have

$$f_i(C^*(\lambda)) \leq C_i(f_i(\lambda)).$$

It implies that

$$C^*(\lambda) \leq f_i^{-1}(f_i(C^*(\lambda))) \qquad \text{(by Lemma 2.1(2))}$$
$$\leq f_i^{-1}(C_i(f_i(\lambda))).$$

Hence

$$C^*(\lambda) \leq inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda))). \qquad (A)$$

We have the followings: for all finite families $\{\lambda_j \mid j=1,\dots,p\}$ such that $\lambda = \vee_{j=1}^{p} \lambda_j$,

$$C(\lambda) = inf\{\bigvee_{j=1}^{p} (inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda_j))))\}$$
$$\geq inf\{\bigvee_{j=1}^{p} (C^*(\lambda_j))\} \qquad \text{(by (A))}$$
$$= inf\{C^*(\bigvee_{j=1}^{p} \lambda_j)\} \qquad \text{((C3) of Definition 2.3)}$$
$$= C^*(\lambda).$$

(2) We will show that $C(C(\lambda)) = C(\lambda)$, for all $\lambda \in I^X$.

For every finite family $\{\lambda_j \mid j=1,\dots,p\}$ such that $\lambda = \vee_{j=1}^{p} \lambda_j$, we have

$$C(\lambda) = inf\{\bigvee_{j=1}^{p} (inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda_j))))\}$$

(Since C_i is a topological fuzzy closure operator,)

$$= inf\{\bigvee_{j=1}^{p} (inf_{i\in\Gamma} f_i^{-1}(C_i(C_i(f_i(\lambda_j)))))\}$$

(Since $f_i(C(\lambda_j)) \leq C_i(f_i(\lambda_j))$ from (1),)

$$\geq inf\{\bigvee_{j=1}^{p} (inf_{i\in\Gamma} f_i^{-1}(C_i(f_i(C(\lambda_j)))))\}$$

(Since $\bigvee_{j=1}^{p} (C(\lambda_j)) = C(\lambda)$ from (C_3) of (1),)

$$\geq C(C(\lambda)).$$

Combined with (C2) of (1), we have $C(C(\lambda)) = C(\lambda)$.

(3) Necessity of the composition condition is clear since the composition of C-maps is a C-map.

Conversely, suppose $f_i \circ f$ is a C-map. Then, for each $\mu \in I^X$, we have

$$f_i(f(C'(f^{-1}(\mu)))) \leq C_i(f_i(f(f^{-1}(\mu))))$$
$$\leq C_i(f_i(\mu)). \qquad \text{(by Lemma 2.1(1))}$$

It follows that for all $i \in \Gamma$,

$$f(C'(f^{-1}(\mu))) \leq f_i^{-1}(f_i(f(C'(f^{-1}(\mu)))))$$
$$\leq f_i^{-1}(C_i(f_i(\mu))).$$

Hence we have

$$f(C'(f^{-1}(\mu))) \leq inf_{i\in\Gamma}(f_i^{-1}(C_i(f_i(\mu)))). \qquad (B)$$

For all finite families $\{\mu_j \mid j=1,\dots,p\}$ such that $f(\lambda) = \vee_{j=1}^{p} \mu_j$, we have

$$f(C'(\lambda)) \leq f(C'(f^{-1}(f(\lambda)))) \qquad \text{(by Lemma 2.1(2))}$$
$$= f(C'(f^{-1}(\bigvee_{j=1}^{p} \mu_j)))$$
$$= f(C'(\bigvee_{j=1}^{p} f^{-1}(\mu_j))) \qquad \text{(by Lemma 2.1(5))}$$
$$= f(\bigvee_{j=1}^{p}(C' f^{-1}(\mu_j))) \qquad \text{(by Definition 2.3(C3))}$$
$$= \bigvee_{j=1}^{p} f(C'(f^{-1}(\mu_j))) \qquad \text{(by Lemma 2.1(7))}$$
$$\leq \bigvee_{j=1}^{p} inf_{i\in\Gamma}(f_i^{-1}(C_i(f_i(\mu_j)))) \qquad \text{(by (B))}$$

Therefore $f(C'(\lambda)) \leq C(f(\lambda))$ for all it $\lambda \in I^X$. Hence $f:(Y, C') \to (X, C)$ is a C-map. □

Theorem 3.3. Let $\{(X_i, C_i)\}_{i\in\{1,2\}}$ be fuzzy closure spaces, X a set and $f_i: X \to X_i$ a function, for each $i \in \{1, 2\}$. For each $\lambda \in I^X$, we define a function $C^*: I^X \to I^X$ on X by

$$C^*(\lambda) = inf\{\bigvee_{j=1}^{2} (inf_{i\in\{1,2\}}(f_i^{-1}(C_i(f_i(\lambda_j)))))\},$$

where the infimum is taken over all family $\{\lambda_1, \lambda_2\}$ such that $\lambda = \lambda_1 \vee \lambda_2$. Then C^* is the coarsest fuzzy closure operator on X which for each $i \in \{1, 2\}$, f_i is a C-map.

Proof. Let C be the coarsest fuzzy closure operator on X which for each $i \in \{1, 2\}$, f_i is a C-map from Theorem 3.2 .

We will show that $C^* = C$.

From the definition of C, it is easily proved $C \leq C^*$.

Suppose $C^*(\lambda) \not\leq C(\lambda)$ for some $\lambda \in I^X$. Then there exists $x \in X$ such that

$C^*(\lambda)(x) > C(\lambda)(x)$.

There exists a finite index $K=\{1, 2,..., q\}$ with $\{\lambda_k \in I^X \mid k \in K\}$ such that $\lambda = \vee_{k=1}^{q} \lambda_k$ with

$$C^*(\lambda)(x) > \bigvee_{k=1}^{q} (f_1^{-1}(C_1(f_1(\lambda_k)))(x) \wedge f_2^{-1}(C_2(f_2(\lambda_k)))(x)).$$

Put an index set

$J=\{k \in K \mid f_1^{-1}(C_1(f_1(\lambda_k)))(x) \leq f_2^{-1}(C_2(f_2(\lambda_k)))(x)\}$

and $L=K-J$. Then we have the followings:

$$C^*(\lambda)(x) > \bigvee_{k=1}^{q} (f_1^{-1}(C_1(f_1(\lambda_k)))(x) \wedge f_2^{-1}(C_2(f_2(\lambda_k)))(x))$$

$$= \bigvee_{k \in J} f_1^{-1}(C_1(f_1(\lambda_k)))(x) \vee (\bigvee_{k \in L} f_2^{-1}(C_2(f_2(\lambda_k)))(x))$$

(By Lemma 2.1(5) and (C3) of Definition 2.3)

$$= f_1^{-1}(C_1(f_1(\bigvee_{k \in J} \lambda_k)))(x) \vee (f_2^{-1}(C_2(f_2(\bigvee_{k \in L} \lambda_k)))(x)).$$

On the other hand, let $\lambda_J = \vee_{k \in J} \lambda_k$ and $\lambda_L = \vee_{k \in L} \lambda_k$ be given. Then there exists a family $\{\lambda_J, \lambda_L\}$ such that $\lambda = \lambda_J \vee \lambda_L$ with

$$C^*(\lambda) \leq \{f_1^{-1}(C_1(f_1(\lambda_J))) \wedge f_2^{-1}(C_2(f_2(\lambda_J)))\}$$
$$\vee \{f_1^{-1}(C_1(f_1(\lambda_L))) \wedge f_2^{-1}(C_2(f_2(\lambda_L)))\}$$
$$\leq f_1^{-1}(C_1(f_1(\lambda_J))) \vee f_2^{-1}(C_2(f_2(\lambda_L))).$$

It is a contradiction. □

Using Theorem 3.2, we can define subspaces and products in the obvious way.

Definition 3.4. Let (X, C_X) be a fuzzy closure space and A a subset of X. The pair (A, C) is said to be a subspace of (X, C_X) if it is endowed with the initial fuzzy closure structure with respect to the inclusion.

From Theorem 3.2, for all finite families $\{\lambda_j \mid j=1,..., p\}$ such that $\lambda = \vee_{j=1}^{p} \lambda_j$, we have

$$C(\lambda) = inf\{\bigvee_{j=1}^{p} i^{-1}(C_X(i(\lambda_j)))\}$$

$$= inf\{i^{-1}(\bigvee_{j=1}^{p} (C_X(i(\lambda_j)))\} \qquad \text{(by Lemma 2.1(5))}$$

$$= inf\{i^{-1}(C_X(i(\bigvee_{j=1}^{p} \lambda_j)))\}$$

(by Lemma 2.1(8) and Definition 2.3 (C3))

$= i^{-1}(C_X(i(\lambda)))$.

Hence it follows Corollary 3.5.

Corollary 3.5. Let C_X be a fuzzy closure operator on X and $i : A \to X$ an inclusion. Define the function $C : I^A \to I^A$ on A by

$C(\lambda) = i^{-1}(C_X(i(\lambda)))$.

Then (A, C) is a subspace of (X, C_X).

Definition 3.6. Let X be the product $\Pi_{i \in \Gamma} X_i$ of the family $\{(X_i, C_i)\}_{i \in \Gamma}$ of fuzzy closure spaces. An initial fuzzy closure structure $C = \otimes C_i$ on X with respect to all the projections $\pi_i: X \to X_i$ is called the product fuzzy closure structure of $\{C_i \mid i \in \Gamma\}$, and $(X, \otimes C_i)$ is called the product *fuzzy closure space.*

Using Theorem 3.2, we have the following corollary.

Corollary 3.7. Let $\{(X_i, C_i)\}_{i \in \Gamma}$ be a family of fuzzy closure spaces. Let $X = \Pi_{i \in \Gamma} X_i$ be the product and, for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection. The structure $C = \otimes C_i$ on X is defined by

$$C(\lambda) = inf\{\bigvee_{j=1}^{p} inf_{i \in \Gamma}\ \pi_i^{-1}(C_i(\pi_i(\lambda_j)))\},$$

where the infimum is taken over all finite families $\{\lambda_j \mid j=1,..., p\}$ such that $\lambda = \vee_{j=1}^{p} \lambda_j$.

Then:

(1) C is the coarsest fuzzy closure operator on X which for each $i \in \Gamma$, π_i is a C-map.

(2) A map $f: (Y, C') \to (X, C)$ is a C-map iff for each $i \in \Gamma$, $\pi_i \circ f : (Y, C') \to (X_i, C_i)$ is a C-map.

Theorem 3.8. Let $\{(X_i, \tau_i)\}_{i \in \Gamma}$ be a family of fuzzy topological spaces. Let X be a set and, for each $i \in \Gamma$, $f_i : X \to X_i$ a function. Define the function $C: I^X \to I^X$ on X by

$$C(\lambda) = inf\{\bigvee_{j=1}^{p} inf_{i \in \Gamma} f_i^{-1}(C_{\tau_i}(f_i(\lambda_j)))\},$$

where the infimum is taken over all finite families $\{\lambda_j \mid j=1,..., p\}$ such that $\lambda = \vee_{j=1}^{p} \lambda_j$. Then we have

$$\tau_\beta = \tau_C$$

where τ_C is the fuzzy topology induced by C and τ_β is the one generated by basis β as in Theorem 2.7.

Proof. If $\lambda \in \tau_C$, by the definition of τ_C from Theorem 2.5, we have

$C(\tilde{1} - \lambda) = \tilde{1} - \lambda$.

Thus, it follows that

$$\lambda = \tilde{1} - C(\tilde{1} - \lambda)$$
$$= \tilde{1} - inf\{\bigvee_{j=1}^{p} inf_{i \in \Gamma} f_i^{-1}(C_{\tau_i}(f_i(\nu_j)))\}$$
$$= sup\{\bigwedge_{j=1}^{p} \{\tilde{1} - inf_{i \in \Gamma} f_i^{-1}(C_{\tau_i}(f_i(\nu_j)))\}\}$$
$$= sup\{\bigwedge_{j=1}^{p} sup_{i \in \Gamma}\{\tilde{1} - f_i^{-1}(C_{\tau_i}(f_i(\nu_j)))\}\}$$
$$= sup\{\bigwedge_{j=1}^{p} sup_{i \in \Gamma}\{f_i^{-1}(\tilde{1} - C_{\tau_i}(f_i(\nu_j)))\}\}$$
$$= \vee\{\bigwedge_{j=1}^{p} \{\bigvee_{i \in \Gamma}\{f_i^{-1}(\tilde{1} - C_{\tau_i}(f_i(\nu_j)))\}\}\},$$

where the first $\vee$ is taken over all finite families $\{\nu_j \mid j=1,..., p\}$ such that $\tilde{1} - \lambda = \vee_{j=1}^{p} \nu_j$.

Since $C_{\pi}(f_i(v_j))=C_{\pi}(C_{\pi}(f_i(v_j)))$, using the fact $\tau_i=\tau_{C\pi}$ from Theorem 2.5, we have
$\tilde{1}-C_{\pi}(f_i(v_j))\in\tau_i$.

Put $\mu_{ij}=f_i^{-1}(\tilde{1}-C_{\pi}(f_i(v_j)))$. By the definition of τ_β, we have
$\bigvee_{i\in\Gamma}\mu_{ij}\in\tau_\beta$.

Hence $\lambda\in\tau_\beta$ from (O2),(O3) of Definition 2.2.

We show that $\mu\in\tau_\beta$ implies $\mu\in\tau_C$, equivalently, the identity function $1:(X,\tau_C)\to(X,\tau_\beta)$ is a fuzzy continuous map. From Theorem 2.7(3) we only show that $f_i\circ 1:(X,\tau_C)\to(X_i,\tau_i)$ is a fuzzy continuous map, that is,
$f_i^{-1}(v_i)\in\tau_C$, for all $v_i\in\tau_i$.

Since $v_i\in\tau_i$, then, by Theorem 2.5, we have
$$C_{\pi}(\tilde{1}-v_i)=\tilde{1}-v_i. \quad \text{(L)}$$
From definition of C, it follows that
$$\begin{aligned} C(\tilde{1}-f_i^{-1}(v_i)) &\le f_i^{-1}(C_{\pi}(f_i(\tilde{1}-f_i^{-1}(v_i))))\\ &= f_i^{-1}(C_{\pi}(f_i(f_i^{-1}(\tilde{1}-v_i))))\\ &\le f_i^{-1}(C_{\pi}(\tilde{1}-v_i))\\ &= f_i^{-1}(\tilde{1}-v_i) \quad \text{(by (L))}\\ &= \tilde{1}-f_i^{-1}(v_i). \end{aligned}$$

Thus, combined with (C2) of Definition 2.3, we have
$C(\tilde{1}-f_i^{-1}(v_i))=\tilde{1}-f_i^{-1}(v_i)$.
Hence, by Theorem 2.5,
$f_i^{-1}(v_i)\in\tau_C$.

□

Let $\Delta(X)$ denote the set of all fuzzy closure operators on X. For each $C_1, C_2\in\Delta(X)$, we define $C_1\sqcap C_2$, $C_1\vee C_2$ as follows:

$C_1\sqcap C_2$ is the coarsest fuzzy closure operator finer than C_1 and C_2,
$(C_1\vee C_2)(\mu)=C_1(\mu)\vee C_2(\mu)\ \forall\mu\in I^X$.

Moreover, we define C^0 and C^1 as follows, for each $\lambda\in I^X$
$$C^0(\lambda)=\begin{cases}\tilde{0}, & \text{if } \lambda=\tilde{0}\\ \tilde{1}, & \text{otherwise,}\end{cases} \quad C^1(\lambda)=\lambda.$$

Then $C^0(C^1)$ is the coarsest (finest) fuzzy closure operator on X.

Denote $C_1<C_2$ if C_2 is finer than C_1, that is, $C_1(\lambda)\ge C_2(\lambda)$ for all $\lambda\in I^X$.

Using Theorem 3.2, we can easily derive the following corollary.

Corollary 3.9. The partially ordered set $(\Delta(X),\sqcap,\vee,<)$ is a complete lattice.

Example 3.10. We define $C_1, C_2\in\Delta(X)$ as follows
$$C_1(\lambda)=\begin{cases}\tilde{0}, & \text{if } \lambda=\tilde{0},\\ \mu_1, & \text{if } \tilde{0}\ne\lambda\le\mu_1,\\ \tilde{1}, & \text{otherwise,}\end{cases}$$
$$C_2(\lambda)=\begin{cases}\tilde{0}, & \text{if } \lambda=\tilde{0},\\ \mu_2, & \text{if } \tilde{0}\ne\lambda\le\mu_2,\\ \tilde{1}, & \text{otherwise,}\end{cases}$$

If $\lambda\le\mu_1$ and $\lambda\not\le\mu_2$, we have the following statements:

For every families $\{\lambda_1,\lambda_2\}$ such that $\lambda=\lambda_1\vee\lambda_2$, from Theorem 3.3, since $\lambda=\lambda_1\vee\lambda_2\le\mu_1$, we have $\lambda_k\le\mu_1$ for all $k=\{1,2\}$.

Furthermore, since $\lambda\not\le\mu_2$, without loss of generality we may assume that $\lambda_2\not\le\mu_2$. Hence we have
$$\begin{aligned}&(C_1(\lambda_1)\wedge C_2(\lambda_1))\vee(C_1(\lambda_2)\wedge C_2(\lambda_2))\\ &=(\mu_1\wedge C_2(\lambda_1))\vee(\mu_1\wedge\tilde{1})\\ &=\mu_1.\end{aligned}$$

It follows $C_1\sqcap C_2(\lambda)=\mu_1$.

If $\lambda\le\mu_1\vee\mu_2$, $\lambda\not\le\mu_1$ and $\lambda\not\le\mu_2$, we have the following statements (A) and (B): (A) Put for each $i\in\{1,2\}$, $\lambda_i=\lambda\wedge\mu_i$. It follows $\lambda_i\le\mu_i$ such that $\lambda_1\not\le\mu_2$ and $\lambda_2\not\le\mu_1$. Hence there exist $\lambda_1,\lambda_2\in I^X$ such that $\lambda=\lambda_1\vee\lambda_2$
$$\begin{aligned}C(\lambda)&\le(C_1(\lambda_1)\wedge C_2(\lambda_1))\vee(C_1(\lambda_2)\wedge C_2(\lambda_2))\\ &=\mu_1\vee\mu_2. \quad (1)\end{aligned}$$

(B) For every families $\{\lambda_1,\lambda_2\}$ such that $\lambda=\lambda_1\vee\lambda_2$, from Theorem 3.3, since $\lambda=\lambda_1\vee\lambda_2\not\le\mu_1$, there exists $k\in\{1,2\}$ such that $\lambda_k\not\le\mu_1$. Hence we have
$$\begin{aligned}C(\lambda)&=\inf\{\textstyle\bigvee_{j=1}^{2}(C_1(\lambda_j)\wedge C_2(\lambda_j))\}\\ &\ge C_1(\lambda_k)\wedge C_2(\lambda_k)\\ &\ge\mu_2.\end{aligned}$$

Since $\lambda=\lambda_1\vee\lambda_2\not\le\mu_2$, there exists $l\in\{1,2\}$ such that $\lambda_l\not\le\mu_2$. Hence we have
$C(\lambda)\ge\mu_1$.

Since $\lambda\le\mu_1\vee\mu_2$, we have $\lambda_k\ne\lambda_l$. It follows that
$$C(\lambda)\ge\mu_1\vee\mu_2. \quad (2)$$
By (1) and (2), we have
$C(\lambda)=\mu_1\vee\mu_2$.

Similarly, for other cases, we can define $C:I^X\to I^X$ as follows:
$$C(\lambda)=\begin{cases}\tilde{0}, & \text{if } \lambda=\tilde{0},\\ \mu_1\wedge\mu_2, & \text{if } \tilde{0}\ne\lambda\le\mu_1\wedge\mu_2,\\ \mu_1, & \text{if } \lambda\le\mu_1,\lambda\not\le\mu_2,\\ \mu_2, & \text{if } \lambda\le\mu_2,\lambda\not\le\mu_1,\\ \mu_1\vee\mu_2, & \text{if } \lambda\le\mu_1\vee\mu_2,\lambda\not\le\mu_1,\lambda\not\le\mu_2,\\ \tilde{1}, & \text{otherwise.}\end{cases}$$

Example 3.11 Let $A=\{x, y\}$ and $X=\{x, y, z\}$ be sets. Let $\nu_1, \nu_2 \in I^X$ be fuzzy sets as follows:

$$\nu_1(x)=\frac{1}{2},\quad \nu_1(y)=\frac{1}{3},\quad \nu_1(z)=\frac{1}{4}$$

and

$$\nu_2(x)=\frac{1}{2},\quad \nu_2(y)=\frac{1}{3},\quad \nu_2(z)=\frac{3}{4}.$$

We define the fuzzy topology τ as follows:
$\tau=\{\tilde{0}, \tilde{1}, \nu_1, \nu_2\}$.
Since $i^{-1}(\nu_1)=i^{-1}(\nu_2)$, put $\mu=i^{-1}(\nu_1)=i^{-1}(\nu_2)$. From Theorem 2.7, we have
$\beta=\{\tilde{1}, \mu\}$.
Then $\tau_\beta=\{\tilde{0}, \tilde{1}, \mu\}$. On the other hand, we have $C_\tau: I^X \to I^X$ as follows:

$$C_\tau(\lambda)=\begin{cases}\tilde{0}, & \text{if } \lambda=\tilde{0}\\ \tilde{1}-\nu_2, & \text{if } \tilde{0}\neq\lambda\leq\tilde{1}-\nu_2,\\ \tilde{1}-\nu_1, & \text{if } \tilde{1}-\nu_2<\lambda\leq\tilde{1}-\nu_1,\\ \tilde{1}, & \text{otherwise.}\end{cases}$$

Since for $j=1, 2$, $i(\rho)\leq \tilde{1}-\nu_j$ iff $\rho\leq(\tilde{1}-i^{-1}(\nu_j))=\tilde{1}-\mu$ from Lemma 2.1, we have the followings:

$$i^{-1}(C_\tau(i(\rho)))=\begin{cases}\tilde{0}, & \text{if } \rho=\tilde{0},\\ \tilde{1}-\mu, & \text{if } \rho\leq\tilde{1}-\mu,\\ \tilde{1}, & \text{otherwise.}\end{cases}$$

Therefore we have
$\tau_c=\{\tilde{0}, \tilde{1}, \mu\}$.
Hence we have $\tau_c=\tau_\beta$. □

Example 3.12 Let X be a set. Define $\tau_1, \tau_2 \subset I^X$ as follows:
$\tau_1=\{\tilde{0}, \tilde{1}, \mu_1\}$, $\tau_2=\{\tilde{0}, \tilde{1}, \mu_2\}$.
From Theorem 2.7, we have
$\beta=\{\tilde{1}, \pi_1^{-1}(\mu_1), \pi_2^{-1}(\mu_2), \pi_1^{-1}(\mu_1)\wedge\pi_2^{-1}(\mu_2)\}$.
We obtain an initial topology τ_β as follows:
$\tau_\beta=\{\tilde{0}, \tilde{1}, \pi_1^{-1}(\mu_1), \pi_2^{-1}(\mu_2), \pi_1^{-1}(\mu_1)\wedge\pi_2^{-1}(\mu_2), \pi_1^{-1}(\mu_1)\vee\pi_2^{-1}(\mu_2)\}$

Define $C_{\tau_1}: I^X \to I^X$ as follows:

$$C_{\tau_1}(\rho_1)=\begin{cases}\tilde{0}, & \text{if } \rho_1=\tilde{0},\\ \tilde{1}-\mu_1, & \text{if } \tilde{0}\neq\rho_1\leq\tilde{1}-\mu_1,\\ \tilde{1}, & \text{otherwise.}\end{cases}$$

From Theorem 2.4, Define $C_{\tau_2}: I^Y \to I^Y$ as follows:

$$C_{\tau_2}(\rho_2)=\begin{cases}\tilde{0}, & \text{if } \rho_2=\tilde{0},\\ \tilde{1}-\mu_2, & \text{if } \tilde{0}\neq\rho_2\leq\tilde{1}-\mu_2,\\ \tilde{1}, & \text{otherwise.}\end{cases}$$

Let $\pi_1: X\times Y\to X$ and $\pi_2: X\times Y\to Y$ be projection maps. Since for $i=1, 2$, $\pi_i(\lambda)\leq\tilde{1}-\rho$ iff $\lambda\leq\tilde{1}-\pi_i^{-1}(\rho)$ from Lemma 2.1, we have the followings:

$$\pi_1^{-1}(C_{\tau_1}(\pi_1(\lambda)))=\begin{cases}\tilde{0}, & \text{if } \lambda=\tilde{0},\\ \tilde{1}-\pi_1^{-1}(\mu_1), & \text{if } \tilde{0}\neq\lambda\leq\tilde{1}-\pi_1^{-1}(\mu_1),\\ \tilde{1}, & \text{otherwise}\end{cases}$$

and

$$\pi_2^{-1}(C_{\tau_2}(\pi_2(\lambda)))=\begin{cases}\tilde{0}, & \text{if } \lambda=\tilde{0},\\ \tilde{1}-\pi_2^{-1}(\mu_2), & \text{if } \tilde{0}\neq\lambda\leq\tilde{1}-\pi_2^{-1}(\mu_2),\\ \tilde{1}, & \text{otherwise.}\end{cases}$$

By a similar method as Example 3.10, we can define $C: I^{X\times Y}\to I^{X\times Y}$ as follows:

$$C(\lambda)=\begin{cases}\tilde{0}, & \text{if } \lambda=\tilde{0},\\ (\tilde{1}-\pi_1^{-1}(\mu_1))\wedge(\tilde{1}-\pi_2^{-1}(\mu_2)), & \text{if } \tilde{0}\neq\lambda\leq (\tilde{1}-\pi_1^{-1}(\mu_1))\wedge(\tilde{1}-\pi_2^{-1}(\mu_2)),\\ \tilde{1}-\pi_1^{-1}(\mu_1), & \text{if } \lambda\leq\tilde{1}-\pi_1^{-1}(\mu_1),\ \lambda\not\leq 1-\pi_2^{-1}(\mu_2)\\ \tilde{1}-\pi_2^{-1}(\mu_2), & \text{if } \lambda\leq\tilde{1}-\pi_2^{-1}(\mu_2),\ \lambda\not\leq\tilde{1}-\pi_1^{-1}(\mu_1),\\ (\tilde{1}-\pi_1^{-1}(\mu_1))\vee(\tilde{1}-\pi_2^{-1}(\mu_2)), & \text{if } \lambda\leq(\tilde{1}-\pi_1^{-1}(\mu_1))\vee(\tilde{1}-\pi_2^{-1}(\mu_2)),\ \lambda\not\leq\tilde{1}-\pi_1^{-1}(\mu_1),\ \lambda\not\leq\tilde{1}-\pi_2^{-1}(\mu_2),\\ \tilde{1}, & \text{otherwise.}\end{cases}$$

We obtain a fuzzy topology $\tau_c\subset I^{X\times Y}$ as follows:
$\tau_c=\{\tilde{0}, \tilde{1}, \pi_1^{-1}(\mu_1), \pi_2^{-1}(\mu_2), \pi_1^{-1}(\mu_1)\wedge\pi_2^{-1}(\mu_2), \pi_1^{-1}(\mu_1)\vee\pi_2^{-1}(\mu_2)\}$.
Hence we have $\tau_c=\tau_\beta$. □

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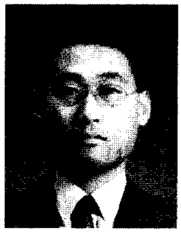

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