Architectures for Arbitrarily Connected Synchronization Networks

William C. Lindsey and Jeng-Hong Chen

Abstract: **In a synchronization (sync) network[1] containing N nodes, it is shown (Theorem 1c) that an *arbitrarily connected sync network* $\mathcal{S}$ is the union of a countable set of *isolated connecting sync networks* $\{\mathcal{S}_i, i = 1, 2, ..., L\}$, i.e., $\mathcal{S} = \bigcup_{i=1}^{L} \mathcal{S}_i$. It is shown (Theorem 2e) that a connecting sync network is the union of a set of *disjoint irreducible subnetworks* having one or more nodes. It is further shown (Theorem 3a) that there exists at least one *closed irreducible subnetwork* in $\mathcal{S}_i$. It is further demonstrated that a connecting sync network is the union of both a *master group* and a *slave group* of nodes. The *master group* is the union of *closed* irreducible subnetworks in $\mathcal{S}_i$. The *slave group* is the union of non-closed irreducible subnetworks in $\mathcal{S}_i$. The relationships between *master-slave (MS), mutual synchronous (MUS)* and *hierarchical MS/MUS* networks are clearly manifested [1]. Additionally, Theorem 5 shows that each node in the slave group is *accessible* by at least one node in the master group. This allows one to conclude that the synchronization information available in the master group can be reliably transported to each node in the slave group.**

Counting and combinatorial arguments are used to develop a recursive algorithm which counts the number A_N of arbitrarily connected sync network architectures in an N-nodal sync network and the number C_N of isolated connecting sync networks in $\mathcal{S}$. Examples for N=2, 3, 4, 5 and 6 are provided. Finally, network examples are presented which illustrate the results offered by the theorems. The notation used and symbol definitions are listed in Appendix A.

Index Terms: **Synchronization Network, Network Partition, Irreducible Network, Master-Slave Network, Mutually Synchronous Network, Hierarchical Network.**

I. INTRODUCTION

A synchronization network contains numerous assets and possesses numerous attributes. In what follows, sync network architectures are specified using *Set Theory* [2], [3]. In this paper, it is convenient to refer to a set or subset as a network or subnetwork respectively. An arbitrarily connected sync network is established by arbitrarily connecting a finite or countably infinite set of N nodes[2]. Each sync node in the arbitrarily connected sync network is assumed to be capable of processing all arriving sync signals so as to align in time the sync information stored locally with that contained in the arriving sync signals. This sync information is required for extracting the transported information (symbols, burst packets) between network nodes or for rerouting the transported information communicated to alternate network nodes. On the other hand, network sync information may be required for the purposes of deriving time-of-arrival measurements needed to support network location, navigation and/or guidance services. In either case, network sync information can be considered to be a commodity and a valuable network resource. A sync network is an *isolated* system in the sense that sync information never enters or leaves the network. In this sense, a sync network can be considered to be a circulatory system.

With regard to digital communication networks, there are two major categories of sync information. One is conveniently referred to as the synchronization information associated with the communications carrier. This sync information includes the carrier's phase and frequency. Carrier synchronization information at a network node implies aligning the locally generated carrier sync information with that contained in the arriving carrier signal or signals. Associated with a modulated communications carrier is the symbol timing (or epoch) associated with the clock; this clock sets the rate of information flow. Epoch (event marking) information contained in the communication burst packets or frame boundaries is also of great significance. These epochs boundaries are recovered from the received frame sync words or other special pattern streams contained in the communication signaling and control channels. Epoch or time synchronization among network clocks is also of interest in location and navigation system design. In what follows, we are not concerned with specifying network sync techniques, [4]–[7], but rather with characterizing architectures for use in sync network design [8]–[12]. The sync performance achievable when employing various sync network techniques and network performance metrics is covered elsewhere [1], [13]–[22].

The connecting relations between arbitrary nodes are defined in Definition 1. It is shown in Truth Table I that there are only two connecting relations which satisfy the three properties: reflexivity, symmetry, and transitivity. In this regard, two disjoint partitions are provided in Definitions 4 and 5 in order to gain insight into the properties of the $A_N = 2^{N(N-1)}$ possible network architectures in an arbitrarily connected sync network. Theorems will be provided which allow for the investigation of the flow of sync signals from node to node or from one subnetwork to another in an arbitrarily connected sync network.

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[1] For brevity throughout this paper, the word *sync* will be used to represent the word *synchronization* and the term *sync network* or simply *network* is used to represent the term *synchronization network*.

[2] In practice, the nodes are connected using network deployment guidelines.

II. NETWORK NODAL CONNECTIONS

Definition 1: A *synchronization link* from node i to node j in an arbitrarily connected sync network exists if a sync signal is transmitted from node i and is received and processed by node j; this is denoted by the directional symbol $i \rightarrow j$, otherwise[3], $i \not\rightarrow j$. In what follows, such a link serves to transport the required sync information discussed in the introduction. Other symbols that will be used to denote connections between any two network nodes i and j are defined as follows: (a) the symbol $i \leftrightarrow j$ will be used to denote *two-way connected* sync links, (b) the symbol $i \Rightarrow j$ will be used to denote *one-way accessible* (perhaps via other nodes) between nodes i and j, (c) the symbol $i \Leftrightarrow j$ denotes *two-way accessible* between nodes i and j, and (d) the symbol $i \sim j$ is used to imply *directly or indirectly connected* network nodes. For mathematical completeness, we will use the symbol $i \rightarrow i$ to imply node i is connected to itself[4]. Networks having only one node are not interesting in practice. This defines all possible connecting relations from a set theory operation perspective. Further discussion of these nodal connection symbols is now given:

(a) Two nodes i and j are said to be *two-way connected* (or $i \leftrightarrow j$) by sync links if $i \rightarrow j$ and $j \rightarrow i$; otherwise, $i \not\leftrightarrow j$. This implies that both nodes transmit sync signals to each other.

(b) If there exists a *direct or indirect* (via intermediate nodes) sync link from node i to node j, we say that node i can *access* node j or $i \Rightarrow j$, i.e.,

$$\exists \text{ nodes } x_1 = i, x_2, ..., x_m, \text{ and } x_{m+1} = j, 1 \leq m \leq N-1, \quad s.t.\ x_l \rightarrow x_{l+1} \forall\, l = 1, 2, ..., m,$$

where N is the total number of network nodes; otherwise, $i \not\Rightarrow j$. We will refer to these connecting paths as *one-way synchronization trails.*

(c) If $i \Rightarrow j$ and $j \Rightarrow i$, then we say that i and j are *two-way accessible* (or $i \Leftrightarrow j$); otherwise, $i \not\Leftrightarrow j$. In such cases, the sync signals are transmitted to and from node i and j.

(d) Two nodes i and j are said to be *directly or indirectly connected* (or $i \sim j$), if there exists a **connecting path** via intermediate nodes between nodes i and j, i.e.,

$$\exists \text{ nodes } x_1 = i, x_2, ..., x_m, \text{ and } x_{m+1} = j, 1 \leq m \leq N-1, \quad s.t.\ (x_l \rightarrow x_{l+1}) \text{ or } (x_{l+1} \rightarrow x_l), \forall\, l = 1, 2, ..., m.$$

Otherwise, $i \not\sim j$. This does not imply that $i \Rightarrow j$ or $j \Rightarrow i$ since there may not exist a *one-way sync trail* to transmit the sync signal from node i to node j or vice versa.

Definition 2: An arbitrarily connected sync network $\mathcal{B}$ is defined and established as follows. ***Step 1:*** Select a countably finite or countably infinite number of nodes and name the selected nodes using distinct non-negative integers from 1 to N. ***Step 2:*** If $N = 1$, stop. A one-nodal network is obtained. If $N > 1$, go to *Step 3*. ***Step 3:*** According to a decision rule, determine whether the sync link from node i to node j is connected $(i \rightarrow j)$ or not connected for each ordered pair of distinct nodes in an arbitrary ordered pair of nodes $(i, j), i \neq j, i, j = 1, 2, 3..., N$. Since there are two choices for each ordered pair, there will be $A_N = 2^{N(N-1)}$ possible network architectures for an arbitrarily connected sync network[5]. This number grows surprisingly fast as the number of network nodes increases. For example, if $N = 10$, there are $A_{10} = 2^{90} > 10^{27}$ possible network architectures, and $A_{10} = 2^{90}$ network performances to establish. Therefore, the problem of selecting a particular network architecture from this set by optimizing pertinent network performance metrics is of great interest [23]–[30] as well as the combining/breaking of networks to form a new network.

Example 1: Suppose 3 nodes are selected and numbered as nodes 1, 2, and 3. In order to establish the network connectivity, one could perform a Bernoulli trial to determine whether the nodal connectivity from node i to node j is established or not for each of the following ordered pair of nodes (i, j): (1,2), (1,3), (2,1), (2,3), (3,1), and (3,2). For this case, there are $2^{3\times 2} = 64$ arbitrarily connected sync network architectures.

Definition 3: Subnetworks within $\mathcal{B}$ and their connectivity are also important. The connecting relations between any two subnetworks E and F in $\mathcal{B}$ are defined as follows.

(a) $E \rightarrow F$(*one-way connected*).
(b) $E \leftrightarrow F$(*two-way connected*).
(c) $E \Rightarrow F$(one-way accessible).
(d) $E \Leftrightarrow F$(*two-way accessible*).
(e) $E \sim F$ (*directly or indirectly connected*); $E \not\sim F$(isolated).
(f) $E \cap F = \emptyset$ (*disjoint*).

From Definition 1 $(i \rightarrow i)$, if $E = F$, then one can show that $E \rightarrow F$[6]. We further elaborate on the connecting relations (a) through (f) in what follows.

(a) We say that $E \rightarrow F$, if $\exists\, i \in E, \exists\, j \in F$, and $i \rightarrow j$; otherwise, $E \not\rightarrow F$. The sync signal can be transmitted from subnetwork E to subnetwork F.

(b) We say that $E \leftrightarrow F$, if $\exists\, i, j \in E, \exists\, k, q \in F, i \rightarrow k$ and $q \rightarrow j$; otherwise, $E \not\leftrightarrow F$. At least one sync link is established from a node in E to a node in F and vice versa from F to E. But this does not imply that any node in E can transmit a sync signal to nodes in F or vice versa.

(c) We say that $E \Rightarrow F$, if $\exists\, i \in E, \exists\, j \in F$, and $i \Rightarrow j$; otherwise, $E \not\Rightarrow F$. The sync signal can be transmitted directly or indirectly from at least one node in E to at least one node in F.

(d) We say that $E \Leftrightarrow F$, if $E \Rightarrow F$ and $F \Rightarrow E$; otherwise, $E \not\Leftrightarrow F$. The sync signal can be transmitted directly or indirectly from at least one node in E to at least one node in F and vice versa from nodes in F to E.

(e) We say that E and F are *directly or indirectly connected* and denoted this by $E \sim F$,
if $\exists\, i \in E, \exists\, j \in F, i \sim$ j; otherwise, E and F are called *isolated* subnetworks or $E \not\sim F$. It is permissible that $E \not\rightarrow F$ and $F \not\rightarrow E$ but $E \sim F$ which means that there are no sync links connected between E and F but they can be connected via intermediate nodes not in E and F.

[3] Herein we are only concerned with whether the nodes are connected or not; however, the "*strength*" of this connection is of great importance in establishing network performance for various performance metrics and will be taken into account in a later paper.

[4] Therefore, one notes the connections: $i \leftrightarrow i, i \Rightarrow i, i \Leftrightarrow i$, and $i \sim i$.

[5] This is equivalent to considering that only one connecting relation exists in the following four cases for each non-ordered pair (i, j) of distinct nodes in an arbitrarily connected sync network: 1) $i \not\rightarrow j$ and $j \not\rightarrow i$, 2) $i \rightarrow j$ and $j \not\rightarrow i$, 3) $i \not\rightarrow j$ and $j \rightarrow i$, 4) $i \rightarrow j$ and $j \rightarrow i$. There are $4^{N(N-1)/2}$ possible network architectures.

[6] Therefore, $E \leftrightarrow E, E \Rightarrow E, E \Leftrightarrow E$, and $E \sim E$.

Table 1. Truth Table I

Connecting Relation: $\oplus$	(i) Reflexivity $i \oplus i$	(ii) Symmetry $i \oplus j$ iff. $j \oplus i$	(iii) Transitivity If $i \oplus j$, $j \oplus k$, then $i \oplus k$.
$\rightarrow$	True	False	False
$\leftrightarrow$	True	True	False
$\Rightarrow$	True	False	True
$\Leftrightarrow$	True	True	True
$\sim$	True	True	True

(f) We say that E and F are *disjoint* if $E \cap F = \emptyset$, i.e., E and F do not have any common nodes.

III. PARTITIONING ARBITRARILY CONNECTED SYNC NETWORKS

In order to identify the primitive building blocks and find suitable partitions for arbitrarily connected sync networks, the five connecting relations given in Definition 1 must be examined from an *equivalent relation* perspective, [2], [31]. In mathematics, if a defined mathematical relation satisfies the three properties, (i) reflexivity, (ii) symmetry and (iii) transitivity, an *equivalent relation* [2] is defined and a disjoint partition of the network nodes based on this relation can be established.

Using Definition 1, Truth Table I can be readily verified. From Truth Table I, one readily observes that there are two disjoint partitions for the network nodes based on the connecting relations $\Leftrightarrow$ and $\sim$ for an arbitrarily connected network $\mathcal{S}$. These two disjoint partitions of $\mathcal{S}$ will be needed when Definitions 4 and 5 are introduced. In particular, a *two-way accessible* ***class***[7] [31] can be defined using the connecting relation $\Leftrightarrow$. On the other hand, a *directly or indirectly connected* ***class*** can be defined using the connecting relation $\sim$. The reflexivity property indicates that the relation is valid for a one-nodal network. The symmetry property indicates that there is no difference in establishing this relation from node i to j or from node j to i. The transitivity property indicates that when establishing this relation from node i to node k, there is no difference in orders to connect the intermediate nodes between them. Therefore, when applying Truth Table I to characterize the building blocks of an arbitrarily connected sync network $\mathcal{S}$, one can show that: 1) each node belongs to a *two-way accessible* (or a *directly or indirectly connected*) subnetwork in $\mathcal{S}$, 2) different two-way accessible subnetworks are disjoint, and 3) the arbitrarily connected sync network is the union of all disjoint two-way accessible (or directly or indirectly connected) subnetworks in $\mathcal{S}$.

Example 2: Two disjoint partitions for the arbitrarily connected sync network $\mathcal{S}$ illustrated in Fig. 1 are demonstrated as follows.

(a) Partition $\mathcal{S}$ according to the two-way accessible relation ($\Leftrightarrow$):

$$\mathcal{S} = \{\text{node } 1\} \cup \{2,3,4,5\} \cup \{6,7,8\} \cup \{9,10\} \cup \{11\} \cup \{12,13\} \cup \{14,15\}.$$

(b) Partition $\mathcal{S}$ according to the directly or indirectly connected relation ($\sim$):

$$\mathcal{S} = \{\text{node } i, i = 1,2,...,11.\} \cup \{12,13,14,15\}.$$

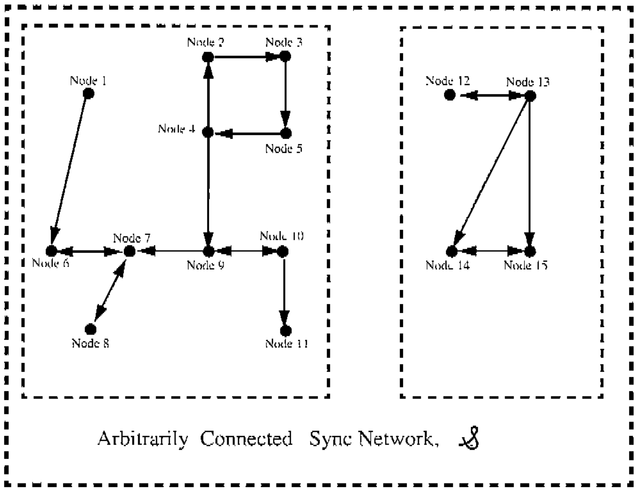

Fig. 1. One architecture for an arbitrarily connected sync network (N=15).

Partitioning of $\mathcal{S}$ is similar to putting distinct balls (nodes) into indistinguishable boxes (subnetworks). The same partition of an arbitrarily connected sync network $\mathcal{S}$ (according to a specified class) is always obtained and there is no difference in the orders of putting balls into indistinguishable boxes. Besides, each subset in (b) is a union of disjoint subsets found in (a). This observation motivates one to apply further partitioning of (b) in later sections. Suppose one tries to partition $\mathcal{S}$ based upon the relation $\leftrightarrow$. In this case, one will have trouble deciding which subset to put node 7 into since node 7 is two-way connected to nodes 6 and 8. However, nodes 6 and 8 are not two-way connected so nodes 6 and 8 cannot be put in the same subset. Similarly, there are no disjoint partitions according to the connection relations $\rightarrow$, $\leftrightarrow$ and $\Rightarrow$.

Definition 4: The ***connecting sync networks*** based upon the connecting relation "$\sim$" that results from partitioning an arbitrarily connected sync network are defined as follows.

(a) A subnetwork $E \subset \mathcal{S}$ is said to be a *connecting sync network* if

(i) If $k \in E, j \in E$ then $k \sim j$.
(ii) If $k \in E, k \sim j$ then $j \in E$.

Property (i) indicates that any two nodes in a connecting sync network are connected. Property (ii) forces subset E to be a *maximal*[8] set containing all nodes in $\mathcal{S}$ which are connected to node k. We will denote the i^{th} connecting sync network by the symbol $\mathcal{S}_i$.

(b) A subnetwork E $\subset \mathcal{S}$ is said to be *closed*, if $k \in E$ and $j \notin E$ imply that $j \not\sim k$. In fact, properties (i) and (ii) can imply (b) and all connecting sync networks are *closed*[9]. Therefore, a

[7] **Class** is a mathematical term; usually, it stands for a subset (referred as a network or subnetwork in this paper) defined by a relation which satisfies the three properties in Truth Table I.

[8] Here a set of nodes $E \subset \mathcal{S}$ is said to be a *maximal* set if E has the largest number of nodes in $\mathcal{S}$ and satisfies properties (i) and (ii).

[9] Hereafter, we will use the term *connecting sync network* to represent a *closed connecting sync network*.

Table 2. Distribution of possible sync network architectures for an arbitrarily connected sync network $\mathfrak{S}$ containing $N = 2$ nodes; $A_2 = 4$.

L	$N_1, N_2, ..., N_L$	Number of Possible Sync Network Architectures for $\mathfrak{S}$ as a Function of L.
2	1,1	$[\frac{2!}{1!1!}][\frac{1}{2!0!}][C_1C_1] = 1$
1	2	$C_2 = A_2 - 1 = 3$

Table 3. Distribution of possible sync network architectures for an arbitrarily connected sync network $\mathfrak{S}$ containing $N = 3$ nodes; $A_3 = 64$.

L	$N_1, N_2, ..., N_L$	Number of Possible Sync Network Architectures for $\mathfrak{S}$ as a Function of L.
3	1,1,1	$[\frac{3!}{1!1!1!}][\frac{1}{3!0!0!}][C_1C_1C_1] = 1$
2	1,2	$[\frac{3!}{1!2!}][\frac{1}{1!1!0!}][C_1C_2] = 9$
1	3	$C_3 = A_3 - 1 - 9 - 54$

connecting sync network E is connected as **one-cluster** without any *isolated* (Definition 3e) subnetworks which have no sync links to or from the rest of nodes in E.

Theorem 1: The connecting relations between two connecting sync networks are:

(a) Two isolated networks are disjoint. Two disjoint networks are not necessarily isolated.

(b) Two different connecting sync networks are isolated.

(c) An arbitrarily connected sync network is the union of isolated connecting sync networks, i.e., $\mathfrak{S} = \bigcup_{i=1}^{L} \mathfrak{S}_i$; where L is the number of connecting sync networks in $\mathfrak{S}$.

Proof:

(a) Suppose that E and F are not disjoint and at least one node (say i) is in E and in F. According to Definition 1, $i \sim i$ and E and F are not isolated.

(b) Suppose that two connecting sync networks E and $F \in \mathfrak{S}$ are not isolated and they are connected directly or indirectly from intermediate nodes. According to the Definition 4a-(ii), we have $E = F$ which is a contradiction.

(c) Each node belongs to a connecting sync network $\mathfrak{S}_i$ in $\mathfrak{S}$ according to a directly or indirectly connected relation. From (b), the connecting sync networks are disjoint and isolated. Therefore, an arbitrarily connected network is the union of isolated connecting sync networks by the mathematical equivalence relation [2] presented in Truth Table I. □

Since an arbitrarily connected sync network $\mathfrak{S}$ is the union of L isolated connecting sync networks ($\mathfrak{S} = \bigcup_{i=1}^{L} \mathfrak{S}_i$) (Theorem 1), it is shown in Appendix B that the number A_N of different sync network architectures for $\mathfrak{S}$ is given by

$$A_N = 2^{N(N-1)} = \sum_{L=1}^{N} \left\{ \sum_{(\sum_{i=1}^{L} N_i = N, 1 \le N_i < N)} \left[\frac{N!}{N_1!N_2!...N_L!}\right] \times \left[\frac{1}{\nu_1!\nu_2!\nu_3!...\nu_N!}\right] [C_{N_1}C_{N_2}...C_{N_L}] \right\}, \quad (1)$$

where N_i is the total number of nodes in $\mathfrak{S}_i$, C_k is the total number of possible sync network architectures for a connecting sync network having k nodes, and ν_j is the number of connecting sync networks $\mathfrak{S}_i$ in $\mathfrak{S}$ which have node number $j, 1 \le j \le N$. Finding the number C_N directly is a difficult problem and there appears to be no closed form for C_N. An easier way to specify C_N is to use a recursion method starting from $C_1 = A_1 = 1$. Since the number C_N is a special case of A_N when all N nodes are directly or indirectly connected (for the case $L = 1$), one can readily show from (1) that if $N \ge 2$, then

$$C_N = A_N - \sum_{L=2}^{N} \left\{ \sum_{(\sum_{i=1}^{L} N_i = N, 1 \le N_i \le N)} \left[\frac{N!}{N_1!N_2!...N_L!}\right] \times \left[\frac{1}{\nu_1!\nu_2!\nu_3!...\nu_N!}\right] [C_{N_1}C_{N_2}...C_{N_L}] \right\}. \quad (2)$$

By using (2) recursively, one can specify C_N from A_N and $C_i, i = 1, 2, ..., N-1$. Several examples of the distribution of possible sync network architectures in an arbitrarily connected

Table 4. Distribution of possible sync network architectures for an arbitrarily connected sync network $\mathfrak{S}$ containing $N = 4$ nodes; $A_4 = 4096$.

L	$N_1, N_2, ..., N_L$	Number of Possible Sync Network Architectures for $\mathfrak{S}$ as a Function of L.
4	1,1,1,1	$[\frac{4!}{1!1!1!1!}][\frac{1}{4!0!0!0!}][C_1^4] = 1$
3	1,1,2	$[\frac{4!}{1!1!2!}][\frac{1}{2!1!0!0!}][C_1^2C_2] = 18$
2	1,3	$[\frac{4!}{1!3!}][\frac{1}{1!0!1!0!}][C_1C_3] = 216$
	2,2	$[\frac{4!}{2!2!}][\frac{1}{0!2!0!0!}][C_2^2] = 54$
1	4	$C_4 = A_4 - 1 - 18 - 216 - 54 = 3807$

Table 5. Distribution of possible sync network architectures for an arbitrarily connected sync network $\mathfrak{S}$ containing $N = 5$ nodes; $A_5 = 1048576$.

L	$N_1, N_2, ..., N_L$	Number of Possible Sync Network Architectures for $\mathfrak{S}$ as a Function of L.
5	1,1,1,1,1	$[\frac{5!}{1!1!1!1!1!}][\frac{1}{5!0!0!0!0!}][C_1^5] = 1$
4	1,1,1,2	$[\frac{5!}{1!1!1!2!}][\frac{1}{3!1!0!0!0!}][C_1^3C_2] = 30$
3	1,1,3	$[\frac{5!}{1!1!3!}][\frac{1}{2!0!1!0!0!}][C_1^2C_3] = 540$
	1,2,2	$[\frac{5!}{1!2!2!}][\frac{1}{1!2!0!0!0!}][C_1C_2^2] = 135$
2	1,4	$[\frac{5!}{1!4!}][\frac{1}{1!0!0!1!0!}][C_1C_4] = 19035$
	2,3	$[\frac{5!}{2!3!}][\frac{1}{0!1!1!0!0!}][C_2C_3] = 1620$
1	5	$C_5 = A_5 - 1 - 30 - 540 - 135 - 19035 - 1620 = 1027215$

Table 6. Some specific values for A_N, C_N and their ratios.

N	A_N	C_N	C_N/A_N	A_{N-1}/A_N	C_{N-1}/C_N
1	1	1	100%	—	—
2	4	3	75%	25%	33.33%
3	64	54	84.38%	6.25%	5.56%
4	4096	3807	92.94%	1.56%	1.42%
5	1048576	1027215	97.96%	0.39%	0.37%
6	1073741824	1067309298	99.40%	0.098%	0.096%

sync network containing N=2,3,...,6 nodes as a function of $L = 1, 2, ...N$, are provided in Tables 2 to 6.

For example, for the case $L = 2$ in Table 3, without loss of generality, one can choose $\mathcal{S}_1 = \{\text{node } 1\}$ and $\mathcal{S}_2 = \{\text{ nodes } 2, 3\}$. This results in three choices for connecting nodes 2 and 3 together (2$\sim$3). Therefore, one has a total 3×3=9 choices for the case $L = 2$ since each node can be placed in $\mathcal{S}_1$. In fact, all possible sync architectures in an arbitrarily connected sync network containing $N = 3$ nodes are illustrated in Fig. 2. The architecture (plesiochronous case) in the top-left-hand corner is the only architecture for the case $L = 3$. For $L = 2$, there are nine architectures indicated by the dark blocks. The remaining 54 architectures are for the case $L = 1$.

Table 7 demonstrates that the case $L = 1$ contains most of the total number A_N of possible sync network architectures. When the total number of nodes N approaches infinity, one readily notes from (2) the interesting limiting ratios:

$$\lim_{N\to\infty} \frac{C_N}{A_N} = 1, \tag{3a}$$

$$\lim_{N\to\infty} \frac{A_K}{A_N} = 0, \forall\, 1 \le K \le N-1, \tag{3b}$$

$$\lim_{N\to\infty} \frac{C_K}{C_N} = 0, \forall\, 1 \le K \le N-1. \tag{3c}$$

IV. PARTITIONING CONNECTING SYNC NETWORKS

Partitioning of arbitrarily connected sync networks into a set of connecting sync networks was presented in Section III. The connecting relations and properties of connecting sync networks are investigated in this section.

Definition 5: The resulting ***irreducible subnetworks*** based upon the connecting relation "$\Leftrightarrow$" for partitioning a connecting sync network are defined as follows.

(a) A subnetwork $E \subset \mathcal{S}$ is said to be an *irreducible subnetwork* of $\mathcal{S}$ if

(i) $k \in E, j \in E$ then $k \Leftrightarrow j$.
(ii) $k \in E, k \Leftrightarrow j$ then $j \in E$.

One should be cautioned that nodes in E are allowed to *access* nodes not in E; the reverse statement is also true. Property (i) indicates that arbitrary nodes k and j in E are *two-way accessible*. Property (ii) forces subnetwork E to be a *maximal* subnetwork containing any node j in $\mathcal{S}$ that can *two-way access* k or $j \Leftrightarrow k$.

(b) An irreducible subnetwork $E \subset \mathcal{S}$ is said to be *closed*, if $k \in E$ and $j \notin E$ imply that $j \not\Rightarrow k$. One should be cautioned that the nodes in E are allowed to *access* nodes not in E; the reverse statement is not true.

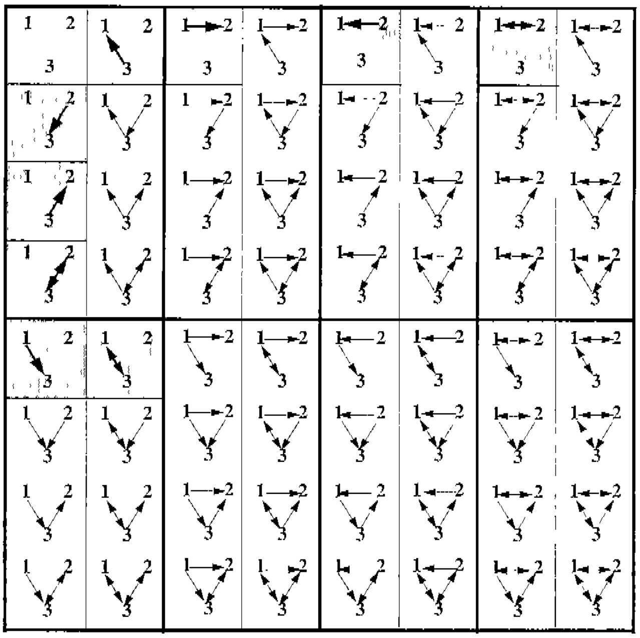

Fig. 2. $A_3 = 2^{3\times 2} = 64$ possible sync network architectures for a three-nodal arbitrarily connected sync network.

Table 7. Distribution of possible sync network architectures for an arbitrarily connected sync network $\mathcal{S}$ containing $N = 6$ nodes; $A_6 = 1073741824$.

L	$N_1, N_2, ..., N_L$	Number of Possible Sync Network Architectures for $\mathcal{S}$ as a Function of L.
6	1,1,1,1,1,1	$[\frac{6!}{1!1!1!1!1!1!}][\frac{1}{6!0!0!0!0!0!}][C_1^6] = 1$
5	1,1,1,1,2	$[\frac{6!}{1!1!1!1!2!}][\frac{1}{4!1!0!0!0!0!}][C_1^4 C_2] = 45$
4	1,1,1,3	$[\frac{6!}{1!1!1!3!}][\frac{1}{3!0!1!0!0!0!}][C_1^3 C_3] = 1080$
	1,1,2,2	$[\frac{6!}{1!1!2!2!}][\frac{1}{2!2!0!0!0!0!}][C_1^2 C_2^2] = 405$
3	1,1,4	$[\frac{6!}{1!1!4!}][\frac{1}{2!0!0!1!0!0!}][C_1^2 C_4] = 57105$
	1,2,3	$[\frac{6!}{1!2!3!}][\frac{1}{1!1!1!0!0!0!}][C_1 C_2 C_3] = 9720$
	2,2,2	$[\frac{6!}{2!2!2!}][\frac{1}{0!3!0!0!0!0!}][C_2^3] = 405$
2	1,5	$[\frac{6!}{1!5!}][\frac{1}{1!0!0!0!1!0!}][C_1 C_5] = 6163290$
	2,4	$[\frac{6!}{2!4!}][\frac{1}{0!1!0!1!0!0!}][C_2 C_4] = 171315$
	3,3	$[\frac{6!}{3!3!}][\frac{1}{0!0!2!0!0!0!}][C_3^2] = 29160$
1	6	$C_6 = A_6 - 1 - 45 - 1080 - ... - 171315 - 29160 = 1067309298$

(c) A *closed* irreducible subnetwork is called a *master irreducible subnetwork*.

(d) A non-closed irreducible subnetwork is called a *slave irreducible subnetwork*.

According to Definition 1, Definitions 5(a)-5(b) are valid for node numbers in E equal to or greater than 1. One should notice that Truth Table I implies that two connecting sync networks in an arbitrarily connected sync network are either disjoint or they are the same.

If E and F are two connecting sync networks in an arbitrarily connected sync network and $E \neq F$, then $E \not\sim F$ since E and F cannot be connected via a third connecting sync network in $\mathfrak{S}$. On the other hand, if E and F are two irreducible subnetworks in a connecting sync network $\mathfrak{S}_i$ and $E \neq F$, then $E \sim F$. Irreducible subnetworks E and F can be connected directly or via a third irreducible subnetwork in $\mathfrak{S}_i$.

Theorem 2: Suppose that E and F are irreducible subnetworks in a connecting sync network $\mathfrak{S}_i$. If $E \neq F$, then

(a) $E \not\leftrightarrow F$: two different irreducible subnetworks cannot be *two-way connected*.
(b) $E \not\Leftrightarrow F$: two different irreducible subnetworks cannot be *two-way accessible*.
(c) $E \sim F$: two different irreducible subnetworks are *directly* or indirectly connected.
(d) $E \cap F = \emptyset$: two different irreducible subnetworks are *disjoint*.
(e) A connecting sync network is a union of disjoint irreducible subnetworks.

Proof:

(a) Suppose that $E \rightarrow F$ and $F \rightarrow E$, then $E \cup F$ is an irreducible subnetwork. This is a contradiction because the irreducible subnetworks E or F are defined as the *maximal* set of a *two-way accessible* relation in Definition 5(a).

(b) Suppose that $E \Rightarrow F$ and $F \Rightarrow E$, then the union of E, F and all intermediate nodes required to establish $E \Rightarrow F$ and $F \Rightarrow E$ is a new *maximal* set defined as an irreducible subnetwork in Definition 5a. This is a contradiction.

(c) Subnetworks $E \sim F$, since arbitrary nodes i and j are connected in a connecting sync network.

(d) Suppose that irreducible subnetworks E and F are not disjoint and there exists node i which is in E and in F, then $E \cup F$ is an irreducible subnetwork since node i can *access* all nodes in E and in F. This is a contradiction.

(e) One should notice that the *irreducible subnetwork* in Definition 5a is a *class* defined by the *two-way access* relation which satisfies Truth Table I. When characterizing all nodes in a connecting sync network $\mathfrak{S}_i$, three results are obtained from Truth Table I: 1) each node in $\mathfrak{S}_i$ belongs to an irreducible subnetwork, 2) all irreducible subnetworks are disjoint in $\mathfrak{S}_i$, and 3) $\mathfrak{S}_i$ is the union of all irreducible subnetworks in $\mathfrak{S}_i$ by the mathematical equivalence relation given in Truth Table I. □

Justification for these two partitions provisioned for in Truth Table I and defined in Definitions 4 and 5 is explained as follows. The first partition of $\mathfrak{S}$ (Definition 4 and Theorem 1) is found by use of intuition since there are no sync signals exchanged between two *isolated* connecting sync networks. In addition, if the sync signals can be transported between any two nodes in a specified subnetwork in $\mathfrak{S}$, then it is not necessary to perform further partitioning in this *irreducible* subnetwork (Definition 5 and Theorem 2). *Irreducible* subnetworks are the smallest building blocks of interest when investigating the flow of sync information in $\mathfrak{S}$.

Theorem 3: If one traces a sync signal path through the network starting from arbitrary node to its originating or source node in a connecting sync network, one will always end up at a *master irreducible subnetwork*. Theorem 3 will prove this existence for finite value of N.

(a) In a connecting sync network, there exists at least one master irreducible subnetwork.

(b) If F is a *closed* subnetwork in a connecting sync network, then there exists at least one master irreducible subnetwork in F.

Comment: Theorem 3 establishes a surprising result. It demonstrates that by connecting sync network nodes with directional sync links into one-cluster (a connecting sync network), at least a part of the connecting sync network[10] will naturally become a master irreducible subnetwork. If one breaks a connecting sync network into several new connecting sync networks, at least one master irreducible subnetwork can be found in each new connecting sync network since each connecting sync network is *closed* (Definition 4b). Such architectures are, no doubt, of great significance in scientific disciplines such as biology and physics as well as in the subjects of religion and architecture of societies[11].

Proof:

(a) Suppose that a connecting sync network is only an irreducible subnetwork, it is *isolated* from all other nodes in $\mathfrak{S}$. Therefore, it is *closed* from Definition 5b. Suppose that a connecting sync network is the union of finite $K \geq 2$ non-closed irreducible subnetworks E_i, $1 \leq i \leq K$. Since all nodes are connected in a connecting sync network, all non-closed irreducible subnetworks E_i must be accessible by at least one irreducible subnetwork E_j, $(i \neq j)$. Without loss of generality, arbitrary irreducible subnetworks E_1 and E_2 are picked first and $E_2 \rightarrow E_1$. Since E_2 is not closed, there exists an E_3 such that $E_3 \rightarrow E_2$. According to Theorem 2b, two irreducible subnetworks cannot be two-way accessible and therefore $E_3 \neq E_1$. By repeating this argument, one has $E_{i+1} \rightarrow E_i, 1 \leq i \leq K-1$. Since K is finite and all other irreducible subnetworks cannot be *two-way accessible* to E_K (Theorem 2b), E_K is *closed*. This is a contradiction.

(b) Since the *two-way accessible* property in Definition 5a is not *closed* (Definition 5b), a closed subnetwork cannot include a subset of an irreducible subnetwork. Thus, a *closed* subnetwork E in a connecting sync network is the union of **disjoint** irreducible subnetworks. Therefore, from similar arguments pro-

[10] If a connecting sync network is only an irreducible network, it is *closed* in $\mathfrak{S}$ since two connecting sync networks in $\mathfrak{S}$ are isolated. The whole network becomes a master group which has only one master irreducible subnetwork.

[11] For example, one may assume that the *master group* represents the *Gods or leaders* and their ***commands*** can be delivered to all other people in the *slave group*.

vided in (a), the theorem is proved. □

In conclusion, connecting sync networks $\mathcal{S}_i$ are the smallest isolated building blocks for an arbitrarily connected sync network $\mathcal{S}$ and there are no isolated subnetworks in $\mathcal{S}_i$. On the other hand, irreducible subnetworks are the smallest building blocks of a connecting sync network $\mathcal{S}_i$ and there are no closed subnetworks in an irreducible subnetwork.

V. DECOMPOSITION OF THE ARCHITECTURE OF ARBITRARILY CONNECTED SYNC NETWORKS

In this section, we succinctly summarize the results developed thus far in this paper. From Theorem 1c, an N-nodal ($N \geq 1$) arbitrarily connected sync network $\mathcal{S}$ is the union of L isolated connecting sync networks, $\mathcal{S}_i$, $i = 1, 2, ..., L$ in which $\mathcal{S}_i$ has N_i nodes. According to Theorems 2e and 3a, there exists at least one master irreducible subnetwork such that the connecting sync network $\mathcal{S}_i$, is the union of: 1) a *master group* of nodes M_i which is the union of K_{im} master irreducible subnetworks, M_{ij}, and 2) the *slave group* of nodes S_i which is the union of K_{is} slave irreducible subnetworks, S_{ij}. Furthermore, there are K_{im1} master irreducible subnetworks in M_i which have only one node and $K_{im2} = K_{im} - K_{im1}$ master irreducible subnetworks in M_i which have at least two nodes. The j^{th} master irreducible subnetwork in M_i has m_{ij} nodes and the total number of nodes in the master group M_i is m_i.

In summary, the architecture of an arbitrarily connected sync network can be characterized in terms of these subnetworks and is concisely set forth using set theoretic notation as (4) shown at the bottom of this page.

Equation (4a) is a disjoint and isolated partition for $\mathcal{S}$. Equation (4b-d) are disjoint but not isolated partitions for $\mathcal{S}_i$. Equation (4) says that a connecting sync network is characterized by the parameter set $P_i = (N_i, m_i, K_{im1}, K_{im2}, K_{is})$ for all $i = 1, 2, ..., L$. The architecture of the i^{th} connecting sync network $\mathcal{S}_i$ in $\mathcal{S}$ is illustrated in Fig. 3.

Definition 6: Two types of connecting sync networks that contain only one master irreducible subnetwork in a connecting sync network are defined as follows:

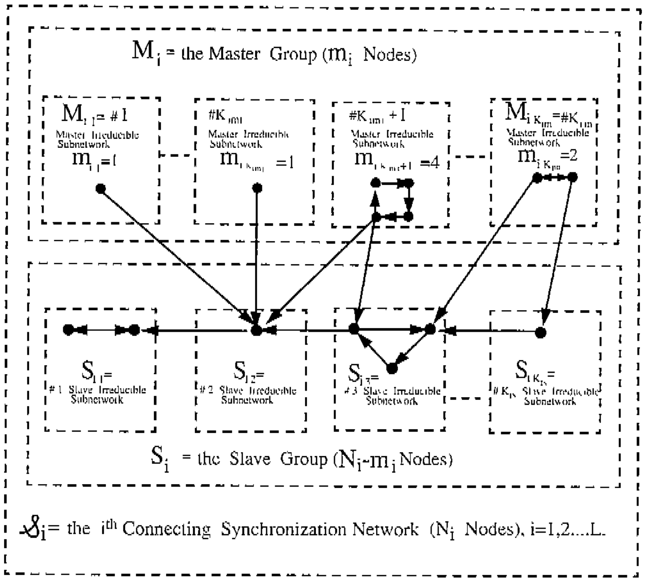

Fig. 3. Architecture of the i^{th} connecting sync network, $\mathcal{S}_i$.

(a) A connecting sync network $\mathcal{S}_i$ is called a ***master slave (MS)*** network, if there is only one master node in the master group ($K_{im} = K_{im1} = 1, K_{im2} = 0$ and $m_{i1} = 1$).

(b) A connecting sync network $\mathcal{S}_i$ is called a ***mutual synchronous (MUS)*** network, if there is only one master irreducible subnetwork having at least two nodes ($K_{im} = K_{im2} = 1, K_{im1} = 0$, and $m_{i1} \geq 2$).

Using Definition 6 and the network model for $\mathcal{S}_i$ in (4), a ***hierarchical MS/MUS*** network architecture is obtained if the slave irreducible subnetwork does not form an empty set. If there exists more than one slave irreducible subnetworks, a multilevel hierarchical MS or MUS network architectures can be realized.

Theorem 4: The connecting relations between two irreducible subnetworks in a connecting sync network are characterized by Theorem 2. The additional connecting relations between the master group and the slave group of nodes and the master and slave irreducible subnetworks in $\mathcal{S}_i$ are character-

$$\mathcal{S} = \bigcup_{i=1}^{L} \mathcal{S}_i, 1 \leq L \leq N = \sum_{i=1}^{L} N_i; \mathcal{S}_i \cap \mathcal{S}_j = \emptyset, \mathcal{S}_i \not\to \mathcal{S}_j, \ \forall\, i \neq j. \tag{4a}$$

$$\mathcal{S}_i = M_i \cup S_i, \ \ i = 1, 2, ..., L; M_j \cap S_k = \emptyset, \ \forall\, j, k. \tag{4b}$$

$$M_i = \bigcup_{j=1}^{K_{im}} M_{ij}, 1 \leq K_{im} = K_{im1} + K_{im2} \leq \ \text{Max}\ \{1, N_i\ \text{-}1\}\ ; M_{ij} \cap M_{kl} = \emptyset, \ \forall\, (i,j) \neq (k,l). \tag{4c}$$

$$S_i = \bigcup_{j=1}^{K_{is}} S_{ij}, 0 \leq K_{is} \leq N_i - 1; S_{ij} \cap S_{kl} = \emptyset, \ \forall\, (i,j) \neq (k,l). \tag{4d}$$

$$1 \leq m_i = \sum_{j=1}^{K_{im}} m_{ij} \leq N_i, 1 \leq m_{ij} \leq N_i. \tag{4e}$$

$$m_{ij} = 1, \ \text{for}\ 1 \leq j \leq K_{im1}, \ \text{and}\ m_{ij} \geq 2, \ \text{for}\ K_{im1} + 1 \leq j \leq K_{im}. \tag{4f}$$

ized as follows: $\forall\, i,\ 1 \le i \le L,$ and $\forall\, j,\ 0 \le j \le N_i - 1$.

(a) $M_i \to S_i$: there exists at least one sync link from the master group to the slave group.

(b) $S_i \not\to M_i$: there are no sync links connecting the slave group to the master group.

(c) $S_i \not\Rightarrow M_i$: the nodes in the slave group cannot access any node in the master group.

(d) $M_i \sim S_i$: the master group and the slave group are *directly or indirectly connected.*

(e) $M_i \cap S_i = \emptyset$: the master and slave groups are disjoint.

(f) $M_{ij} \not\to M_{ik}$: there are no sync link connections between different master irreducible subnetworks.

(g) $M_{ij} \not\Rightarrow M_{ik}$: the nodes in any master irreducible subnetwork cannot access those nodes in all other master irreducible subnetworks.

(h) $S_{ij} \not\to M_{ij}$: there are no sync links connecting any slave irreducible subnetwork to any master irreducible subnetwork.

(i) $S_{ij} \not\Rightarrow M_{ij}$: the nodes in any slave irreducible subnetwork cannot *access* nodes in any master irreducible subnetwork.

(j) $M_{ij} \to S_i$: there exists at least one sync link which connects each master irreducible subnetwork to at least one node in the slave group.

Proof: Theorem 4 can be easily proved using all of the previous definitions and theorems. □

Theorem 5: For any node j in the slave group S_i, there exists at least one node, say i, in the master group M_i such that $i \Rightarrow j$.

Proof: Assuming that there exists one node j in the slave group of nodes which is not *accessible* from all nodes in the master group, i.e.,

$$\exists \text{ node } j \in S_i,\ \forall \text{ node } i \in M_i, i \not\Rightarrow j.$$

The connecting network $\mathfrak{S}_i$ can be partitioned into three subnetworks, i.e.,

$$\mathfrak{S}_i = M_i \cup F_2 \cup F_3,$$

where F_2 is the subnetwork of all nodes in S_i which can *access* node j, and F_3 contains the remaining nodes in S_i which cannot *access* j. In addition,

$$\forall\, i \in M_i \cup F_3, \forall\, k \in F_2, i \not\Rightarrow k.$$

Since there are no sync links from the slave group to the master group (Theorem 4b) and all network nodes in $\mathfrak{S}_i$ are directly or indirectly connected, one can readily conclude that

$$M_i \to F_3 \text{ and } F_2 \to F_3, \text{ i.e., } \exists\, r \in F_2, \exists\, q \in F_3, r \to q.$$

Therefore, from Definition 5b, F_2 is a closed subnetwork. From Theorem 3b, there exists a master irreducible subnetwork in F_2 which gives a contradiction. □

Theorem 6: If there is only one master irreducible subnetwork (K_{im}=1) in $\mathfrak{S}_i$, then each node in the master group can *access* all other network nodes in $\mathfrak{S}_i$.

Proof: Theorem 6 can be proved by following the same partitions and derivations used to prove Theorem 5. □

VI. BREAKING AND/OR COMBINING CONNECTING SYNC NETWORKS

In practice, it is of interest to investigate the effect on network performance when breaking (failures of network nodes) or combining (adding new network nodes) connecting sync networks. A few significant facts are readily observable from the theorems, viz.,

(1) There exists at least one master irreducible subnetwork in each new connecting sync network when breaking sync links or adding new sync links or network nodes.

(2) If a connection in a connecting sync network is broken or a network node fails, new connecting sync networks and/or new master irreducible subnetworks may result with new network sync architectures.

(3) Without breaking the original connections, when two connecting sync networks are connected together so as to create a new or combined network, the slave groups from both connecting sync networks will never become a subnetwork of the new master group. Therefore, when combining connecting sync networks, it is recommended that this be accomplished by connecting the master groups. This avoids creating a higher-level network architecture [11], [12] than what appeared in the original network.

Example 3: Consider an arbitrarily connected sync network $\mathfrak{S}$ consisting of the union of the two isolated connecting sync networks $\mathfrak{S}_1$ and $\mathfrak{S}_2$ illustrated in Fig. 1. One can readily show from (4) that

(a)

$$\begin{gathered}\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2,\\ \mathfrak{S}_i = M_i \cup S_i, i = 1, 2,\\ M_1 = \textstyle\bigcup_{j=1}^{2} M_{ij} = \{1\} \cup \{2, 3, 4, 5\},\\ S_1 = \textstyle\bigcup_{j=1}^{3} S_{ij} = \{6, 7, 8\} \cup \{9, 10\} \cup \{11\},\\ M_2 = M_{21} = \{12, 13\},\\ S_2 = S_{21} = \{14, 15\}.\end{gathered}$$

(b) If the sync link from node 4 to node 9 in Fig. 1 is broken, then the resulting arbitrarily connected sync network is broken into three connecting sync networks. In particular, one can show that:

$$\begin{gathered}\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \mathfrak{S}_3,\\ \mathfrak{S}_i = M_i \cup S_i, i = 1, 2, 3,\\ M_1 = \{1\} \cup \{9, 10\},\\ S_1 = \{6, 7, 8\} \cup \{11\},\\ M_3 = \{2, 3, 4, 5\},\\ S_3 = \emptyset.\end{gathered}$$

and $\mathfrak{S}_2$ is the same as in (a). A new connecting sync network $\mathfrak{S}_3$ (which is a MUS network) and a master irreducible subnetwork $\{9, 10\}$ is generated.

(c) If one sync link from node 5 to node 12 is added to Fig. 1, then the resulting arbitrarily connected sync network becomes a connecting sync network. In particular, one can show that

$$\begin{gathered}\mathfrak{S} = \mathfrak{S}_1 = M_1 \cup S_1,\\ M_1 = \{1\} \cup \{2, 3, 4, 5\},\\ S_1 = \{6, 7, 8\} \cup \{9, 10\} \cup \{11\} \cup \{12, 13\} \cup \{14, 15\}.\end{gathered}$$

The original master group M_2 in (a) becomes a subnetwork of the new slave group.

(d) If one sync link from node 10 to node 14 is added to Fig. 1, then the resulting arbitrarily connected sync network becomes a connecting sync network. In particular, one can show that

$$\mathfrak{S} = \mathfrak{S}_1 = M_1 \cup S_1,$$
$$M_1 = \{1\} \cup \{2,3,4,5\} \cup \{12,13\},$$
$$S_1 = \{6,7,8\} \cup \{9,10\} \cup \{11\} \cup \{14,15\}.$$

The new master/slave group is the union of the original master/slave group given in (a).

VII. CONCLUSIONS

This paper has presented a mathematical characterization summarized in (4) for the architectures of arbitrarily connected synchronization networks $\mathfrak{S}$ in terms of a set of disjoint isolated connecting synchronization networks of size C_N in (2); N is the number of network nodes. The isolated connecting sync networks are further partitioned into the number of a set of a master and a slave group of nodes. Moreover, it is demonstrated that there exists at least one closed irreducible subnetwork in each connecting sync network. It is succinctly demonstrated how master-slave (MS), mutual synchronous (MUS) and hierarchical MS/MUS network architectures are related by applying the definitions and theorems proven in this paper. The authors believe that decomposing and partitioning of an arbitrarily connected sync network into disjoint fundamental building blocks provides much needed and new insights into problems associated with architecting synchronization networks and characterizing their performance metrics [11], [12]. Applications to Personal Communication Networks (PCN) such as DCS-1800, PCS-1900, Global Systems for Mobile Communications (GSM), IS-95/CDMA, etc. are forthcoming.

APPENDIX A : GLOSSARY OF NOTATION

Sets:

$\mathfrak{S}$= Arbitrarily connected sync network, $\mathfrak{S} = \bigcup_{i=1}^{L} \mathfrak{S}_i$;
$\quad \mathfrak{S}_i \cap \mathfrak{S}_j = \emptyset, \mathfrak{S}_i \not\sim \mathfrak{S}_j,\ \forall\, i \neq j$.

$\mathfrak{S}_i$= Connecting sync network, $\mathfrak{S}_i = M_i \cup S_i$;
$\quad M_j \cap S_k = \emptyset,\ \forall\, j, k$.

M_i= the master group of nodes, $M_i = \bigcup_{j=1}^{K_{im}} M_{ij}$;
$\quad M_{ij} \cap M_{kl} = \emptyset,\ \forall\, (i,j) \neq (k,l)$.

M_{ij}=the j^{th} master irreducible subnetwork in $\mathfrak{S}_i$.

S_i= the slave group of nodes, $S_i = \bigcup_{j=1}^{K_{is}} S_{ij}$;
$\quad S_{ij} \cap S_{kl} = \emptyset,\ \forall\, (i,j) \neq (k,l)$.

S_{ij}= the j^{th} slave irreducible subnetwork in $\mathfrak{S}_i$;
$\quad M_{ij} \cap S_{kl} = \emptyset,\ \forall\, i,j,k,l$.

Numbers:

N = the total number of nodes in $\mathfrak{S}$, $N = \sum_{i=1}^{L} N_i$.
L = the total number of $\mathfrak{S}_i$ in $\mathfrak{S}$.
N_i = the total number of nodes in $\mathfrak{S}_i$.
m_i = the total number of nodes in M_i, $m_i = \sum_{j=1}^{K_{im}} m_{ij}$.
m_{ij} = the total number of nodes in M_{ij}.
K_{im} = the total number of M_{ij} in $\mathfrak{S}_i$, $K_{im} = K_{im1} + K_{im2}$.
K_{im1} = the total number of M_{ij} in $\mathfrak{S}_i$ which has only one node.
K_{im2} = the total number of M_{ij} in $\mathfrak{S}_i$ which has at least two nodes.
K_{is} = the total number of S_{ij} in S_i.
A_N = the number of possible sync architectures in an arbitrarily connected sync network containing N nodes.
C_N = the number of possible sync architectures in a connecting sync network containing N nodes.

The connecting relations between two nodes i and j:

$i \rightarrow j$: one-way connected.
$i \leftrightarrow j$: two-way connected.
$i \Rightarrow j$: one-way accessible. (directions do matter)
$i \Leftrightarrow j$: two-way accessible. (directions do matter)
$i \sim j$: directly or indirectly connected. (not necessarily accessible, directions do not matter)

The connecting relations between subnetworks E and F:

$E \rightarrow F$: one-way connected.
$E \leftrightarrow F$: two-way connected.
$E \Rightarrow F$: one-way accessible. (directions do matter)
$E \Leftrightarrow F$: two-way accessible. (directions do matter)
$E \sim F$: directly or indirectly connected. (not necessarily accessible, directions do not matter)

APPENDIX B : THE NUMBER OF ARCHITECTURES IN AN ARBITRARILY CONNECTED SYNC NETWORK

In this appendix, the number of possible network architectures for an arbitrarily connected sync network $\mathfrak{S}$ containing N nodes is found. As shown in Theorem 1c, $\mathfrak{S}$ is the union of L isolated connecting sync networks, i.e., $\mathfrak{S} = \bigcup_{i=1}^{L} \mathfrak{S}_i$. Assume that there are N_i nodes in $\mathfrak{S}_i$. According to Definition 2, all nodes are numbered from 1 to N in an arbitrarily connected sync network[12]. However, from Fig. 1, there is no difference in naming either one of the two connecting sync networks as $\mathfrak{S}_1$ or $\mathfrak{S}_2$. Therefore, the problem of counting the number of possible sync network architectures in $\mathfrak{S}$ is equivalent to asking the question: How many choices are available for distributing N distinct nodes (balls) in L ($N \geq L \geq 1$) indistinguishable connecting networks (boxes) where each connecting sync network has at least one node [3], [32], [33] and all distinct nodes in each connecting sync networks are *directly or indirectly connected.* To find the answer, one needs to: (i) *group* the number of nodes in each $\mathfrak{S}_i$, (ii) remove the repeated choices in (i) since all *boxes* are assumed indistinguishable, and (iii) connect all N_i nodes in each $\mathfrak{S}_i$ *directly or indirectly* for all i=1,2,...L. By performing these three tasks, one can readily show that

$$A_N = 2^{N(N-1)} = \sum_{L=1}^{N} \sum_{(\sum_{i=1}^{L} N_i = N, 1 \leq N_i \leq N)} \left[\frac{N!}{N_1!N_2!...N_L!}\right] \times \left[\frac{1}{\nu_1!\nu_2!\nu_3!...\nu_N!}\right] [C_{N_1} C_{N_2} ... C_{N_L}], \tag{B-1}$$

[12] All nodes are assumed to be distinct. For example, a node located in the United States is different from a node located in Taiwan.

where the symbols A_N, C_{N_i}, ν_j are defined in (1). The first summation indicates that by arbitrarily connecting N nodes (as described in Definition 2), the resulting sync network architectures may contain any number L≤N of connecting sync networks, i.e., L=1,...,N. The second summation indicates that there may exist more than one choice for distributing N distinct balls into L indistinguishable boxes ($N \geq L \geq 1$). The first bracketed terms in (B-1) indicates that the number of choices for a fixed L is given by

$$\binom{N}{N_1}\binom{N-N_1}{N_2}\cdots\binom{N_L}{N_L} = \frac{N!}{N_1!N_2!\cdots N_L!}. \tag{B-2}$$

Since all boxes are indistinguishable, the second bracketed terms in (B-1) are applied to resolve the repeated choices in the first bracketed terms of (B-1). Once a specific partitioning of the nodes is finished, the number of possible choices for connecting network nodes is calculated in the third bracketed terms of (B-1).

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