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Change-Point Estimation and Bootstrap Confidence Regions in Weibull Distribution†

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ABSTRACT

We considered a change-point hazard rate model generalizing constant hazard rate model. This type of model is very popular in the sense that the Weibull and exponential distributions formulating survival time data are the special cases of it. Maximum likelihood estimation and the asymptotic properties such as the consistency and its limiting distribution of the change-point estimator were discussed. A parametric bootstrap method for finding confidence intervals of the unknown change-point was also suggested and the proposed method is explained through a practical example.

Keywords: Change-point model; Hazard rate; Limiting distribution; Parametric bootstrap; Weibull distribution.

1. INTRODUCTION

In classical change-point problem the main concern has been on the mean changes in a sequence of random variables. If the functional forms of distributions are known parametric methods such as the maximum likelihood estimation(MLE) and the likelihood ratio test(LRT) are usually used. Hinkley(1970), Worsley(1986) and Siegmund(1988), among others, are the researches of this type. On the other hand Bhattacharyya and Johnson(1968), Darkhovskh(1976), Carlstein(1988), and Boukai(1993), Chang, Chen and Hsiung(1994) studied the change-point problem in a nonparametric set up.

We mainly concentrate on the change-point hazard rate model. We may expect early failures with one hazard rate and next another hazard rate after a specific time point. For a data set of survival times of patients or mechanical components there exists high initial risk but it settles down to lower long term

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risk. This type of change-point problem assuming constant hazard rate before and after a certain time point is treated by, for example, Nguyen, Rogers and Walker(1984), Yao(1986), Pham and Nguyen(1990), and Loader(1991). Asymptotic properties such as consistency of MLEs and their limiting distributions have been the main interests in change-point hazard rate models. For the case of constant hazard rate model Yao(1986), and Pham and Nguyen(1990) independently discussed these asymptotic properties. In particular Pham and Nguyen(1991) showed that the parametrically bootstrapped estimator also has the same limiting distribution as that of MLE.

We extend the constant hazard rate model to more general ones so that it can be applicable to wider problems. We similary suggest MLEs and also examine the consistency and its limiting distribution under the assumed change-point model. Both complete data and censored data will be considered. Because of the complexity and non-normality of the limiting distribution we cannot directly use it in finding a confidence interval of change-point. In this respect a bootstrap approximation can be a good alternative to finding confidence interval of change-point. The proposed method will be explained through a practical example and also will be compared with that obtained by Loader(1991) under the constant hazard rate model.

2. ESTIMATION OF CHANGE-POINT

2.1. Change-Point Hazard Rate Models

Let T be a random variable from a distribution function $F(t)$ and probability density function (pdf) $f(t)$. The hazard rate $\lambda(t)$ is defined by

$$\lambda(t) = \frac{f(t)}{1 - F(t)}$$

with the corresponding cummulative hazard function $\Lambda(t)$ given by

$$\Lambda(t) = \int_0^t \lambda(u)du = -\log[1 - F(t)] .$$

From this relationship we can express $f(t)$ as

$$f(t) = \lambda(t)\exp(-\Lambda(t)) . \tag{2.1}$$

A constant hazard rate change-point model which corresponds to the exponential distribution is of the form

$$\lambda(t) = \begin{cases} 1/\beta_1 , & t \leq \tau \\ 1/\beta_2 , & t > \tau , \end{cases} \tag{2.2}$$

where β_1 and β_2 are unknown constants, and τ is an unknown change-point. Until now a simple model of this type has been mainly studied. If hazard rates are not constant but functions of time we need to generalize the model (2.2). In particular we confine our attention to a model of the form

$$\lambda(t) = \begin{cases} \frac{\gamma_1}{\beta_1} t^{\gamma_1 - 1} , & t \leq \tau \\ \\ \frac{\gamma_2}{\beta_2} t^{\gamma_2 - 1} , & t > \tau , \end{cases} \tag{2.3}$$

which corresponds to a Weibull distribution that is very popular in survival time data. The parameters γ_1, γ_2 and β_1, β_2 denote shape and scale parameters, respectively. If $\gamma_1 = \gamma_2 = 1$ the model (2.3) is reduced to a constant hazard rate model (2.2) of exponential distribution. It also includes some other useful distributions as special cases.

2.2. Maximum Likelihood Estimation

Let $T_1, T_2, \cdots, T_n$ be independently observed failure times with hazard rate defined by (2.3) and let $\boldsymbol{\theta} = (\gamma_1, \gamma_2, \beta_1, \beta_2)$. Hereafter we assume the observations are complete with no censoring until otherwisely stated. The log-likelihood function $l_n(\tau, \boldsymbol{\theta})$ can be written, from (2.1), as

$$\begin{aligned} l_n(\tau, \boldsymbol{\theta}) &\propto \int \log f_{\boldsymbol{\theta}}(t) \, dF_n(t) \\ &\propto F_n(\tau) \log\left(\frac{\gamma_1}{\beta_1}\right) + (1 - F_n(\tau)) \left\{ \log\left(\frac{\gamma_2}{\beta_2}\right) - \left(\frac{\tau^{\gamma_1}}{\beta_1} - \frac{\tau^{\gamma_2}}{\beta_2}\right) \right\} \\ &+ \frac{1}{n} \sum_{T_i \leq \tau} \left\{ \gamma_1 \log T_i - \frac{T_i^{\gamma_1}}{\beta_1} \right\} + \frac{1}{n} \sum_{T_i > \tau} \left\{ \gamma_2 \log T_i - \frac{T_i^{\gamma_2}}{\beta_2} \right\} . \end{aligned} \tag{2.4}$$

By differentiating $l_n(\tau, \boldsymbol{\theta})$ with respect to β_i and equating to zero we find that

$$\begin{aligned} \hat{\beta}_1(\tau, \gamma_1) &= \frac{\int_0^\tau (t^{\gamma_1} - \tau^{\gamma_1}) dF_n(t) + \tau^{\gamma_1}}{F_n(\tau)} , \\ \hat{\beta}_2(\tau, \gamma_2) &= \frac{\int_\tau^\infty (t^{\gamma_2} - \tau^{\gamma_2}) \, dF_n(t)}{1 - F_n(\tau)} . \end{aligned} \tag{2.5}$$

For simplicity of notations we denote $\hat{\beta}_i(\tau,\gamma_i)$ as $\hat{\beta}_i$. If $T_{(r)} \le \tau < T_{(r+1)}$ then $\hat{\beta}_i$ can be expressed as

$$\hat{\beta}_1 = \frac{\sum_1^r T_i^{\gamma_1} + (n-r)\tau^{\gamma_1}}{r}, \quad \hat{\beta}_2 = \frac{\sum_{r+1}^n T_i^{\gamma_2} - (n-r)\tau^{\gamma_2}}{n-r}. \tag{2.6}$$

By substituting $\hat{\beta}_i$ into (2.4) we obtain the pseudo-likelihood function $\tilde{l_n}(\tau,\boldsymbol{\gamma})$

$$\begin{aligned} \tilde{l_n}(\tau,\boldsymbol{\gamma}) \quad &\propto \quad F_n(\tau)\log\left(\frac{\gamma_1}{\hat{\beta}_1}\right) + (1-F_n(\tau))\log\left(\frac{\gamma_2}{\hat{\beta}_2}\right) \\ &+ \quad \frac{1}{n}\sum_{T_i\le\tau}\gamma_1\log T_i + \frac{1}{n}\sum_{T_i>\tau}\gamma_2\log T_i\ , \end{aligned} \tag{2.7}$$

where $\boldsymbol{\gamma} = (\gamma_1,\gamma_2)$.

Next the MLE of $(\tau,\boldsymbol{\gamma})$ is found as the maximizer of $\tilde{l_n}(\tau,\boldsymbol{\gamma})$, that is,

$$(\hat{\tau},\hat{\boldsymbol{\gamma}}) = \arg\max{}_{\tau_1\le\tau\le\tau_2}\sup{}_{\boldsymbol{\gamma}>0}\ \tilde{l_n}(\tau,\boldsymbol{\gamma})\ ,$$

where the interval $[\tau_1,\tau_2]$ is usually taken to be a random interval $[B_n^{'}, B_n^{''}]$ with $B_n^{'} = B_n^{'}(T_1,\cdots,T_n)$ and $B_n^{''} = B_n^{''}(T_1,\cdots,T_n)$ defined on the given sample. The random interval $[B_n^{'}, B_n^{''}]$ can be taken in various ways, for example, $[T_{(1)}, T_{(n-1)}]$, and we refer to Worsley(1988), Loader(1991), and Chang, et al.(1994) for further discussions on it.

For any fixed τ such that $T_{(r)} \le \tau < T_{(r+1)}$ the likelihood equations of γ_i are obtained as follows by differentiating $\tilde{l_n}(\tau,\boldsymbol{\gamma})$ with respect to γ_1, γ_2, respectively.

$$\begin{aligned} \frac{r}{\gamma_1} - (n-r)\frac{1}{\hat{\beta}_1}\tau^{\gamma_1}\log\tau &= \frac{1}{\hat{\beta}_1}\sum_{T_i\le\tau}T_i^{\gamma_1}\log T_i - \sum_{T_i\le\tau}\log T_i\ , \\ \frac{n-r}{\gamma_2} + (n-r)\frac{1}{\hat{\beta}_2}\tau^{\gamma_2}\log\tau &= \frac{1}{\hat{\beta}_2}\sum_{T_i>\tau}T_i^{\gamma_2}\log T_i - \sum_{T_i>\tau}\log T_i\ . \end{aligned} \tag{2.8}$$

Iterative methods such as Newton-Raphson may be used to find simultaneous solutions of (2.6) and (2.8). Usually the MLE of τ is taken over the set $\{T_{(1)}-, T_{(1)}, \cdots, T_{(n)}-, T_{(n)}\}$ of order statistics $T_{(i)}$s , where $T_{(i)}-$ means the MLE is attained as τ approaches $T_{(i)}$ from below. As was noted by Nguyen, et al.(1984) in constant hazard rate model as $\tau \to T_{(n)}$, $\tilde{l_n}(\tau,\ \boldsymbol{\gamma})$ is unbounded.

For the case of constant hazard rate change-point model the consistency of $\hat{\tau}$ was established by Nguyen, et al.(1984), Yao(1986), and Pham and Nguyen(1990). We extend Lemma 1 of Pham and Nguyen(1990) to a parameter vector case to derive the consistency of $\hat{\tau}$ under the assumed change-point model (2.3).

Lemma 2.1. *Let* $\tilde{l}\,(\tau,\gamma)$ *be a true pseudo-likelihood function for the change-point model (2.3). Then the following facts*

(i) $\tilde{l_n}\,(\tau,\gamma)$ *converges a.s. to* $\tilde{l}\,(\tau,\gamma)$ *uniformly in* τ *on* $[B_n^{'}, B_n^{''}]$ *and* γ_i

(ii) $\lim\sup B_n^{'} < \tau_0 < \lim\,\inf B_n{''}$ *a.s.*

(iii) $\tilde{l}\,(\tau,\gamma)$ *is continuous and also has a unique maximum at* (τ_0, γ_0)

hold, and hence the MLE $(\hat{\tau},\hat{\gamma})$ *is strongly consistent.*

Proof We may regard γ_i as nuisance parameters and briefly check the conditions in a similar manner to Pham and Nguyen(1990). First, the a.s. convergence follows from the strong law of large numbers and the uniformity can be shown as was done for the constant hazard rate model. Next, the second condition states on the methods of choosing intervals. Finally, $\hat{\beta}_i(\tau,\gamma_i)$ is continuous at γ_0 and similarly the continuity of $\tilde{l}(\tau,\gamma)$ follows by transforming the time T_i.

3. LIMITING DISTRIBUTION

For the special case of constant hazard rate change-point model the asymptotic distribution of $\hat{\tau}$ was independently established by Yao(1986), and Pham and Nguyen(1990) in a little different but in fact identical form.

To discuss the limiting distribution of $\hat{\tau}$ we define

$$R_i = \begin{cases} -\frac{1}{\mu(\tau_0,\theta_0)} \sum_{j=i}^{0} Z_j \,, & \text{for } i \le 0 \\ \frac{1}{\nu(\tau_0,\theta_0)} \sum_{j=1}^{i} Z_j \,, & \text{for } i > 0 \,, \end{cases}$$

where Z_js are independent exponential random variables with unit mean, and

$$\begin{aligned} \mu(\tau_0, \boldsymbol{\theta}_0) &= \frac{\gamma_{10}}{\beta_{10}} \tau_0^{\gamma_{10}-1} \exp\left(-\frac{\tau_0^{\gamma_{10}}}{\beta_{10}}\right) \\ \nu(\tau_0, \boldsymbol{\theta}_0) &= \frac{\gamma_{20}}{\beta_{20}} \tau_0^{\gamma_{20}-1} \exp\left(-\frac{\tau_0^{\gamma_{20}}}{\beta_{10}}\right) \end{aligned}$$

with τ_0, $\boldsymbol{\theta}_0$ denoting true parameters.

For the pdf $f(t,\tau,\boldsymbol{\theta})$ corresponding to the hazard rate of (2.3) we check the following regularity conditions C_1 through C_3 of Chernoff and Rubin(1956).

C_1 : For $\boldsymbol{\theta}$ in some neighborhood about $\boldsymbol{\theta}_0$

$$\lim_{\substack{t\to\tau_0\\ \tau\to\tau_0\\ t<\tau}} f(t,\tau,\boldsymbol{\theta}) = \mu(\tau_0,\boldsymbol{\theta}) > 0\ ,\quad \lim_{\substack{t\to\tau_0\\ \tau\to\tau_0\\ t>\tau}} f(t,\tau,\boldsymbol{\theta}) = \nu(\tau_0,\boldsymbol{\theta}) > 0$$

hold uniformly in some interval of $\boldsymbol{\theta}_0$ and also continuous at $\boldsymbol{\theta}_0$.

C_2 : For t not belongs to between τ and τ_0 the continuity at τ_0 implies the relation

$$\frac{\log f(t,\tau,\boldsymbol{\theta}) - \log f(t,\tau_0,\boldsymbol{\theta})}{\tau-\tau_0} = \frac{\partial \log f(t,\tau_0,\boldsymbol{\theta})}{\partial\tau} + H(t)\cdot o(1)$$

with $E[|H(T)|\,|\ \tau_0,\boldsymbol{\theta}_0] < \infty$, and $\partial\log f(t,\tau_0,\boldsymbol{\theta})/\partial\tau$ is bounded.

C_3 : For some interval about $\boldsymbol{\theta}_0$ we use strong law of large number and the uniformity of empirical distribution function to show that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\partial\log f(T_i,\tau_0,\boldsymbol{\theta})}{\partial\tau}$$

converges uniformly in probability to

$$\delta(\boldsymbol{\theta}) = E\left\{\frac{\partial\log f(T_i,\tau_0,\boldsymbol{\theta})}{\partial\tau}|\tau_0,\boldsymbol{\theta}_0\right\} = \nu(\tau_0,\boldsymbol{\theta}) - \mu(\tau_0,\boldsymbol{\theta})\ ,$$

where $\delta(\boldsymbol{\theta})$ is continuous at $\boldsymbol{\theta}_0$.

Then for the change-point model (2.3) we obtain the following theorem.

Theorem 3.1. *Under the regularity conditions C_1 through C_3, as n goes to infinity, we obtain*

$$n(\hat{\tau}-\tau_0) \xrightarrow{d} R_I,$$

where $\xrightarrow{d}$ denotes the convergence in distribution, and I is the index realizing the maximum of

$$S_i = i\log\left(\frac{\mu(\tau_0,\boldsymbol{\theta}_0)}{\nu(\tau_0,\boldsymbol{\theta}_0)}\right) + \left(\nu(\tau_0,\boldsymbol{\theta}_0) - \mu(\tau_0,\boldsymbol{\theta}_0)\right) R_i\ .$$

4. CONFIDENCE INTERVAL OF CHANGE-POINT

4.1. Confidence Interval Using Bootstrap Distribution

Various methods for finding confidence regions of the change-point have been treated by many authors, in particular by Siegmund(1988). For the special change-point model of constant hazard rates Loader(1991) applied a large deviation technique to determine LRT based confidence regions. But this technique cannot easily be applied to the model of our interest, to which a bootstrap technique is a nice alternative.

A parametric bootstrap method for investigating the limiting distribution of $\hat{\tau}$ was suggested by Pham and Nguyen(1991). We briefly review a parametric bootstrap procedure. Let $f_{\theta_n}(t)$ be a pdf parametrized by $\boldsymbol{\theta}_n$, a parameter vector of our interest. Firstly we estimate $\hat{\boldsymbol{\theta}}_n$ using the given data. From the distribution $f_{\hat{\theta}_n}(t)$ with $\hat{\boldsymbol{\theta}}_n$ plugged in, we generate a bootstrap sample T_i^*, $i = 1, 2, \cdots, n$. Finally a bootstrap version $\hat{\boldsymbol{\theta}}_n^{\,*}$ is computed using resampled T_i^*, $i = 1, 2, \cdots, n$.

According to Pham and Nguyen(1990) the following two conditions

D_1: $\lim_{n\to\infty} P_{\theta_n}[\tau_0 \in [B_n^{'}(T_1^*, \cdots, T_n^*), B_n^{''}(T_1^*, \cdots, T_n^*)] = 1$,

D_2: $\frac{1}{n}\log[T_n^* - B_n^{''}(T_1^*, \cdots, T_n^*)] \xrightarrow{P_{\theta_n}} 0$

guarantee $\hat{\tau}_n^* - \tau_n \xrightarrow{P_{\theta_n}} 0$, where $\xrightarrow{P_{\theta_n}}$ denotes the convergence in P_{θ_n} probability and τ_n is used instead of τ_0 to denote the dependence on n. This fact is similar to Lemma 1 on the strong consistency of $\hat{\tau}$.

We are interested in the consistency of the bootstrap distribution of $n(\hat{\tau}_n - \tau_n)$. Let $n(\hat{\tau}_n - \tau_n)$ be represented by a functional $U_n(T_1, \cdots, T_n, \tau_n)$, that is,

$$n(\hat{\tau}_n - \tau_n) = U_n(T_1, \cdots, T_n, \tau_n).$$

Then the distribution of bootstrapped estimator $n(\hat{\tau}_n^* - \hat{\tau}_n)$ is identical to that of $U_n(T_1^*, T_2^*, \cdots, T_n^*, \hat{\tau}_n)$.

Using the asymptotic property discussed before the conditional law of $U_n(T_1^*, T_2^*, \cdots, T_n^*, \hat{\tau}_n)$ converges weakly to the law of R_I for almost all sample sequences $T_1, T_2, \cdots$ with R_I defined in Theorem 3.1.

Theorem 4.1. *If we take $B_n^{'}(T_1^*, \cdots, T_n^*)$ and $B_n^{''}(T_1^*, \cdots, T_n^*)$ appropriately to satisfy D_1 and D_2, we obtain the limiting distribution of $\hat{\tau}_n^*$*

$$n(\hat{\tau}_n^* - \tau_n) \xrightarrow{d} R_I,$$

where $\xrightarrow{d}$ *denotes the convergence in distribution, and* I *is the index defined in Theorem 1.*

The empirical probability distribution of $n(\hat{\tau}_n^* - \hat{\tau})$, repeated B times, can be used to approximate the upper(or lower) $\alpha-$point of the limiting distribution. A bootstrap counterpart for the LRT based confidence region is not treated in this paper because of its heavy computational burden in perfoming parametric bootstrap.

4.2. Estimation Under Random Censoring

For the randomly censored data $(X_1, \delta_1), \cdots, (X_n, \delta_n)$, where the censoring indicator δ_i equals 0 or 1 according as whether the i^{th} survival time is censored or not, the likelihood function up to a constant is of the form

$$\prod_{i=1}^{n} \lambda(t_i)^{\delta_i} e^{-\Lambda(t_i)} .$$

The MLE $\hat{\beta}_i$ is the same except that r and n in (2.6) are replaced by the corresponding number of uncensored observations. Similarly the likelihood equations for γ_i are given by

$$\sum_{T_i \le \tau} \delta_i \left(\frac{1}{\gamma_1} + \log T_i \right) - \frac{1}{\hat{\beta}_1} \sum_{T_i \le \tau} T_i^{\gamma_1} \log T_i - (n-r) \frac{1}{\hat{\beta}_1} \tau^{\gamma_1} \log \tau = 0 ,$$

$$\sum_{T_i > \tau} \delta_i \left(\frac{1}{\gamma_2} + \log T_i \right) - \frac{1}{\hat{\beta}_2} \sum_{T_i > \tau} T_i^{\gamma_2} \log T_i + (n-r) \frac{1}{\hat{\beta}_2} \tau^{\gamma_2} \log \tau = 0 .$$

We also note that these are very similar to the equations in (2.8) and the iterative method can be used to solve them. Under moderate censorship the consistency and other properties for the complete data can be applied. We may refer to Matthews and Farewell(1982), and Loader(1991) for the discussions on this point.

4.3. A Practical Example

We explain the proposed procedure through a real data set of survival times for 184 persons who actually received heart transplant, which was obtained from the Stanford Heart Transplantation Program (Miller and Halpern, 1982). Among $n = 184$ persons 65 patients were still alive until the end of experiment and so they are assumed to be censored. For computational convenience the original data is scaled by dividing with 100.

The plot of $\log \Lambda(t)$ against $\log t$ depicted in Figure 4.1 seems to be approximately piecewise linear with different slopes before and after one specific time point. This fact justifies the validity of hazard rate change-point model given in (2.3).

We find the change-point estimator by varying over the failure times which maximizing the log-likelihood after 5% truncation at both ends of the ordered values. This is the rule suggested by Worsley(1986) and Loader(1991). The estimated change-point is $\hat{\tau} = 0.68$ for the given data set and we comment that this estimate is the same whether we take 5% or 10% truncation. Given the estimated change-point $\hat{\tau} = 0.68$ the other parameters $\hat{\gamma}_i$ and $\hat{\beta}_i$ are routinely obtained by iterative method solving equations (2.6) and (2.8) simultaneously. The MLEs $\hat{\gamma}_1 = 1.0178$, $\hat{\gamma}_2 = 0.6822$ of shape parameters before and after the estimated change-point are a little more different than assumed to be a common value. This fact may be one reason to consider a more general model with no assumption on common shape parameters. We may expect a great improvement over the constant hazard rate model assuming $\gamma_1 = \gamma_2 = 1$. We also note that the MLEs $\hat{\beta}_1 = 2.2562$, $\hat{\beta}_2 = 7.2020$ of scale parameters are greatly different from each other. The estimated hazard rate function of (2.3) obtained by substituting the estimated parameters is also depicted in Figure 4.1.

To perform a parametric bootstrap using the estimated parameter values we firstly generate uniform random variates via IMSL(International Mathematical and Statistical Libraries) on Workstation and then transform them to Weibull random variates. We also assumed uniformly distributed censoring times. From $B = 5,000$ bootstrap iterations we find a 95% confidence interval of τ to be (0.6803, 0.7555), and this interval approximately coincides with the LRT based confidence region consisting of two intervals (0.66, 0.67) and (0.68, 0.751), which was found by Loader(1991) using large deviation technique under the constant hazard rate model.

On the other hand if we assume $\gamma_1 = \gamma_2 = \gamma$(a common value) in (2.3), the MLEs of the unknown parameters are $\hat{\tau} = 0.68$, $\hat{\gamma} = 0.792$, $\hat{\beta}_1 = 2.4866$, $\hat{\beta}_2 = 10.1471$. We note that the common shape parameter estimate is greately different from $\hat{\gamma}_1 = 1.0178$ and $\hat{\gamma}_2 = 0.6822$, which are obtained under more general assumption on the hazard rate model with no common shapes. And the

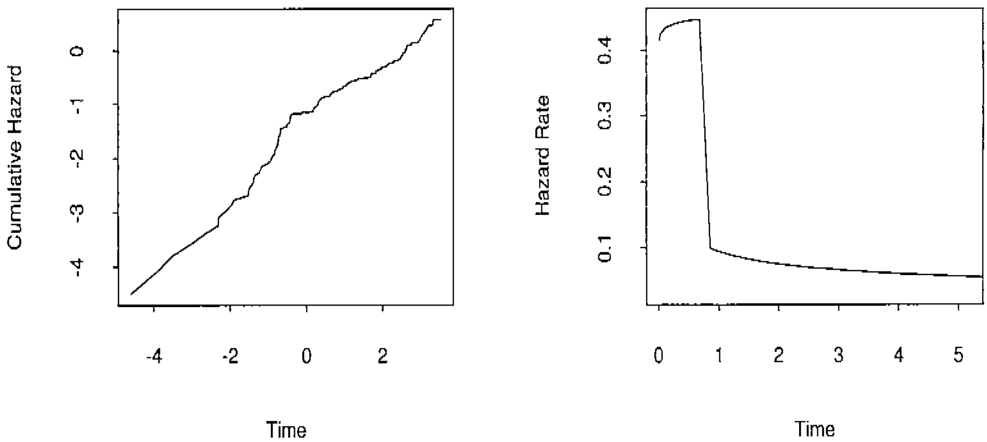

Figure 4.1: Cumulative hazard in log scales and hazard rate against time

95% confidence interval of τ is (0.4265, 0.8832). These results are quitely different from those obtained under the assumption of no common shape parameters.

5. SUMMARY AND FURTHER STUDIES

Under the change-point hazard rate model corresponding to the Weibull distribution the MLE of change-point and other parameters were considered. In particular the MLEs of shape parameters cannot be expressed in explicit forms under the assumed model, and hence an iterative method of Newton-Raphson algorithm was used to simultaneously solve the likelihood equations. We also investigated the consistency and its limiting distribution of change-pont estimator by checking the regularity conditions of Chernoff and Rubin(1956).

The limiting distribution of change-point estimator is complicated and non-normal, and hence we cannot directly use it in practical applications. A parametric bootstrap method was suggested to approximate the limiting distribution and further we used this one to find a confidence interval of change-point. We explained the proposed procedure through a practical example of the well-known Stanford heart transplant data set.

We haven't considered an appropriate goodness-of-fit criterion to compare two models, i.e., a common shape model versus unequal shapes model. This may also be an interesting topic in the data analytic aspect. Finally, we comment on the computational intensity incurred from both Newton-Raphson iterations and bootstrapping. A modified estimation procedure instead of MLE will be a nice alternative to tackle this problem. Generalizations to other change-point models such as Cox's regression model are remained as further researches.

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