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Sequential Confidence Intervals for Quantiles Based on Recursive Density Estimators

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Abstract

A sequential procedure of fixed-width confidence intervals for quantiles satisfying a condition of coverage probability is provided based on recursive density estimators. It is shown that the proposed sequential procedure is asymptotically efficient. In addition, the asymptotic normality for the proposed stopping time is derived.

Keywords: Sequential confidence intervals; Stopping time; Recursive density estimator; Asymptotic normality

1. INTRODUCTION

Let $\{X_n : n = 1,\ 2,\ \cdots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with a common distribution function F and let ξ_p, $0 < p < 1$, be the p-th quantile. It is well-known that the order statistic $X_{n,[np]+1}$, $0 < p < 1$, is asymptotically normally distributed:

$$\sqrt{n}\left(X_{n,[np]+1} - \xi_p\right) \xrightarrow{\mathcal{L}} N\left(0,\ p(1-p)/f^2\left(\xi_p\right)\right), \tag{1.1}$$

under some conditions about the derivative f of F. $[x]$ means the largest integer not exceeding x.

Consider the problem of fixed-width confidence intervals for ξ_p of the form

$$I_n(d) = \left[X_{n,[np]+1} - d,\ X_{n,[np]+1} + d\right], \quad d > 0,$$

satisfying the coverage probability condition:

$$P\left(\xi_p \in I_n(d)\right) \geq 1 - 2\alpha, \quad 0 < \alpha < 1/2. \tag{1.2}$$

Since $f(\xi_p)$ is unknown in (1.1) we have to construct a sequential procedure in the problem above.

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For the problem of interval estimation for quantiles, Serfling (1980) described various methods of determining a confidence interval for a quantile in Section 2.6. Geertsema (1970) introduced a method for constructing bounded length confidence intervals and Gijbels and Veraverbeke (1989) proposed a sequential procedure for quantiles in the presence of censoring by the method of Geertsema (1970).

In this paper we provide sequential fixed-width confidence intervals for quantiles using recursive density estimators developed in density estimation. In Section 2 we first investigate some asymptotic properties of an estimator of $f(\xi_p)$. In Section 3 we propose a stopping time N based on recursive density estimators, and prove that the sequential confidence interval $I_N(d)$ using the stopping time N satisfies the asymptotic coverage condition (1.2) as $d \to 0$ and that the proposed sequential procedure is asymptotically efficient. Theorem 3.3 shows that the sequential procedure proposed here has the same asymptotic efficiency in the sense of average sample size as the comparable sequential procedures such as the procedure based on the sign test of Geertsema (1970) when f is symmetric about 0 and $p = 1/2$, and the sequential procedure of Gijbels and Veraverbeke (1989) in the i.i.d. case. In particular, the asymptotic normality for the proposed stopping time N is derived.

2. CONVERGENCE OF RECURSIVE DENSITY ESTIMATORS

Now we introduce the recursive density estimator

$$\widehat{f_n}(x) = n^{-1}\sum_{j=1}^{n}(2b_j)^{-1}I_{\{|X_j-x|\le b_j\}}.$$

Recursive density estimators with a kernel function have been studied by several authors (Caroll (1976), Basu and Sahoo (1989), and so on) because of the advantage that $\widehat{f_n}(x)$ can be computed from $\widehat{f}_{n-1}(x)$, b_n and the observation X_n. But the estimator $\widehat{f_n}\left(X_{n,[np]+1}\right)$ of $f(\xi_p)$ does not have such advantage since $X_{n,[np]+1}$ is not fixed. Nevertheless, in this paper, the estimator will be used to derive the asymptotic normality of the stopping time to be defined later.

Throughout this paper, the following conditions about f and b_n are assumed:

(A1) f is differentiable on R and $f(\xi_p) > 0$.

(A2) $\sup_{x\in R} f'(x) < B < +\infty$.

(A3) $b_n = n^{-a}$, $1/3 < a < 1/2$.

In (A3), the sequence $\{b_n\}$ doesn't need to be of such special form. We can obtain the results of this section under a necessary condition of (A3) as follows: (1) $b_n \to 0$, (2) $nb_n^k \to \infty$ for some k $(2 < k < 3)$, (3) $nb_n^3 = O(1)$, (4) $n^{-1}\sum_{j=1}^n b_n/b_j \to \gamma$ for some constant $\gamma > 0$ and (5) $b_n^{1/2}n^{-1/2}\sum_{j=1}^n b_j \to 0$. However, (A3) is enforced to give further information (Theorems 3.3 and 3.6) on the stopping time to be proposed in Section 3. The next lemma is easily checked by (A3).

Lemma 2.1. *Under assumption (A3), as $n \to \infty$, (1) $n^{-1}\sum_{j=1}^n b_n/b_j \to (1+a)^{-1}$, (2) $n^{-1}\sum_{j=1}^n (b_n/b_j)^2 \to (1+2a)^{-1}$ and (3) $b_n^{1/2}n^{-1/2}\sum_{j=1}^n b_j \to 0$.*

The following lemma is proved along the lines of the proof of Bahadur's lemma (Serfling (1980, p97)). The notation $u(n) \sim v(n)$, $n \to \infty$, stands for $\lim_{n\to\infty} u(n)/v(n) = 1$.

Lemma 2.2. *Let $\{a_n\}$ be a sequence of positive real numbers such that $a_n \sim c_0 n^{-1/2}(\log n)^{1/2}$, $n \to \infty$, for some constants $c_0 > 0$ and put*

$$\widehat{g}_{n+}(t) = n^{-1}\sum_{j=1}^n (2b_j)^{-1} I_{\{X_j \le t+b_j\}}, \qquad \widehat{g}_{n-}(t) = n^{-1}\sum_{j=1}^n (2b_j)^{-1} I_{\{X_j \le t-b_j\}}$$

and

$$\widehat{H}_{pn} = \sup_{|x|\le a_n} \left|\widehat{g}_n(\xi_p + x) - \widehat{g}_n(\xi_p) - \{E\widehat{g}_n(\xi_p+x) - E\widehat{g}_n(\xi_p)\}\right|,$$

where $\widehat{g}_n$ is either $\widehat{g}_{n+}$ or $\widehat{g}_{n-}$. Then w.p.1

$$\widehat{H}_{pn} = O\left(b_n^{-1}\left(n^{-1}\log n\right)^{3/4}\right), \quad n \to \infty.$$

Proof: We will prove the result only in the case of $\hat{g}_{n+}$, because the other case is proved by the similar method. Put $\hat{g}_n = \hat{g}_{n+}$. Note that $\widehat{g}_n(t)$ is a nondecreasing function of t and $E\widehat{g}_n(t) = (nb_n)^{-1}\sum_{j=1}^n b_n(2b_j)^{-1}F(t+b_j)$. Let $\{d_n\}$ be a sequence of positive integers such that $d_n \sim c_0 n^{1/4}(\log n)^{1/2}$, $n \to \infty$. For integers $r = -d_n, \cdots, d_n$, put

$$\eta_{r,n} = \xi_p + a_n d_n^{-1} r, \qquad \widehat{\alpha}_{r,n} = E\widehat{g}_n(\eta_{r+1,n}) - E\widehat{g}_n(\eta_{r,n}),$$

and

$$\widehat{G}_{r,n} = \left|\widehat{g}_n(\eta_{r,n}) - \widehat{g}_n(\xi_p) - \{E\widehat{g}_n(\eta_{r,n}) - E\widehat{g}_n(\xi_p)\}\right|.$$

Using the monotonicity of $\widehat{g}_n(x)$ and $E\widehat{g}_n(x)$, for $\eta_{r,n} \le \xi_p + x \le \eta_{r+1,n}$, $r = -d_n, \cdots, d_n - 1$, we have

$$\begin{aligned} &|\widehat{g}_n(\xi_p + x) - \widehat{g}_n(\xi_p) - \{E\widehat{g}_n(\xi_p + x) - E\widehat{g}_n(\xi_p)\}| \\ \le\ &\max_{j=r,\, r+1} |\widehat{g}_n(\eta_{j,n}) - \widehat{g}_n(\xi_p) - \{E\widehat{g}_n(\eta_{j,n}) - E\widehat{g}_n(\xi_p)\}| + \widehat{\alpha}_{r,n}. \end{aligned}$$

Therefore $\widehat{H}_{pn} \le \widehat{K}_n + \widehat{\beta}_n$, where $\widehat{K}_n = \max\{\widehat{G}_{r,n} : |r| \le d_n\}$ and $\widehat{\beta}_n = \max\{\widehat{\alpha}_{r,n} : |r| \le d_n\}$. By (A1), there exists $M > 0$ such that

$$\sup_{|x| \le \max a_n + \max b_n} f(\xi_p + x) < M. \tag{2.1}$$

Since $\eta_{r+1,n} - \eta_{r,n} = a_n d_n^{-1} = n^{-3/4}$, $-d_n \le r \le d_n - 1$, we have by the mean value theorem that

$$\widehat{\alpha}_{r,n} = (nb_n)^{-1} \sum_{j=1}^{n} b_n (2b_j)^{-1} f(c_{r,n,j})\, n^{-3/4} \le (M/2) b_n^{-1} n^{-3/4} \left(n^{-1} \sum_{j=1}^{n} b_n / b_j \right),$$

where $\eta_{r,n} b_j < c_{r,n,j} < \eta_{r+1,n} b_j$ for $1 \le j \le n$. By Lemma 2.1, we have $\widehat{\beta}_n = O\left(b_n^{-1} n^{-3/4}\right)$, $n \to \infty$.

We now establish that w.p.1 $\widehat{K}_n = O\left(b_n^{-1}\left(n^{-1}\log n\right)^{3/4}\right)$. It suffices by Borel-Cantelli's lemma to show that $\sum_{n=1}^{\infty} P\left(\widehat{K}_n \ge x_n\right)$ converges, where $x_n = c_1 b_n^{-1}\left(n^{-1}\log n\right)^{3/4}$ for a constant $c_1 > 0$ to be specified later. Note that

$$P\left(\widehat{K}_n \ge x_n\right) \le \sum_{r=-d_n}^{d_n} P\left(\widehat{G}_{r,n} \ge x_n\right),$$

and that $n\widehat{G}_{r,n}$ is distributed as $\left|\sum_{j=1}^{n} (2b_j)^{-1}(Y_j - EY_j)\right|$, in which the Y_j's are independent binomial $(1, z_{r,n,j})$ together with $z_{r,n,j} = |F(\eta_{r,n} + b_j) - F(\xi_p + b_j)|$. By Bernstein's inequality (Serfling (1980, p95, Lemma A)), we have

$$P\left(nb_n\widehat{G}_{r,n} \ge nb_n x_n\right) \le 2e^{-\theta_{r,n}},$$

where $\theta_{r,n} = n^2 b_n^2 x_n^2 / \left\{\sum_{j=1}^{n} (b_n/b_j)^2 z_{r,n,j} + nb_n x_n\right\}$. But there exists N^{*1} such that for all $n \ge N^{*1}$, $z_{r,n,j} \le f(\widehat{c}_{r,n,j})\,|\eta_{r,n} - \xi_p| \le Ma_n$, where $\widehat{c}_{r,n,j}$'s $(1 \le j \le n)$ are some suitable numbers between $\eta_{r,n} + b_j$ and $\xi_p + b_j$. It then follows that for $|r| \le d_n$ and $n \ge N^{*1}$

$$\begin{aligned} \theta_{r,n} &\ge nb_n^2 x_n^2 / \left\{ Ma_n n^{-1} \sum_{j=1}^{n} (b_n/b_j)^2 + b_n x_n \right\} \\ &\ge nb_n^2 x_n^2 / \{Ma_n + b_n x_n\} = \delta_n, \quad \text{say.} \end{aligned}$$

Note that $\delta_n \geq c_1^2 (2c_0 M)^{-1} \log n$ for n sufficiently large. Given c_0 and M, we may choose c_1 large enough that $c_1^2 (2c_0 M)^{-1} > 2$. It implies that $P\left(\widehat{G}_{r,n} \geq x_n\right) \leq 2n^{-2}$, $|r| \leq d_n$, for n sufficiently large. Thus, noting that $n^{-1/4} (\log n)^{1/2} \to 0$ as $n \to \infty$, we can see that there exists $N^{*2} (> N^{*1})$ such that for all $n \geq N^{*2}$

$$P\left(\widehat{K}_n \geq x_n\right) \leq \sum_{r=-d_n}^{d_n} P\left(\widehat{G}_{r,n} \geq x_n\right) \leq 4 d_n n^{-2} \leq c_0 n^{-3/2}.$$

Hence $\sum_{n=1}^{\infty} P\left(\widehat{K}_n \geq x_n\right)$ converges and w.p.1 $\widehat{K}_n = O\left(b_n^{-1}\left(n^{-1}\log n\right)^{3/4}\right)$ as $n \to \infty$. □

Lemma 2.3. *It holds that w.p.1*

$$\widehat{f}_n\left(X_{n,[np]+1}\right) - \widehat{f}_n(\xi_p) = O\left(n^{a-3/4} (\log n)^{3/4}\right).$$

Proof: First, note under assumption (A1) that w.p.1

$$\left|X_{n,[np]+1} - \xi_p\right| \leq (2/f(\xi_p))\left(n^{-1}\log n\right)^{1/2}, \tag{2.2}$$

for all n sufficiently large. From Lemma 2.2 and (2.2), we have w.p.1

$$\begin{aligned} \widehat{f}_n\left(X_{n,[np]+1}\right) - \widehat{f}_n(\xi_p) &= \left\{\widehat{f}_n\left(X_{n,[np]+1}\right) - \widehat{f}_n(\xi_p)\right\} + \Gamma_n \\ &= O\left(n^{a-3/4} (\log n)^{3/4}\right) + \Gamma_n, \end{aligned}$$

where

$$\begin{aligned} \Gamma_n &= n^{-1}\sum_{j=1}^{n} (2b_j)^{-1}\left\{F\left(X_{n,[np]+1} + b_j\right) - F(\xi_p + b_j)\right\} \\ &\quad - n^{-1}\sum_{j=1}^{n} (2b_j)^{-1}\left\{F\left(X_{n,[np]+1} - b_j\right) - F(\xi_p - b_j)\right\}. \end{aligned}$$

Let $Y_n = X_{n,[np]+1} - \xi_p$. Using Young's formula of Taylor's theorem for F at $\xi_p \pm b_j$, we get

$$F\left(X_{n,[np]+1} \pm b_j\right) - F(\xi_p \pm b_j) = f(\xi_p \pm b_j) Y_n + o\left(Y_n^2\right).$$

Furthermore, there are $\widehat{c_j}$'s $(1 \leq j \leq n)$ for which

$$|f(\xi_p + b_j) - f(\xi_p - b_j)| = |f'(\widehat{c_j})|(2b_j) \leq B(2b_j)$$

by (A2). Hence we have w.p.1

$$|\Gamma_n| \le B\,|Y_n| + \left(n^{-1}\sum_{j=1}^{n} b_n/b_j\right) o\left(b_n^{-1}Y_n^2\right). \tag{2.3}$$

From (2.2), the first term on the right hand of (2.3) is bounded by $O((\frac{\log n}{n})^{1/2})$ w.p.1 and from Lemma 2.1, the second term is bounded by $o\left(b_n^{-1}n^{-1}\log n\right)$ w.p.1. Since $b_n^{-1}n^{-1}\log n \le b_n^{-1}(n^{-1}\log n)^{3/4}$ and $(n^{-1}\log n)^{1/2} \le n^{a-(3/4)}(\log n)^{3/4} = b_n^{-1}\left(n^{-1}\log n\right)^{3/4}$, $1/3 < a < 1/2$, the conclusion is valid. □

Theorem 2.1. *(1)* $\widehat{f}_n\left(X_{n,[np]+1}\right) \to f\left(\xi_p\right)$, *w.p.1.*

(2) $(2nb_n)^{1/2}\left\{\widehat{f}_n\left(X_{n,[np]+1}\right) - f\left(\xi_p\right)\right\} / \left(\gamma f\left(\xi_p\right)\right)^{1/2} \xrightarrow{\mathcal{L}} N\left(0,1\right)$, *where* $\gamma = (1+a)^{-1}$.

Proof: By Lemma 2.3, we have w.p.1

$$(2nb_n)^{1/2}\left\{\widehat{f}_n\left(X_{n,[np]+1}\right) - \widehat{f}_n\left(\xi_p\right)\right\} = O\left(n^{\{a-(1/2)\}/2}\left(\log n\right)^{3/4}\right) \to 0. \tag{2.4}$$

By the definition of $\widehat{f}_n\left(\xi_p\right)$, $E\widehat{f}_n\left(\xi_p\right) = n^{-1}\sum_{j=1}^{n} p_j/\left(2b_j\right)$ and $Var\widehat{f}_n\left(\xi_p\right) = n^{-2}\sum_{j=1}^{n}\left(2b_j\right)^{-2}p_j\left(1-p_j\right)$, where $p_j = F\left(\xi_p + b_j\right) - F\left(\xi_p - b_j\right)$. Using Taylor's theorem, we get

$$p_j = f\left(\xi_p\right)2b_j + 2^{-1}\left[f'(c_j) - f'(d_j)\right]b_j^2$$

for some c_j and d_j with $0 \le c_j - \xi_p \le b_j$ and $0 \le \xi_p - d_j \le b_j$. Therefore from assumption (A2) and Lemma 2.1

$$(2nb_n)^{1/2}\left|E\widehat{f}_n\left(\xi_p\right) - f\left(\xi_p\right)\right| \le Bb_n^{1/2}n^{-1/2}\sum_{j=1}^{n} b_j \to 0 \tag{2.5}$$

and

$$2nb_nVar\widehat{f}_n\left(\xi_p\right) = f\left(\xi_p\right)\left(n^{-1}\sum_{j=1}^{n} b_n/b_j\right) + O\left(b_n\right) \to \gamma f\left(\xi_p\right). \tag{2.6}$$

Hence, from (2.4)-(2.6) and Kolmogorov's strong law of large numbers, we can prove easily the strong consistency of $\widehat{f}_n\left(X_{n,[np]+1}\right)$ for the estimator of $f\left(\xi_p\right)$.

To complete the proof of the theorem, it suffices by (2.4)-(2.6) to show that

$$\left\{\widehat{f}_n\left(\xi_p\right) - E\widehat{f}_n\left(\xi_p\right)\right\} / \sqrt{Var\widehat{f}_n\left(\xi_p\right)} \xrightarrow{\mathcal{L}} N\left(0,\ 1\right).$$

By Berry-Esséen's theorem (Serfling (1980, p33)), we have

$$\left|P\left(\left\{\widehat{f}_n(\xi_p) - E\widehat{f}_n(\xi_p)\right\}/\sqrt{Var\widehat{f}_n(\xi_p)} \le t\right) - \Phi(t)\right| \le D_n,$$

where $\Phi(t)$ is the standard normal distribution function and

$$D_n = C\sum_{j=1}^{n}(2b_j)^{-3}p_j\left(\sum_{j=1}^{n}(2b_j)^{-2}p_j(1-p_j)\right)^{-3/2}$$

for a universal constant C. Note that

$$(n^{-1}b_n)\sum_{j=1}^{n}(2b_j)^{-2}p_j(1-p_j) \to (\gamma/2)f(\xi_p)$$

and

$$(n^{-1}b_n^2)\sum_{j=1}^{n}(2b_j)^{-3}p_j = \frac{f(\xi_p)}{4}\left(n^{-1}\sum_{j=1}^{n}\left(\frac{b_n}{b_j}\right)^2\right) + O(b_n) \to \frac{f(\xi_p)}{4(1+2a)}.$$

Hence we have $D_n = O\left((nb_n)^{-1/2}\right) = O\left(n^{-(1-a)/2}\right)$, $1/3 < a < 1/2$.

Thus, by Slutsky's theorem, $\widehat{f}_n\left(X_{n,[np]+1}\right)$ is asymptotically normally distributed with centering $f(\xi_p)$ and scale $(\gamma f(\xi_p)/(2nb_n))^{1/2}$. □

3. SEQUENTIAL CONFIDENCE INTERVALS AND ASYMPTOTIC PROPERTIES OF THE PROPOSED STOPPING TIME

Let $I_n(d) = \left[X_{n,[np]+1} - d,\ X_{n,[np]+1} + d\right]$, $d > 0$. By the asymptotic normality of $X_{n,[np]+1}$, we know that for large n

$$P(\xi_p \in I_n(d)) = P\left(\left|\frac{\sqrt{n}f(\xi_p)}{\sqrt{p(1-p)}}\left(X_{n,[np]+1} - \xi_p\right)\right| \le \frac{\sqrt{n}d\,f(\xi_p)}{\sqrt{p(1-p)}}\right) \approx 1 - 2\alpha.$$

If $f(\xi_p)$ were known, we get approximately

$$\sqrt{n}d\,f(\xi_p)/\sqrt{p(1-p)} = z_\alpha, \tag{3.1}$$

where $z_\alpha = \Phi^{-1}(1-\alpha)$, $\Phi(x)$ is the standard normal distribution function. Since $f(\xi_p)$ is unknown, we propose a stopping rule based on (3.1) with $f(\xi_p)$ replaced by its estimator $f_n\left(X_{n,[np]+1}\right)$:

$$N = N(d) = \inf\left\{n \ge n_0 : n\widehat{f}_n^2\left(X_{n,[np]+1}\right) \ge p(1-p)z_\alpha^2/d^2\right\}, \tag{3.2}$$

and the sequential confidence interval based on the stopping rule (3.2) is given by

$$I_N(d) = \left[X_{N,[Np]+1} - d,\ X_{N,[Np]+1} + d\right].$$

We have the following lemmas and theorems for the above stopping rule N and the sequential confidence interval $I_N(d)$.

Lemma 3.1. *(1)* $N(d) < +\infty$ *w.p.1 for each* d.

(2) $N(d) \to \infty$ *w.p.1 as* $d \to 0$.

(3) $d^2 N(d) \to p(1-p) z_\alpha^2 / f^2(\xi_p)$ *w.p.1 as* $d \to 0$.

Proof: Lemma 3.1 can be easily proved by lemmas of Chow and Robbins (1965). □

Theorem 3.1. *(Asymptotic consistency) For given* α, $0 < \alpha < 1/2$,

$$\lim_{d \to 0} P(\xi_p \in I_N(d)) = 1 - 2\alpha.$$

Proof: Since the normalized partial sum of i.i.d. random variables is u.c.i.p. (Gut (1988, p15), Woodroofe (1982, pp10-11)), we can see that

$$\sqrt{n}\,(F_n(\xi_p) - p)\,/\sqrt{p(1-p)} = n^{-1/2} \sum_{j=1}^{n} Y_j$$

is u.c.i.p., where $Y_j = (I_{\{X_j \le \xi_p\}} - p)/\sqrt{p(1-p)}$. By Anscombe's theorem (Gut (1988, Section I.3), Woodroofe (1982, p11, Theorem 1.4)), we have

$$\sqrt{N}\,(F_N(\xi_p) - p)\,/\sqrt{p(1-p)} \xrightarrow{\mathcal{L}} N(0,\ 1) \quad \text{as } d \to 0.$$

Bahadur's representation (Serfling (1980, p91, p93)) and Lemma 3.1 yield that

$$\sqrt{N} f(\xi_p) \left(X_{N,[Np]+1} - \xi_p\right) / \sqrt{p(1-p)} \xrightarrow{\mathcal{L}} N(0,\ 1) \quad \text{as } d \to 0,$$

which implies that $P(\xi_p \in I_N(d)) \to 1 - 2\alpha$ as $d \to 0$. □

Theorem 3.2. *(Asymptotic efficiency)*

$$d^2 EN(d) \to p(1-p) z_\alpha^2 / f^2(\xi_p) \quad as\ d \to 0.$$

Proof: By Theorem 1.5 in Woodroofe (1983, p13), it is sufficient to show that $G(y) = \sup_{d\leq d_0} P\left(d^2 N(d) > y\right)$ is integrable with respect to Lebesgue measure over $(0,\infty)$. Let $\widehat{p}_{nj} = F(\xi_p + b_j - c_n) - F(\xi_p - b_j + c_n)$, where $c_n = (2/f(\xi_p))\left(n^{-1}\log n\right)^{1/2}$. By the definition of N, we have for n sufficiently large

$$\begin{aligned}
P\left(d^2 N(d) > y\right) &\leq P\left(\widehat{f}_n\left(X_{n,[np]+1}\right) \leq z_\alpha \sqrt{p(1-p)}/\left(\sqrt{n}d\right),\ n = \left[y/d^2\right]\right) \\
&\leq P\left(\sum_{j=1}^{n}\left(\widehat{Y}_j - E\widehat{Y}_j\right) \leq -nx_n,\ n = \left[y/d^2\right]\right) = \Gamma, \quad \text{say,}
\end{aligned}$$

where $\widehat{Y}_j = b_n(2b_j)^{-1} I_{\{|X_j-\xi_p|\leq b_j-c_n\}}$ and

$$-x_n = b_n\left\{z_\alpha\sqrt{p(1-p)}/\left(\sqrt{n}d\right) - n^{-1}\sum_{j=1}^{n}(2b_j)^{-1}\widehat{p}_{nj}\right\}.$$

$[x]$ denotes the largest integer not exceeding x. Under assumptions (A1)-(A3),

$$n^{-1}\sum_{j=1}^{n}(2b_j)^{-1}\widehat{p}_{nj} = f(\xi_p)\left\{1 - (c_n/b_n)\left(n^{-1}\sum_{j=1}^{n} b_n/b_j\right)\right\} + O\left(n^{-1}\sum_{j=1}^{n} b_j\right),$$

in which the right term converges to $f(\xi_p)$, and

$$\begin{aligned}
(nb_n)^{-1}\sum_{j=1}^{n} Var\widehat{Y}_j &\leq (nb_n)^{-1}\sum_{j=1}^{n} b_n^2 (2b_j)^{-2}\widehat{p}_{nj} \\
&\leq \frac{f(\xi_p)}{2}\left(n^{-1}\sum_{j=1}^{n}\frac{b_n}{b_j}\right) + O(b_n) \to \frac{f(\xi_p)}{2}. \qquad (3.3)
\end{aligned}$$

Choose a_0 $(a_0 > 1)$ such that $z_\alpha\sqrt{p(1-p)}/\sqrt{a_0 - 1} < f(\xi_p)/4$ and for such a_0 there is d_0 $(d_0 < 1)$ such that for all $n \geq [a_0/d_0^2]$

$$n^{-1}\sum_{j=1}^{n}(2b_j)^{-1}\widehat{p}_{nj} \geq f(\xi_p)/2, \qquad (nb_n)^{-1}\sum_{j=1}^{n} Var\widehat{Y}_j \leq f(\xi_p).$$

It then follows that $x_n/b_n \geq f(\xi_p)/4$ for all $n = [y/d^2]$, $y \geq a_0$, $d \leq d_0$. Thus, using Bernstein's inequality, we get that for all $n = [y/d^2]$, $y \geq a_0$, $d \leq d_0$,

$$\begin{aligned}
\Gamma &\leq P\left(\widehat{f}_n\left(X_{n,[np]+1}\right) \leq z_\alpha\sqrt{p(1-p)}/\left(\sqrt{n}d\right),\ n = \left[y/d^2\right]\right) \\
&\leq P\left(\sum_{j=1}^{n}\left(\widehat{Y}_j - E\widehat{Y}_j\right) \leq -n\left(b_n f(\xi_p)/4\right),\ n = \left[y/d^2\right]\right) \\
&\leq 2\exp\left(-\theta_n\right),
\end{aligned}$$

where $\theta_n = n^2 b_n^2 f^2(\xi_p) / \left\{16\left(2\sum_{j=1}^n VarY_j + nb_n f(\xi_p)\right)\right\}$. By (3.3),

$$\theta_n \geq (f(\xi_p)/48)\, nb_n, \quad \text{for all } n = \left[y/d^2\right],\ y \geq a_0,\ d \leq d_0.$$

Consequently, we have for all $n = [y/d^2]$, $y \geq a_0$, $d \leq d_0$,

$$\begin{aligned} P\left(d^2 N(d) > y\right) &\leq 2\exp\{-(f(\xi_p)/48)\, nb_n\} \\ &\leq 2\exp\left\{-(f(\xi_p)/48)\, d_0^{-2(1-a)}\left(y - d_0^2\right)^{1-a}\right\}, \end{aligned}$$

in which the last term is independent of $d < d_0$, and so $G_2(y)$ is integrable over $[a_0, \infty)$ if $a < 1$. □

As mentioned in the proof of Theorem 3.1, the normalized partial sum of i.i.d. random variables is u.c.i.p. However we need to show that the partial sum of independent but not identically distributed random variables is u.c.i.p. as in the following lemma.

Lemma 3.2. *Let $\{X_j\}$ be independent random variables with $E(X_j) = \mu_j$ and $Var(X_j) = \sigma_j^2\ (> 0)$ and put $S_n = \sum_{j=1}^n X_j$ and $S_n^* = (S_n - ES_n)/\sqrt{\sum_{j=1}^n \sigma_j^2}$. Suppose that S_n^* converges in distribution and that*

$$\left(\sum_{j=n+1}^{n+k} \sigma_j^2 / \sum_{j=1}^{n} \sigma_j^2\right) \leq h(k/n) \quad \text{for all } n,\ k \geq 1,$$

where $h(x)$ is a nondecreasing function such that $\lim_{x\to 0} h(x) = 0$. Then S_n^*, $n \geq 1$, is u.c.i.p.

Proof: Without loss of generality, we may suppose that $\mu_j = 0$, $j \geq 1$. Then

$$\begin{aligned} |S_{n+k}^* - S_n^*| &\leq \left(\sum_{j=1}^n \sigma_j^2\right)^{-1/2} |S_{n+k} - S_n| \\ &\quad + \left\{1 - \left(1 + \left(\sum_{j=n+1}^{n+k} \sigma_j^2 / \sum_{j=1}^n \sigma_j^2\right)\right)^{-1/2}\right\} |S_n^*| \qquad (3.4) \end{aligned}$$

for all n, $k \geq 1$. Let ϵ, $\delta > 0$ and $k \leq n\delta$. For the first term on the right hand of (3.4), Kolmogorov's inequality yields

$$P\left(\max_{k \leq n\delta} |S_{n+k} - S_n| \geq \frac{\epsilon}{2}\left(\sum_{j=1}^n \sigma_j^2\right)^{1/2}\right) \leq \frac{4}{\epsilon^2} \frac{\sum_{j=n+1}^{n+[n\delta]} \sigma_j^2}{\sum_{j=1}^n \sigma_j^2} \leq \frac{4}{\epsilon^2} h(\delta),$$

which tends to 0 as $\delta \to 0$. For the second term,

$$\left\{1+\left(\sum_{j=n+1}^{n+k}\sigma_j^2/\sum_{j=1}^{n}\sigma_j^2\right)\right\}^{-1/2} \geq \{1+h(\delta)\}^{-1/2} \equiv 1-C(\delta)$$

and $|S_n^*|$, $n \geq 1$, are stochastically bounded since its limiting distribution exists. Hence it follows that

$$P\left(\left\{1-\left(1+\left(\sum_{j=n+1}^{n+k}\sigma_j^2/\sum_{j=1}^{n}\sigma_j^2\right)\right)^{-1/2}\right\}|S_n^*| > \frac{\epsilon}{2}\right) \leq P\left(|S_n^*| > \frac{\epsilon}{2C(\delta)}\right),$$

which tends to 0 as $\delta \to 0$ uniformly in $n \geq 1$. Therefore S_n^*, $n \geq 1$, is u.c.i.p. □

Theorem 3.3. *Let $\gamma = (1+a)^{-1}$. Then as $d \to 0$*

$$\widehat{Z}_{N(d)} = (2Nb_N)^{1/2}\left\{\widehat{f}_N\left(X_{N,[Np]+1}\right) - f(\xi_p)\right\}/(\gamma f(\xi_p))^{1/2} \xrightarrow{\mathcal{L}} N(0,\ 1).$$

Proof: We shall show that $\sqrt{2nb_n}\left\{\widehat{f}_n\left(X_{n,[np]+1}\right) - f(\xi_p)\right\}$, $n \geq 1$, is u.c.i.p. Then $\widehat{Z}_{N(d)} \xrightarrow{\mathcal{L}} N(0,\ 1)$ as $d \to 0$ by Anscombe's theorem and Theorem 2.1. It suffices by (2.4)-(2.6) to show that $\left\{\widehat{f}_n(\xi_p) - E\widehat{f}_n(\xi_p)\right\}/\sqrt{Var\widehat{f}_n(\xi_p)}$, $n \geq 1$, is u.c.i.p., where $n^2 Var\widehat{f}_n(\xi_p) = \sum_{j=1}^{n}(2b_j)^{-2}p_j(1-p_j)$ and $p_j = F(\xi_p+b_j)-F(\xi_p-b_j)$. If X_n, $n \geq m$ for some fixed m, is u.c.i.p., then so is X_n, $n \geq 1$. Therefore we may suppose that $VarY_j > 0$ for all j, where $Y_j = (2b_j)^{-1}(I_{\{|X_j-\xi_p|\leq b_j\}} - p_j)$. Since $p_j(1-p_j)/(2b_j) \to f(\xi_p)$, $j \to \infty$, there exist c_0, c_1 such that $0 < c_0 \leq p_j(1-p_j)/(2b_j) \leq c_1$ for all j. Hence we get that for all n, k,

$$\left(\sum_{j=n+1}^{n+k} VarY_j/\sum_{j=1}^{n} VarY_j\right) \leq \frac{c_1}{c_0}\left(\sum_{j=n+1}^{n+k} b_j^{-1}/\sum_{j=1}^{n} b_j^{-1}\right), \quad b_j = j^{-a},$$

in which the right term is bounded by $(4c_1/c_0)\{(1+(k/n))^a - 1\}$ through a simple calculation. Therefore $\left\{\widehat{f}_n(\xi_p) - E\widehat{f}_n(\xi_p)\right\}/\sqrt{Var\widehat{f}_n(\xi_p)}$, $n \geq 1$, is u.c.i.p. by Lemma 3.2. □

Theorem 3.4. *(Asymptotic normality of N) Let $\gamma = (1+a)^{-1}$ and $N_0 = N_0(d)$ $= p(1-p)z_\alpha^2/\left(d^2 f^2(\xi_p)\right)$. Then as $d \to 0$*

$$Z_{N(d)} = (2\gamma f(\xi_p))^{-1/2}(N_0(d))^{-1/(2\gamma)}\{N(d) - N_0(d)\} \xrightarrow{\mathcal{L}} N(0,\ 1).$$

Proof: For convenience's sake, put $\widehat{f}_N = \widehat{f}_N\left(X_{N,[Np]+1}\right)$. Let

$$P_1(d) = P\left(\frac{p(1-p)z_\alpha^2}{d^2 \widehat{f}_{N-1}^2} + 1 \leq (2\gamma f(\xi_p))^{1/2} N_0^{1/(2\gamma)} t + N_0\right)$$

and

$$P_2(d) = P\left(\frac{p(1-p)z_\alpha^2}{d^2 \widehat{f}_N^2} \leq [2\gamma f(\xi_p)]^{1/2} N_0^{1/(2\gamma)} t + N_0\right).$$

Then by the definition of N, $P_1(d) \leq P\left(Z_{N(d)} \leq t\right) \leq P_2(d)$. $P_2(d)$ can be rewritten by

$$P\left(\sqrt{\frac{N_0^a}{b_N}\frac{N_0}{N}}\frac{f(\xi_p)\left\{\widehat{f}_N + f(\xi_p)\right\}}{2\widehat{f}_N^2}\frac{\sqrt{2Nb_N}\left\{\widehat{f}_N - f(\xi_p)\right\}}{\sqrt{\gamma f(\xi_p)}} \leq t\right).$$

Observe that as $d \to 0$

$$b_N N_0^a \xrightarrow{p} 1, \quad b_N/b_{N-1} \xrightarrow{p} 1, \quad \widehat{f}_N\left(X_{N,[Np]+1}\right) \xrightarrow{w.p.1} f(\xi_p). \tag{3.5}$$

By Slutsky's theorem with Theorem 3.3 and (3.5), we have $P_2(d) \to \Phi(t)$ as $d \to 0$, so that $\limsup_{d\to 0} P\left(Z_{N(d)} \leq t\right) \leq \Phi(t)$, where $\Phi(t)$ is the standard normal distribution function. Similarly, using the facts that

$$\sqrt{\frac{b_N}{b_{N-1}}\frac{N_0^a}{b_N}\frac{N_0}{N-1}}\frac{f(\xi_p)\left\{\widehat{f}_{N-1} + f(\xi_p)\right\}}{2\widehat{f}_{N-1}^2} \xrightarrow{w.p.1} 1$$

and

$$\frac{\sqrt{2(N-1)b_{N-1}}\left\{\widehat{f}_{N-1} - f(\xi_p)\right\}}{\sqrt{\gamma f(\xi_p)}} \xrightarrow{\mathcal{L}} N(0,\ 1),$$

we get $P_1(d) \to \Phi(t)$ as $d \to 0$, so that $\liminf_{d\to 0} P\left(Z_{N(d)} \leq t\right) \geq \Phi(t)$. Therefore $Z_{N(d)} \xrightarrow{\mathcal{L}} N(0,\ 1)$ as $d \to 0$. □

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