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Moments of Order Statistics from Doubly Truncated Linear-Exponential Distribution

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ABSTRACT

In this paper we establish some recurrence relations for both single and product moments of order statistics from a doubly truncated linear- exponential distribution with increasing hazard rate. These recurrence relations would enable one to compute all the higher order moments of order statistics for all sample sizes from those of the lower order in a simple recursive way. In addition, percentage points of order statistics are also discussed. These generalize the corresponding results for the linear- exponential distribution with increasing hazard rate derived by Balakrishnan and Malik (1986).

Keywords: Order statistics; Single moments; Product moments; Recurrence relations; Linear- exponential distribution; Percentage points; Exponential distribution; Rayleigh distribution.

1. INTRODUCTION

Bain (1974), Gross and Clark (1975) and Lawless (1982) have made certain suggestions regarding the usage of a distribution with its hazard function being a lower- order polynomial in the fields of life- testing and reliability. The linear- exponential distribution with its hazard rate varying as a linear function is one such distribution. Also, the potential of the linear- exponential distribution as a survival model has been demonstrated by Broadbent (1958) and Carbone, Kellerhouse and Gehan (1967).

Balakrishnan and Malik (1986) considered the linear- exponential distribution with an increasing hazard rate with p.d.f.

$$f(x) = (\lambda + \nu x)\, exp\{-(\lambda x + \nu x^2/2)\}, \;\; 0 \le x < \infty \,;\; \lambda, \nu > 0, \qquad (1.1)$$

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and c.d.f.

$$F(x) = 1 - exp\{-(\lambda x + \nu x^2/2)\}, \quad 0 \le x < \infty, \tag{1.2}$$

and established some recurrence relations satisfied by single and product moments of order statistics.

In this paper, we consider the doubly truncated linear- exponential distribution (with an increasing hazard rate) with p.d.f.

$$f(x) = \begin{cases} (\lambda + \nu x) exp\{-(\lambda x + \nu x^2/2)\}/(P-Q) & , \quad Q_1 \le x \le P_1 \\ 0 & , \quad otherwise, \end{cases}$$

$$\lambda, \nu > 0 \,, \tag{1.3}$$

and c.d.f.

$$F(x) = \frac{1}{P-Q}\left[exp\{-(\lambda Q_1 + \nu Q_1^2/2)\} - exp\{-(\lambda x + \nu x^2/2)\}\right], Q_1 \le x \le P_1, \tag{1.4}$$

where Q and $1-P$ $(0 < Q < P < 1)$ are, respectively, the proportions of truncation on the left and the right of the standard linear- exponential distribution in (1.1), and

$$Q_1 = \frac{\lambda}{\nu}\left[\left(1 - \frac{2\nu log(1-Q)}{\lambda^2}\right)^{1/2} - 1\right] \tag{1.5}$$

and

$$P_1 = \frac{\lambda}{\nu}\left[\left(1 - \frac{2\nu log(1-P)}{\lambda^2}\right)^{1/2} - 1\right] \tag{1.6}$$

are, respectively, the points of truncation on the left and the right. The truncated form of survival models are often of great interest in reliability studies, for example, see Cohen (1991).

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the doubly truncated linear- exponential distribution given in (1.3) and $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics. Let us denote the single moments $E(X_{r:n}^k)$ by $\mu_{r:n}^{(k)}$ for $1 \le r \le n$ and $k = 0, 1, 2, \ldots$, and the product moments $E(X_{r:n}^i X_{s:n}^j)$ by $\mu_{r,s:n}^{(i,j)}$ for $1 \le r < s \le n$ and $i, j = 0, 1, 2, \ldots$. For convenience, let us also use $\mu_{r:n}$ for $\mu_{r:n}^{(1)}$ and $\mu_{r,s:n}$ for $\mu_{r,s:n}^{(1,1)}$.

Denoting $(1-Q)/(P-Q)$ by Q_2 and $(1-P)/(P-Q)$ by P_2, it is easy to see that the characterizing differential equation for the doubly truncated linear-exponential distribution is

$$f(x) = (\lambda + \nu x) \ (Q_2 - F(x)), \tag{1.7}$$

or, equivalently,

$$f(x) = (\lambda + \nu x)\ P_2 + (\lambda + \nu x)\ (1 - F(x)), \tag{1.8}$$

for $Q_1 \le x \le P_1$. We shall use equations (1.7) and (1.8) in the following sections to establish several recurrence relations satisfied by the single and the product moments of order statistics. These relations will enable one to compute all the single and the product moments of all order statistics for all sample sizes in a simple recursive manner.

The results established in this paper generalize the corresponding results for the linear-exponential distribution with increasing hazard rate proved by Balakrishnan and Malik (1986). Similar recurrence relations for moments of order statistics from logistic and truncated logistic distributions were derived by Shah (1966, 1970), Balakrishnan and Joshi (1983) and Balakrishnan and Kocherlakota (1986). Results of this nature are also available for a number of other distributions, and interested reader may refer to the monograph on this topic by Arnold and Balakrishnan (1989) and also the survey paper by Balakrishnan, Malik and Ahmed (1988).

2. RELATIONSHIPS FOR SINGLE MOMENTS

The density function of $X_{r:n}$ is given by (David (1981), p.9, Arnold, Balakrishnan and Nagaraja (1992), p. 10)

$$f_{r:n}(x) = C_{r:n}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x), Q_1 \le x \le P_1, \tag{2.1}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!},$$

and $f(x), F(x), Q_1$ and P_1 are as given in equations (1.3), (1.4), (1.5) and (1.6), respectively. Then, by making use of the characterizing differential equation (1.8), we establish in this section several recurrence relations for the single moments of order statistics.

Theorem 2.1. *For $k = 0, 1, 2, \ldots,$*

$$\nu\mu_{1:1}^{(k+2)} = (k+2)\left[\mu_{1:1}^{(k)} - \frac{\lambda}{k+1}\mu_{1:1}^{(k+1)}\right] - P_2 P_1^{k+1}\left[\nu P_1 + \lambda\frac{(k+2)}{(k+1)}\right]$$

$$+Q_2 Q_1^{k+1}\left[\nu Q_1+\lambda\frac{(k+2)}{(k+1)}\right], \tag{2.2}$$

and, for $n \geq 2$,

$$\nu\mu_{1:n}^{(k+2)} = \frac{(k+2)}{n}\mu_{1:n}^{(k)} - \lambda P_2\frac{(k+2)}{(k+1)}\left[\mu_{1:n-1}^{(k+1)} - Q_1^{k+1}\right] - \lambda\frac{(k+2)}{(k+1)}$$

$$\cdot\left[\mu_{1:n}^{(k+1)} - Q_1^{k+1}\right] - \nu P_2\left[\mu_{1:n-1}^{(k+2)} - Q_1^{k+2}\right] + \nu Q_1^{k+2}. \tag{2.3}$$

Proof: Relations in (2.2) and (2.3) may be proved by following exactly the same steps as those in proving Theorem 2.2, which is presented next. □

Theorem 2.2. *For* $2 \leq r \leq n-1$ *and* $k = 0, 1, 2, \ldots,$

$$\nu\mu_{r:n}^{(k+2)} = \frac{(k+2)}{(n-r+1)}\mu_{r:n}^{(k)} - \frac{n\lambda P_2}{(n-r+1)}\frac{(k+2)}{(k+1)}\left[\mu_{r:n-1}^{(k+1)} - \mu_{r-1:n-1}^{(k+1)}\right]$$

$$-\lambda\frac{(k+2)}{(k+1)}\left[\mu_{r:n}^{(k+1)} - \mu_{r-1:n}^{(k+1)}\right] - \frac{n\nu P_2}{(n-r+1)}\left[\mu_{r:n-1}^{(k+2)} - \mu_{r-1:n-1}^{(k+2)}\right] + \nu\mu_{r-1:n}^{(k+2)}. \tag{2.4}$$

Proof: For $n \geq 1$ and $k = 0, 1, 2, \ldots,$ let us consider

$$\begin{aligned}
\mu_{r:n}^{(k)} &= C_{r:n}\int_{Q_1}^{P_1} x^k[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx \\
&= C_{r:n}\int_{Q_1}^{P_1} x^k(\lambda+\nu x)P_2[F(x)]^{r-1}[1-F(x)]^{n-r}dx \\
&\qquad + C_{r:n}\int_{Q_1}^{P_1} x^k(\lambda+\nu x)[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx \\
&= C_{r:n}\left[\lambda P_2\ E(k,n-r) + \nu P_2\ E(k+1,n-r)\right. \\
&\qquad \left. +\lambda\ E(k,n-r+1) + \nu\ E(k+1,n-r+1)\right],
\end{aligned} \tag{2.5}$$

upon using (1.8), where

$$E(a,b) = \int_{Q_1}^{P_1} x^a[F(x)]^{r-1}[1-F(x)]^b dx\,.$$

Integration by parts directly gives for $a = k, k+1$ and $b = n-r, n-r+1$,

$$E(a,b) = \frac{b}{(a+1)}\frac{\mu_{r:b+r-1}^{(a+1)}}{C_{r:b+r-1}} - \frac{(r-1)}{(a+1)}\frac{\mu_{r-1:b+r-1}^{(a+1)}}{C_{r-1:b+r-1}}, 2 \leq r \leq n-1\,.$$

Now substituting for $E(k,n-r), E(k+1,n-r), E(k,n-r+1)$ and $E(k+1,n-r+1)$ in equation (2.5) and simplifying the resulting expression, we derive the relation in (2.4). □

Proceeding exactly on similar lines, one can easily derive the following recurrence relation.

Theorem 2.3. *For $n \geq 2$ and $k = 0, 1, 2, \ldots,$*

$$\nu\mu_{n:n}^{(k+2)} = (k+2)\,\mu_{n:n}^{(k)} - n\lambda P_2 \frac{(k+2)}{(k+1)}\left[P_1^{k+1} - \mu_{n-1:n-1}^{(k+1)}\right]$$

$$-\lambda\frac{(k+2)}{(k+1)}\left[\mu_{n:n}^{(k+1)} - \mu_{n-1:n}^{(k+1)}\right] - n\nu P_2\left[P_1^{k+2} - \mu_{n-1:n-1}^{(k+2)}\right] + \nu\mu_{n-1:n}^{(k+2)} . \qquad (2.6)$$

Remark 2.1. By letting both the proportions of truncation Q and $1 - P \to 0$ ($\Rightarrow Q_2 \to 1, P_2 \to 0$) in Theorems 2.1 - 2.3, we deduce the recurrence relations established by Balakrishnan and Malik (1986) for the single moments of order statistics from the standard linear- exponential distribution with increasing hazard rate.

Remark 2.2. Letting $\nu \to 0$ (doubly truncated exponential distribution case) in Theorems 2.1 - 2.3, we deduce the recurrence relations established by Joshi (1979) for the single moments of order statistics from the doubly truncated exponential distribution.

Remark 2.3. Setting $\lambda \to 0$ (doubly truncated Rayleigh distribution case) in Theorems 2.1 - 2.3, we deduce the following recurrence relations for the single moments of order statistics from the doubly truncated Rayleigh distribution:

$$\mu_{1:1}^{(k)} = Q_2 Q_1^k - P_2 P_1^k + \frac{k}{\nu}\mu_{1:1}^{(k-2)}, \qquad (2.7)$$

$$\mu_{1:n}^{(k)} = Q_2 Q_1^k - P_2 \mu_{1:n-1}^{(k)} + \frac{k}{n\nu}\mu_{1:n}^{(k-2)}, \qquad (2.8)$$

$$\mu_{r:n}^{(k)} = \mu_{r-1:n}^{(k)} + \frac{k}{(n-r+1)\nu}\mu_{r:n}^{(k-2)} - \frac{nP_2}{(n-r+1)}\left[\mu_{r:n-1}^{(k)} - \mu_{r-1:n-1}^{(k)}\right], \qquad (2.9)$$

$$\mu_{n:n}^{(k)} = \mu_{n-1:n}^{(k)} + \frac{k}{\nu}\mu_{n:n}^{(k-2)} - P_2\left[P_1^k - \mu_{n-1:n-1}^{(k)}\right]. \qquad (2.10)$$

These results generalize the corresponding results obtained by Balakrishnan and Malik (1986, p.186) for the case of untruncated Rayleigh distribution.

3. RECURSIVE COMPUTATIONAL ALGORITHM FOR EVALUATION OF SINGLE MOMENTS

Since the values of $\mu_{r:n}^{(1)} \equiv \mu_{r:n}, 1 \le r \le n$, are needed as initial values for the recursive process, we first derive an exact expression for $\mu_{n:n}$. Consider

$$\mu_{n:n} = n\int_{Q_1}^{P_1} x[F(x)]^{n-1}f(x)dx$$
$$= n\int_{Q_1}^{P_1} x\left[\frac{A-e^{-(\lambda x+\nu x^2/2)}}{P-Q}\right]^{n-1} \frac{(\lambda+\nu x)e^{-(\lambda x+\nu x^2/2)}}{P-Q}dx,$$

upon using (1.3) and (1.4), where

$$A = e^{-(\lambda Q_1+\nu Q_1^2/2)}\ .$$

Now expanding binomially the factor $[A - exp\{-(\lambda x+\nu x^2/2)\}]^{n-1}$ gives

$$\mu_{n:n} = \frac{n}{(P-Q)^n}\sum_{t=0}^{n-1}\binom{n-1}{t}A^{n-1-t}(-1)^t.I_t\ , \tag{3.1}$$

where

$$I_t = \int_{Q_1}^{P_1} x(\lambda+\nu x)e^{-(t+1)(\lambda x+\nu x^2/2)}dx.$$

Integration by parts now yields

$$I_t = B + \frac{1}{(t+1)}\int_{Q_1}^{P_1} e^{-(t+1)(\nu/2)[(x+(\lambda/\nu))^2-(\lambda^2/\nu^2)]}dx$$
$$= B + \frac{e^{(t+1)\lambda^2/2\nu}}{(t+1)}\int_{Q_1}^{P_1} e^{-(t+1)(\nu/2)(x+(\lambda/\nu))^2}dx$$
$$= B + \frac{e^{(t+1)\lambda^2/2\nu}}{(t+1)\sqrt{(t+1)\nu}}\int_{(Q_1+(t/\nu))\sqrt{(t+1)\nu}}^{(P_1+(\lambda/\nu))\sqrt{(t+1)\nu}} e^{-y^2/2}dy$$
$$= B + \frac{\sqrt{2\pi}e^{(t+1)\lambda^2/2\nu}}{(t+1)\sqrt{(t+1)\nu}}\left[\Phi\left((P_1+\frac{\lambda}{\nu})\sqrt{(t+1)\nu}\right) - \Phi\left((Q_1+\frac{\lambda}{\nu})\sqrt{(t+1)\nu}\right)\right], \tag{3.2}$$

where

$$B = \frac{1}{(t+1)}\left[Q_1e^{-(t+1)(\lambda Q_1+\nu Q_1^2/2)} - P_1e^{-(t+1)(\lambda P_1+\nu P_1^2/2)}\right], \tag{3.3}$$

and $\Phi(\cdot)$ is the c.d.f. of a standard normal distribution. Now substituting the value of I_t from (3.2) in (3.1), we derive the exact expression for $\mu_{n:n}$.

The values of $\mu_{n:n}$ for $n \ge 1$ can be computed from the above result either by using the extensive tables of normal distribution function prepared by Owen (1962), Pearson and Hartley (1966, 1972), Rao, Mitra and Mathai (1966), or by using the computational algorithms given by Ibbetson (1963), Adams (1969) and Hill (1969). Hence, the values of $\mu_{r:n}(1 \le r \le n-1)$ can easily be obtained by using the following well-known recurrence relation (David (1981), p.48):

$$\mu_{r:n} = \sum_{j=r}^{n} (-1)^{j-r} \begin{pmatrix} j-1 \\ r-1 \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} \mu_{j:j}, 1 \le r \le n-1 .$$

Now, by using recurrence relations given in equations (2.2), (2.3), (2.4) and (2.6) in a simple recursive way, one can easily obtain all the single moments $\mu_{r:n}^{(k)}$ of all order statistics for any sample size and for any value of $k = 1, 2, \dots$.

4. RELATIONSHIPS FOR PRODUCT MOMENTS

The joint density function of $X_{r:n}$ and $X_{s:n}(1 \le r < s \le n)$ is given by (David (1981), p.10, Arnold, Balakrishnan and Nagaraja (1992), p.16)

$$f_{r,s:n}(x,y) = C_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x)\ f(y),$$

$$Q_1 \le x < y \le P_1, \tag{4.1}$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} ,$$

and $f(x), F(x), Q_1$ and P_1 are as given in equations (1.3), (1.4), (1.5) and (1.6), respectively. Then, by making use of the characterizing differential equations in (1.7) and (1.8), we establish in this section several recurrence relations for the product moments of order statistics.

Theorem 4.1. *For $1 \le r \le n-2$ and $i, j \ge 0$,*

$$\mu_{r,r+1:n}^{(i,j+2)} = \mu_{r:n}^{(i+j+2)} + \frac{(j+2)}{\nu} \left[\frac{1}{(n-r)} \mu_{r,r+1:n}^{(i,j)} - \frac{\lambda}{(j+1)} \left\{ \mu_{r,r+1:n}^{(i,j+1)} - \mu_{r:n}^{(i+j+1)} \right\} \right]$$

$$- \frac{nP_2}{(n-r)} \left[\frac{\lambda}{\nu} \cdot \frac{(j+2)}{(j+1)} \left\{ \mu_{r,r+1:n-1}^{(i,j+1)} - \mu_{r:n-1}^{(i+j+1)} \right\} + \left\{ \mu_{r,r+1:n-1}^{(i,j+2)} - \mu_{r:n-1}^{(i+j+2)} \right\} \right], \tag{4.2}$$

and, for $1 \le r < s \le n-1, s-r \ge 2$ and $i, j \ge 0$,

$$\mu_{r,s:n}^{(i,j+2)} = \mu_{r,s-1:n}^{(i,j+2)} + \frac{(j+2)}{\nu} \left[\frac{1}{(n-s+1)} \mu_{r,s:n}^{(i,j)} - \frac{\lambda}{(j+1)} \left\{ \mu_{r,s:n}^{(i,j+1)} - \mu_{r,s-1:n}^{(i,j+1)} \right\} \right]$$

$$- \frac{nP_2}{(n-s+1)} \left[\frac{\lambda}{\nu} \cdot \frac{(j+2)}{(j+1)} \left\{ \mu_{r,s:n-1}^{(i,j+1)} - \mu_{r,s-1:n-1}^{(i,j+1)} \right\} + \left\{ \mu_{r,s:n-1}^{(i,j+2)} - \mu_{r,s-1,n-1}^{(i,j+2)} \right\} \right] . \tag{4.3}$$

Proof: From equation (4.1), we have for $1 \leq r < s \leq n$ and $i, j \geq 0$

$$\mu_{r,s:n}^{(i,j)} = C_{r,s:n} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^i y^j [F(x)]^{r-1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(x) f(y) dy dx$$

$$= C_{r,s:n} \int_{Q_1}^{P_1} x^i [F(x)]^{r-1} f(x) K_1(x) dx, \tag{4.4}$$

where

$$K_1(x) = \int_{x}^{P_1} y^j [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s} f(y) dy.$$

Making use of the relation in (1.8) and splitting the integral accordingly into four, we have

$$K_1(x) = \lambda P_2 K_{j,0}(x) + \lambda K_{j,1}(x) + \nu P_2 K_{j+1,0}(x) + \nu K_{j+1,1}(x), \tag{4.5}$$

where

$$K_{a,b}(x) = \int_{x}^{P_1} y^a [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+b} dy.$$

Integration by parts yields, for $s = r + 1$,

$$K_{a,b}(x) = -\frac{x^{a+1}[1 - F(x)]^{n-s+b}}{(a+1)} + \frac{(n-s+b)}{(a+1)} \int_{x}^{P_1} y^{a+1} [1 - F(y)]^{n-s+b-1} f(y) dy,$$

and, for $s - r \geq 2$,

$$K_{a,b}(x) = -\frac{(s-r+1)}{(a+1)} \int_{x}^{P_1} y^{a+1} [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s+b} f(y) dy$$

$$+ \frac{(n-s+b)}{(a+1)} \int_{x}^{P_1} y^{a+1} [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+b-1} f(y) dy.$$

Upon substituting for $K_{j,0}(x), K_{j,1}(x), K_{j+1,0}(x)$ and $K_{j+1,1}(x)$ in (4.5) and then substituting the resulting expression for $K_1(x)$ in equation (4.4) and simplifying, we derive the relations in (4.2) and (4.3). □

Proceeding exactly on similar lines, one can easily derive the following recurrence relation.

Theorem 4.2. *For $1 \leq r \leq n-1$ and $i, j \geq 0$,*

$$\mu_{r,n:n}^{(i,j+2)} = \mu_{r,n-1:n}^{(i,j+2)} + \frac{(j+2)}{\nu} \left[\mu_{r,n:n}^{(i,j)} - \frac{\lambda}{(j+1)} \left\{ \mu_{r,n:n}^{(i,j+1)} - \mu_{r,n-1:n}^{(i,j+1)} \right\} \right]$$

$$-nP_2 \left[\frac{\lambda}{\nu} \cdot \frac{(j+2)}{(j+1)} \left\{ P_1^{j+1} \mu_{r:n-1}^{(i)} - \mu_{r,n-1:n-1}^{(i,j+1)} \right\} + \left\{ P_1^{j+2} \mu_{r:n-1}^{(i)} - \mu_{r,n-1:n-1}^{(i,j+2)} \right\} \right]. \tag{4.6}$$

Now, we derive some more recurrence relations for product moments.

Theorem 4.3. *For $n \geq 2$ and $i, j \geq 0$,*

$$\mu_{1,2:n}^{(i+2,j)} = \frac{(i+2)}{\nu}\left[\mu_{1,2:n}^{(i,j)} - \frac{n\lambda Q_2}{(i+1)}\left\{\mu_{1:n-1}^{(i+j+1)} - Q_1^{i+1}\mu_{1,n-1}^{(j)}\right\}\right]$$

$$-nQ_2\left[\mu_{1:n-1}^{(i+j+2)} - Q_1^{i+2}\mu_{1:n-1}^{(j)}\right] + \frac{\lambda}{\nu}\cdot\frac{(i+2)}{(i+1)}\left[\mu_{2:n}^{(i+j+1)} - \mu_{1,2:n}^{(i+1,j)}\right] + \mu_{2:n}^{(i+j+2)}, \quad (4.7)$$

and, for $2 \leq r \leq n-1$ and $i, j \geq 0$,

$$\mu_{r,r+1:n}^{(i+2,j)} = \frac{(i+2)}{r\nu}\left[\mu_{r,r+1:n}^{(i,j)} - \frac{n\lambda Q_2}{(i+1)}\left\{\mu_{r:n-1}^{(i+j+1)} - \mu_{r-1,r:n-1}^{(i+1,j)}\right\}\right]$$

$$-\frac{nQ_2}{r}\left[\mu_{r:n-1}^{(i+j+2)} - \mu_{r-1,r:n-1}^{(i+2,j)}\right] + \frac{\lambda}{\nu}\cdot\frac{(i+2)}{(i+1)}\left[\mu_{r+1:n}^{(i+j+1)} - \mu_{r,r+1:n}^{(i+1,j)}\right] + \mu_{r+1:n}^{(i+j+2)}. \quad (4.8)$$

Proof: From equation (4.1), we have for $1 \leq r \leq n-1$ and $i, j \geq 0$

$$\mu_{r,r+1:n}^{(i,j)} = C_{r,r+1:n}\int_{Q_1}^{P_1}\int_{Q_1}^{y} x^i y^j [F(x)]^{r-1}[1-F(y)]^{n-r-1} f(x)f(y)dxdy$$

$$= C_{r,r+1:n}\int_{Q_1}^{P_1} y^j[1-F(y)]^{n-r-1} f(y)K_2(y)dy, \quad (4.9)$$

where

$$K_2(y) = \int_{Q_1}^{y} x^i[F(x)]^{r-1}f(x)dx\ .$$

Making use of the relation in (1.7) and splitting the integral in $K_2(y)$ accordingly into four, and then following the similar steps as those used earlier in proving Theorem 4.1, one can easily establish the relations in (4.7) and (4.8). □

Proceeding exactly on similar lines, one can establish the following recurrence relation.

Theorem 4.4. *For $1 \leq r < s \leq n$, $s - r \geq 2$ and $i, j \geq 0$,*

$$\mu_{r,s:n}^{(i+2,j)} = \frac{(i+2)}{r\nu}\left[\mu_{r,s:n}^{(i,j)} - \frac{n\lambda Q_2}{(i+1)}\left\{\mu_{r:s-1:n-1}^{(i+1,j)} - \mu_{r-1,s-1:n-1}^{(i+1,j)}\right\}\right]$$

$$-\frac{nQ_2}{r}\left[\mu_{r:s-1:n-1}^{(i+2,j)} - \mu_{r-1,s-1:n-1}^{(i+2,j)}\right] + \frac{\lambda}{\nu}\cdot\frac{(i+2)}{(i+1)}\left[\mu_{r+1,s:n}^{(i+1,j)} - \mu_{r,s:n}^{(i+1,j)}\right] + \mu_{r+1,s:n}^{(i+2,j)}.$$

(4.10)

Remark 4.1. By letting both the proportions of truncation Q and $1-P \to 0$ $(\Rightarrow Q_2 \to 1, P_2 \to 0)$ in Theorems 4.1 - 4.4, we deduce the recurrence relations established by Balakrishnan and Malik (1986) for the product moments of order statistics from the standard linear-exponential distribution with increasing hazard rate.

Remark 4.2. Letting $\nu \to 0$, as in the case of single moments, we would deduce the recurrence relations established by Balakrishnan and Joshi (1984) for the product moments of order statistics from the doubly truncated exponential distribution.

Remark 4.3. Setting $\lambda \to 0$ (doubly truncated Rayleigh distribution case) in Theorems 4.1 - 4.4, we deduce the relations for doubly truncated Rayleigh distribution.

5. RECURSIVE COMPUTATIONAL ALGORITHM FOR EVALUATION OF PRODUCT MOMENTS

Since the values of $\mu_{r,s:n}^{(1,1)} \equiv \mu_{r,s:n}, 1 \le r < s \le n$ are needed as initial values for the recursive process, we first derive an exact expression for $\mu_{r,r+1:n}$. Consider

$$\mu_{r,r+1:n} = C_{r,r+1:n} \int_{Q_1}^{P_1} \int_{x}^{P_1} xy[F(x)]^{r-1}[1-F(y)]^{n-r-1} f(x) f(y)\, dy dx$$
$$= \frac{C_{r,r+1:n}}{(P-Q)^n} \int_{Q_1}^{P_1} \int_{x}^{P_1} xy[Q_3 - e^{-(\lambda x+\nu x^2/2)}]^{r-1}$$
$$\cdot[(P-Q-Q_3) + e^{-(\lambda y+\nu y^2/2)}]^{n-r-1}$$
$$.(\lambda+\nu x)(\lambda+\nu y) e^{-(\lambda x+\nu x^2/2)} e^{-(\lambda y+\nu y^2/2)}\, dy dx\,, \tag{5.1}$$

upon using (1.3) and (1.4), where

$$Q_3 = e^{-(\lambda Q_1+\nu Q_1^2/2)}\,.$$

Now expanding $\left[Q_3 - e^{-(\lambda x+\nu x^2/2)}\right]^{r-1}$ and $\left[(P-Q-Q_3) + e^{-(\lambda y+\nu y^2/2)}\right]^{n-r-1}$ binomially, we get

$$\mu_{r,r+1:n} = \frac{C_{r,r+1:n}}{(P-Q)^n} \sum_{t=0}^{r-1} \sum_{u=0}^{n-r-1} \binom{r-1}{t} \binom{n-r-1}{u} Q_3^{r-1-t}$$
$$.[P-Q-Q_3]^{n-r-1-u}(-1)^t \int_{Q_1}^{P_1} x(\lambda+\nu x) e^{-(\lambda x+\nu x^2/2)(t+1)} I_1(x)\, dx, \tag{5.2}$$

where

$$I_1(x) = \int_x^{P_1} y(\lambda + \nu y) e^{-(u+1)(\lambda y + \nu y^2/2)} dy.$$

Integration by parts yields

$$I_1(x) = -\frac{P_1 P_3^{u+1}}{(u+1)} + \frac{x e^{-(u+1)(\lambda x + \nu x^2/2)}}{(u+1)} + \frac{I_2(x)}{(u+1)}, \tag{5.3}$$

where

$$P_3 = e^{-(\lambda P_1 + \nu P_1^2/2)}$$

and

$$I_2(x) = \int_x^{P_1} e^{-(u+1)(\lambda y + \nu y^2/2)} dy.$$

We now have

$$\begin{aligned} I_2(x) &= \int_x^{P_1} exp\left[-\frac{(u+1)\nu}{2}\left\{\left(y+\frac{\lambda}{\nu}\right)^2 - \frac{\lambda^2}{\nu^2}\right\}\right] dy \\ &= e^{(u+1)\lambda^2/2\nu} \int_x^{P_1} e^{-(u+1)\nu(y+\lambda/\nu)^2/2} dy \\ &= \frac{e^{(u+1)\lambda^2/2\nu}}{\{\nu(u+1)\}^{1/2}} \int_{x^*}^{P_1^*} e^{-z^2/2} dz \\ &= \frac{e^{(u+1)\lambda^2/2\nu}}{\{\nu(u+1)\}^{1/2}} [\Phi(P_1^*) - \Phi(x^*)]\sqrt{2\pi}, \end{aligned} \tag{5.4}$$

where $x^* = \{(u+1)\nu\}^{1/2}(x + \lambda/\nu)$ and $P_1^* = \{(u+1)\nu\}^{1/2}(P_1 + \lambda/\nu)$, and $\Phi(\cdot)$ is the c.d.f. of a standard normal distribution. Substituting this expression for $I_2(x)$ from (5.4) in equation (5.3), we get

$$I_1(x) = -\frac{P_1 P_3^{u+1}}{(u+1)} + \frac{x e^{-(u+1)(\lambda x + \nu x^2/2)}}{(u+1)} + \left\{\frac{2\pi}{\nu(u+1)}\right\}^{1/2} \frac{e^{(u+1)\lambda^2/2\nu}}{(u+1)} [\Phi(P_1^*) - \Phi(x^*)].$$

Upon substituting this expression for $I_1(x)$ in equation (5.2) and splitting the integral accordingly into three, we get

$$\begin{aligned} \mu_{r,r+1:n} = \frac{C_{r,r+1:n}}{(P-Q)^n} \sum_{t=0}^{r-1} \sum_{u=0}^{n-r-1} \binom{r-1}{t} \binom{n-r-1}{u} Q_3^{r-1-t} [P-Q-Q_3]^{n-r-1-u} \\ .(-1)^t \left[-\frac{P_1 P_3^{u+1}}{(u+1)} + \left\{\frac{2\pi}{(u+1)\nu}\right\}^{1/2} \frac{e^{(u+1)\lambda^2/2\nu}}{(u+1)} \Phi(P_1^*)\right] J_1 \\ + \frac{J_2}{(u+1)} - \left[\left\{\frac{2\pi}{(u+1)\nu}\right\}^{1/2} \frac{e^{(u+1)\lambda^2/2\nu}}{(u+1)}\right] J_3, \end{aligned} \tag{5.5}$$

where

$$J_1 = \int_{Q_1}^{P_1} x(\lambda+\nu x)e^{-(t+1)(\lambda x+\nu x^2/2)}dx,$$

$$J_2 = \int_{Q_1}^{P_1} x^2(\lambda+\nu x)e^{-(t+u+2)(\lambda x+\nu x^2/2)}dx,$$

$$J_3 = \int_{Q_1}^{P_1} x\Phi(x^*)(\lambda+\nu x)e^{-(t+1)(\lambda x+\nu x^2/2)}dx.$$

First, consider J_1. Integration by parts immediately yields

$$J_1 = \left[\frac{Q_1Q_3^{t+1} - P_1P_3^{t+1}}{(t+1)}\right] + \frac{1}{(t+1)}\int_{Q_1}^{P_1} e^{-(t+1)(\lambda x+\nu x^2/2)}dx,$$

where $Q_3 = e^{-(\lambda Q_1+\nu Q_1^2/2)}$. Solving the integral as in $I_2(x)$, we get

$$J_1 = \frac{1}{(t+1)}[Q_1Q_3^{t+1} - P_1P_3^{t+1}] + \left\{\frac{2\pi}{(t+1)\nu}\right\}^{1/2}\frac{e^{(t+1)\lambda^2/2\nu}}{(t+1)}[\Phi(P_1') - \Phi(Q_1')], \quad (5.6)$$

where

$$P_1' = \{(t+1)\nu\}^{1/2}(P_1+\lambda/\nu) \text{ and } Q_1' = \{(t+1)\nu\}^{1/2}\{Q_1+\lambda/\nu\}.$$

Next, consider J_2. Integration by parts yields

$$\begin{aligned} J_2 &= \frac{1}{(t+u+2)}[Q_1^2Q_3^{t+u+2} - P_1^2P_3^{t+u+2}] \\ &\quad + \frac{2}{\nu(t+u+2)}\int_{Q_1}^{P_1}\{(\lambda+\nu x)-\lambda\}e^{-(t+u+2)(\lambda x+\nu x^2/2)}dx \\ &= \frac{1}{(t+u+2)}[Q_1^2Q_3^{t+u+2} - P_1^2P_3^{t+u+2}] + \frac{2}{\nu(t+u+2)}(H_1 - \lambda H_2), \end{aligned} \quad (5.7)$$

where

$$H_1 = \int_{Q_1}^{P_1}(\lambda+\nu x)e^{-(t+u+2)(\lambda x+\nu x^2/2)}dx$$

$$= \frac{1}{(t+u+2)}[Q_3^{t+u+2} - P_3^{t+u+2}] \quad (5.8)$$

and

$$H_2 = \int_{Q_1}^{P_1} e^{-(t+u+2)(\lambda x+\nu x^2/2)}dx$$

$$= e^{(t+u+2)(\lambda^2/2\nu)}\left\{\frac{2\pi}{\nu(t+u+2)}\right\}^{1/2}[\Phi(P_4) - \Phi(Q_4)], \quad (5.9)$$

where

$$P_4 = \{\nu(t+u+2)\}^{1/2}(P_1+\lambda/\nu) \text{ and } Q_4 = \{\nu(t+u+2)\}^{1/2}(Q_1+\lambda/\nu).$$

Substituting these expressions for H_1 and H_2 in equation (5.7), we get

$$J_2 = \frac{1}{(t+u+2)}[Q_1^2 Q_3^{t+u+2} - P_1^2 P_3^{t+u+2}] + \frac{2}{\nu(t+u+2)^2}(Q_3^{t+u+2} - P_3^{t+u+2})$$

$$- \lambda\sqrt{\pi}\left\{\frac{2}{\nu(t+u+2)}\right\}^{3/2} e^{(t+u+2)\lambda^2/2\nu}[\Phi(P_4) - \Phi(Q_4)]. \tag{5.10}$$

Finally, consider J_3. Integration by parts yields

$$J_3 = \frac{1}{(t+1)}[Q_1\Phi(Q_1^*)Q_3^{t+1} - P_1\Phi(P_1^*)P_3^{t+1}] + \frac{1}{(t+1)}[\{(u+1)\nu\}^{1/2}G_1 + G_2], \tag{5.11}$$

where

$$G_1 = \int_{Q_1}^{P_1} x\phi(x^*)e^{-(t+1)(\lambda x+\nu x^2/2)}dx$$

and

$$G_2 = \int_{Q_1}^{P_1} \Phi(x^*)e^{-(t+1)(\lambda x+\nu x^2/2)}dx \ ;$$

$x^* = \{(u+1)\nu\}^{1/2}(x+\lambda/\nu)$, $Q_1^* = \{(u+1)\nu\}^{1/2}(Q_1+\lambda/\nu)$, $P_1^* = \{(u+1)\nu\}^{1/2}(P_1+\lambda/\nu)$ and $\phi(\cdot)$ and $\Phi(\cdot)$ are, respectively, the p.d.f. and c.d.f. of a standard normal distribution. First, we have

$$\begin{aligned} G_1 &= \frac{1}{\sqrt{2\pi}}\int_{Q_1}^{P_1} xe^{-(u+1)\nu(x+\lambda/\nu)^2/2}e^{-(t+1)(\lambda x+\nu x^2/2)}dx \\ &= \frac{e^{-(u+1)\lambda^2/2\nu}}{\sqrt{2\pi}}\int_{Q_1}^{P_1} xe^{-(t+u+2)(\lambda x+\nu x^2/2)}dx \\ &= \frac{e^{-(u+1)\lambda^2/2\nu}}{\nu\sqrt{2\pi}}[H_1 - \lambda H_2], \end{aligned}$$

where H_1 and H_2 are defined in (5.8) and (5.9), respectively. Thus, we have

$$G_1 = \frac{e^{-(u+1)\lambda^2/2\nu}}{\nu\sqrt{2\pi}(t+u+2)}[Q_3^{t+u+2} - P_3^{t+u+2}] - \frac{\lambda e^{(t+1)\lambda^2/2\nu}}{\nu\{(t+u+2)\nu\}^{1/2}}[\Phi(P_4) - \Phi(Q_4)]. \tag{5.12}$$

Next, we have

$$\begin{aligned} G_2 &= \int_{Q_1}^{P_1} \Phi(x^*)e^{-(t+1)(\lambda x+\nu x^2/2)}dx \\ &= e^{(t+1)\lambda^2/2\nu}\int_{Q_1}^{P_1} \Phi(x^*)e^{-(t+1)\nu(x+\lambda/\nu)^2/2}dx \end{aligned}$$

$$= e^{(t+1)\lambda^2/2\nu}\left\{\frac{2\pi}{\nu(t+1)}\right\}^{1/2}\int_{Q_1'}^{P_1'}\Phi(gy)\phi(y)dy\ , \tag{5.13}$$

where $g = \{(u+1)/(t+1)\}^{1/2}$ and P_1' and Q_1' are the same as defined in (5.6). Thus,

$$G_2 = e^{(t+1)\lambda^2/2\nu}\left\{\frac{2\pi}{\nu(t+1)}\right\}^{1/2}\left[\int_0^{P_1'}\Phi(gy)\phi(y)dy - \int_0^{Q_1'}\Phi(gy)\phi(y)dy\right]$$

$$= e^{(t+1)\lambda^2/2\nu}\left\{\frac{2\pi}{\nu(t+1)}\right\}^{1/2}\left[\left(V(P_1', P_1'g) + \frac{1}{2}\Phi(P_1') - \frac{1}{4}\right)\right.$$

$$\left. - \left(V(Q_1', Q_1'g) + \frac{1}{2}\Phi(Q_1') - \frac{1}{4}\right)\right]$$

$$= e^{(t+1)\lambda^2/2\nu}\left\{\frac{2\pi}{\nu(t+1)}\right\}^{1/2}\left[(V(P_1', P_1'g) - V(Q_1', Q_1'g)) + \frac{1}{2}\left(\Phi(P_1') - \Phi(Q_1')\right)\right], \tag{5.14}$$

where

$$V(h,k) = \frac{1}{2\pi}\int_0^h\int_0^{kx/h} exp\{-(x^2+y^2)/2\}dydx\ .$$

The quantities $V(h,k)$ have been very extensively tabulated by Nicholson (1943), National Bureau of Standards (1959) and Yamauti (1972).

Upon substituting for G_1 and G_2 in (5.11) and then substituting the resulting expressions for J_1, J_2 and J_3 in (5.5), we derive the exact expression for $\mu_{r,r+1:n}(1 \le r \le n-1)$. Remaining product moments $\mu_{r,s:n}(s-r \ge 2)$ can be computed by making use of the well-known relation (David, 1981, pp. 48-49)

$$(r-1)\mu_{r,s:n} + (s-r)\mu_{r-1,s:n} + (n-s+1)\mu_{r-1,s-1:n} = n\mu_{r-1,s-1:n-1}. \tag{5.15}$$

Now, by using recurrence relations given in equations (4.2), (4.6), (4.7), (4.8) and (4.10) in a simple recursive way, one can easily obtain all the product moments $\mu_{r,s:n}^{(j,k)}$ of all order statistics for any sample size and for any values of $j,k = 1,2,3,\ldots$.

6. PERCENTAGE POINTS OF ORDER STATISTICS

The c.d.f. of $X_{r:n}(1 \le r \le n)$ is given by (David, 1981, p.8)

$$F_{r:n}(x) = I_{F(x)}(r, n-r+1), \tag{6.1}$$

where $I_q(a,b)$ is the incomplete beta function defined by

$$I_q(a,b) = \frac{1}{B(a,b)}\int_0^q t^{a-1}(1-t)^{b-1}dt, \quad (a,b>0),$$

and $F(x)$ is the c.d.f. of the doubly truncated linear- exponential distribution given in equation (1.4). Therefore, the $100p$ - percentage points of $X_{r:n}$ can be obtained by solving the equation

$$I_{F(x)}(r, n-r+1) = p. \tag{6.2}$$

The percentage points of $X_{r:n}(1 \leq r \leq n)$ can be calculated from (6.2), in general, by using either the extensive tables for the incomplete beta function by Karl Pearson (1968) or by using the computational algorithm given by Cran et al. (1977).

However, explicit expressions for the percentage points of the extreme order statistics can be obtained from equation (6.2). For $r = 1$, we have from (6.2)

$$1 - [1 - F(x)]^n = p$$

$$\Rightarrow \quad (1-p)^{1/n} = H(Q_1) + \frac{e^{-(\lambda x + \nu x^2/2)}}{(P-Q)},$$

where

$$H(Q_1) = 1 - \frac{e^{-(\lambda Q_1 + \nu Q_1^2/2)}}{(P-Q)}.$$

$$\Rightarrow \quad \frac{\nu}{2}x^2 + \lambda x + c = 0 \ ,$$

where

$$c = log[(1-p)^{1/n} - H(Q_1)](P-Q).$$

$$\Rightarrow \quad x = -\frac{\lambda}{\nu} + \frac{\sqrt{\lambda^2 - 2\nu c}}{\nu},$$

which is the $100p$- percentage point of the smallest order statistic $X_{1:n}$. Similarly, the explicit expression for the percentage point of the largest order statistic $X_{n:n}$ can also be obtained.

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