Journal of the Korean Statistical Society (1999), **28**: *2, pp 265–277*

Calibration by Median Regression†

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ABSTRACT

Classical and inverse estimation methods are two well known methods in statistical calibration problems. When there are outliers, both methods have large MSE's and could not estimate the input value correctly. We sugest median calibration estimation based on the LD-statistics. To investigate the robust performances, the influence function of the median calibration estimator is calculated and compared with other methods. When there are outliers in the response variables, the influence function is found to be bounded. In simulation studies, the MSE's for each calibration methods are compared. The estimated inputs as well as the performance of the influence functions are calculated.

Key Words: calibration; influence function; break down point; outlier; robustness; median calibration; LD estimation

1. INTRODUCTION

There were several approaches on the statistical calibration problem. The classical method by Eisenhart(1939) is setting up the usual regression model and inverting that model for calibration. On the other hand, the inverse estimator is setting a direct regression model by switching the response variable and regressors.(Krutchkoff(1967)). Based on the Monte Carlo studies, Krutchkoff concluded that the MSE(Mean Squared Error) of the classical estimator is uniformly greater than that of the inverse estimator. But in fact, this is true only when the estimating value is near the center of the input variables.

When there are outliers in the data, the corresponding calibration model can be deeply influenced by these outliers. Since the inferences based on the least squares methodology turned out to be sensitive to the existance of even one single outlier, it becomes important to find a stable (i.e. robust) procedure against

†The research of the first author was supported in part by Korean Reseach Foundation ,1997 and the second author was supported in part by the Research Funds of Ajou University,1997

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to these outliers. Statistics which are robust against traditional assumptions on the underlying population (normality, no gross errors, symmetricity. etc) are called robust statistics. To improve the robustness in the calibration problems, we propose the median calibration based on the LD(Least Distance) estimation method.

2. MEDIAN CALIBRATION ESTIMATION AND INFLUENCE FUNCTION

Consider the following linear models:

$$y_i = x_i\beta + \epsilon_i, \qquad i = 1, 2, \ldots, n \tag{2.1}$$

$$z_j = x_0\beta + e_j, \qquad j = 1, 2, \ldots, m \tag{2.2}$$

where x_i is the i^{th} input, y_i is the i^{th} output with p-components, β is the regression parameter in R^p, ϵ_i, e_j's are i.i.d. random errors , z_j is the j^{th} output corresponding to the unknown fixed input x_0; x_i and ϵ_i are independent each other. For the controlled calibration, x_i's are fixed and in that case, outliers occur only in y_i or z_j's.

Let $G(x, y)$ be the joint distribution function of (x, y) and $H(z)$ be the distribution function of z_1 under the input x_0. Denote $G_n(x, y)$ and $H_m(z)$ for the empirical distribution function of $G(x, y)$ and $H(z)$ respectively when $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$; $z_1|x_0, z_2|x_0, \ldots, z_m|x_0$ are observed.

Let $\beta(G)$ be the LD-estimate of β when G is the underlying distribution function. It can be shown that under certain regularity conditions, $\beta(G)$ exists, is unique and a functional on a set of distribution functions.(See Huber(1981)). Assume $\beta(G) \neq 0$. The median calibration estimation in (2.1)-(2.2) can be obtained by minimizing the following functional form

$$\int \|z - x\beta(G_n)\| dH_m(z). \tag{2.3}$$

We define the median calibration estimator $x(G; H) = \hat{x}$ for x_0 in (2.2) as followings.

Definition 2.1. *$x(G; H)$ is the estimator of x satisfying*

$$\int \|z - x(G; H)\beta(G)\| dH(z) = \min_x \int \|z - x\beta(G)\| dH(z) \tag{2.4}$$

Remark 2.1. Here $x(G;H)$ is defined in the classical calibration model (2.1)-(2.2). It can be defined similarly for the case of the inverse calibration model. For the univariate case, $x(G;H)$ becomes the median of z's devided by $\beta(G)$ in (2.4).

The influence function(Hampel(1974)) was widely adopted for measuring the influence of outliers to the statistical estimation. Define distribution functions G_ϵ and H_ϵ by

$$G_\epsilon(x,y) = (1-\epsilon)G(x,y) + \epsilon\tau(X_0, Y_0) \tag{2.5}$$

$$H_\epsilon(z) = (1-\epsilon)H(z) + \epsilon\pi(Z_0) \tag{2.6}$$

where ϵ is a small fraction of positive number less than 1. τ and π are point masses concentrated on the contaminated data (X_0, Y_0) and Z_0 respectively. The influence function for $x(G;H)$ is

$$IF(X_0, Y_0, Z_0; \hat{x}) = \left.\frac{dx(G_\epsilon, H_\epsilon)}{d\epsilon}\right|_{\epsilon=0}. \tag{2.7}$$

If

$$H(z = x\beta(G)) = 0 \text{ for all } x \in R \tag{2.8}$$

then $x(G;H)$ can be defined by taking the derivative of (2.4) w.r.t. x;

$$\frac{d}{dx}\int \|z - x\beta(G)\| dH(z) = \int \frac{-\beta'(z - x\beta)}{\|z - x\beta\|} dH(z) = 0 \tag{2.9}$$

For the sake of simplicity, let's denote $\beta_0 = \beta(G), \beta_\epsilon = \beta(G_\epsilon)$, $x_0 = x(G;H), x_\epsilon = x(G_\epsilon; H_\epsilon)$.
With these notations, the influence function for the median calibration estimation can be obtained.

Theorem 2.1. *If condition (2.8) holds with $Z_0 \neq x_0\beta_0$,then the influence function of the median calibration estimation for x_0 is given by*

$IF_{LD}(X_0, Y_0, Z_0; \hat{x})$

$$= \begin{cases} M_1^{-1}\{\frac{\beta_0'(Z_0 - x_0\beta_0)}{\|Z_0 - x_0\beta_0\|} - M_2' M^{-1} X_0 \frac{Y_0 - X_0\beta_0}{\|Y_0 - X_0\beta_0\|}\} & (p > 1) \\ \frac{\beta_0 sign(Z_0 - x_0\beta_0) - \frac{2x_0\beta_0 h(x_0\beta_0)}{Ex^2 e(x\beta_0|x)} X_0 sign(Y_0 - X_0\beta_0)}{2\beta_0^2 h(x_0\beta_0)} & (p = 1) \end{cases} \tag{2.10}$$

(where

$$M_1 = \int \frac{\beta_0'\beta_0}{\|z - x_0\beta_0\|} - \frac{[\beta_0'(z - x_0\beta_0)]^2}{\|z - x_0\beta_0\|^3} dH(z) \tag{2.11}$$

$$M_2 = \int \frac{\beta_0'(z - x_0\beta_0)x_0(z - x_0\beta_0)}{\|z - x_0\beta_0\|^3} dH(z) - \int \frac{z - 2x_0\beta_0}{\|z - x_0\beta_0\|} dH(z) \tag{2.12}$$

$$M = \int \frac{x^2[I_{(p\times p)} - \frac{(y - x\beta_0)(y - x\beta_0)'}{\|y - x\beta_0\|^2}]}{\|y - x\beta_0\|} dG \tag{2.13}$$

,$e(y)$ is a density function of y, $h(z)$ is a density function of z.)

Proof: Let's rewrite (2.9) w.r.t. H_ϵ and G_ϵ,

$$-\frac{d}{dx}\int \|z - x\beta(G_\epsilon)\| dH_\epsilon(z) = (1-\epsilon)\int \frac{\beta_\epsilon'(z - x\beta_\epsilon)}{\|z - x\beta_\epsilon\|} dH(z) + \epsilon \frac{\beta_\epsilon'(Z_0 - x\beta_\epsilon)}{\|Z_0 - x\beta_\epsilon\|} \tag{2.14}$$

In order to use (2.14), we need to show that $Z_0 - x\beta_\epsilon \neq 0$ for every ϵ in $(0,1)$. Suppose β_ϵ is the LD-estimate at G_ϵ and denote $x_1 = x_\epsilon, x_t = x_0 + t(x_1 - x_0)$ for $t \in (0,1]$.
Set

$$a(t) = \int \|z - x_t\beta_\epsilon\| dH_\epsilon. \tag{2.15}$$

Suppose $Z_0 = x_1\beta_\epsilon$. It is clear that $a(t)$ is convex and takes minimum value at $t = 1$ with $\frac{da(t)}{dt} < 0$ for $t \in [0,1)$.
Rewrite $\frac{da(t)}{dt}$ as

$$\begin{aligned} \frac{da(t)}{dt} &= \int \frac{(x_0 - x_1)\beta_\epsilon'(z - x_t\beta_\epsilon)}{\|z - x_t\beta_\epsilon\|} dH_\epsilon \\ &= (1-\epsilon)\int \frac{(x_0 - x_1)\beta_\epsilon'(z - x_t\beta_\epsilon)}{\|z - x_t\beta_\epsilon\|} dH - \epsilon\|(x_0 - x_t)\beta_\epsilon\| \end{aligned} \tag{2.16}$$

On the other hand, $\int \|z - x_t\beta_\epsilon\| dH$ is convex w.r.t. t and has a minimum at $t = 0$ with

$$\frac{d}{dt}\int \|z - x_t\beta_\epsilon\| dH = \int \frac{(x_0 - x_1)\beta_\epsilon'(z - x_t\beta_\epsilon)}{\|z - x_t\beta_\epsilon\|} dH > 0 \quad \text{for } t \in (0, 1]. \tag{2.17}$$

By the strong consistency results of the LD-estimate, we can choose ϵ small enough so that $\frac{da(t)}{dt} > 0$ for $t \in (0, 1)$. Since $x_1 = x(H_\epsilon; G_\epsilon)$, we have a contradiction to the convexity of $a(t)$. Now, the result follows by differentiating (2.14) implicitly w.r.t. ϵ. □

Remark 2.2. If we denote $\beta(G)$ for the least squares estimation with M_1, M_2 and M given as above the influence function becomes

$$IF_{LS}(X_0, Y_0, Z_0; \hat{x}) = M_1^{-1}\{\frac{\beta_0'(Z_0 - x_0\beta_0)}{\|Z_0 - x_0\beta_0\|} + M_2' M^{-1}\frac{X_0(Y_0 - X_0\beta_0)}{\int x^2 dG}\}. \tag{2.18}$$

The influence due to the point (X_0, Y_0) which appears in the second term shows a strong sensitivity to both output and input variables. For the median calibration estimation, the influence due to the output variable is bounded in the form of,

$$\frac{Y_0 - X_0\beta_0}{\|Y_0 - X_0\beta_0\|} \tag{2.19}$$

In (2.10), the term $\dfrac{\beta_0'(Z_0 - x_0\beta_0)}{\|Z_0 - x_0\beta_0\|}$ represents the effect of outliers in the output variable conditional to x. So in the case of median calibration, the influence of outliers in the prediction stage as well as in the conditional output variables is bounded.

Remark 2.3. When the outliers exist in the output variable only, the corresponding influence functions of the median, classical and inverse calibration methods are given as followings:

$$IF_{LD}(Y_0) = -\frac{x_0}{2\beta_0 E[e(x\beta_0|x)x^2]}\int \frac{x(Y_0 - x\beta_0)}{|Y_0 - x\beta_0|} dF(x) \tag{2.20}$$

$$IF_{CL}(Y_0) = \frac{1}{\beta_0^2}\frac{Ez - 2x_0\beta_0}{Ex^2}(Y_0 Ex - \beta_0 Ex^2) \tag{2.21}$$

$$IF_{INV}(Y_0) = \frac{Ez}{Ey^2}(Y_0 Ex - \gamma_0 Y_0^2) \tag{2.22}$$

Note that the influence function is bounded in the median calibration. For the classical calibration, it is represented as a linear term " $Y_0 Ex$ " and for the inverse calibration, as a quadratic term " $Y_0 Ex - \gamma_0 Y_0^2$ ".

Usually, the input data is not random, so in many practical cases we have a bounded influence function for the median calibration estimation. This can be summarized in the following Theorem.

Theorem 2.2. *Suppose there are no outliers in the design matrix. Then the influence function of $\hat{x} = x(G;H)$ is given by*

$$IF(Y_0, Z_0; \hat{x}) = \begin{cases} M_1^{-1}\left[\frac{\beta_0'(Z_0 - x\beta)}{\|Z_0 - x\beta_0\|} - M_2' M^{-1} K\right] & (p > 1) \\ \\ \dfrac{\beta_0 sign(Z_0 - x_0\beta_0) - \frac{2x_0\beta_0 h(x_0\beta_0)}{E(x^2 e(x\beta_0|x))} K}{2\beta_0^2 h(x_0\beta_0)} & (p = 1) \end{cases} \tag{2.23}$$

where M_1, M_2, M are given as before and

$$K = \int \frac{x(Y_0 - x\beta_0)}{\|Y_0 - x\beta_0\|} dF(x) \tag{2.24}$$

Proof: The defining relation of β_ϵ is in this case:

$$(1-\epsilon)\int \frac{x(y - x\beta_\epsilon)}{\|y - x\beta_\epsilon\|} dG(x,y) + \epsilon \int \frac{x(Y_0 - x\beta_\epsilon)}{\|Y_0 - x\beta_\epsilon\|} dF(x) = 0 \tag{2.25}$$

The result follows similarly as in Theorem 2.1 using (2.25). □

3. GLOBAL ROBUSTNESS OF MEDIAN CALIBRATION ESTIMATION

For the measurement of the global robustness, Hampel(1971) introduced the break down point by measuring the maximum portion of contamination without breaking down the corresponding estimator. Let Ψ and Υ be sets of distribution functions for (x, y) and $z|x$ respectively. We define

$$\Lambda_\epsilon = \{G_\epsilon; G_\epsilon(x,y) = (1-\epsilon)G(x,y) + \epsilon P(x,y), P \in \Psi,\ 0 \le \epsilon \le 1\} \tag{3.1}$$

$$\Omega_\epsilon = \{H_\epsilon; H_\epsilon(z) = (1-\epsilon)H(z) + \epsilon Q(z), Q \in \Upsilon,\ 0 \le \epsilon \le 1\} \tag{3.2}$$

The breakdown point ϵ_b is defined as

$$\epsilon_b = \max\{\epsilon; x(\epsilon) < \infty\} \quad \text{,where } x(\epsilon) = \operatorname*{Sup}_{\Lambda_\epsilon, \Omega_\epsilon} \{\|x(G_\epsilon; H_\epsilon) - x(G;H)\|\} \tag{3.3}$$

Theorem 3.1. *Let Ψ and Υ be sets of distributions including all the potential contaminated distributions. Then the median calibration estimation has* 0 *break-down point.*

Proof: Suppose there is an $\epsilon_0 > 0$ such that $x(\epsilon_0) < \infty$. We will derive a contradiction. Choose a sequence of $(G_{\epsilon_0}, H_{\epsilon_0})$ such that

$$(G_{\epsilon_0}, H_{\epsilon_0}) = (1-\epsilon_0)(G(x,y), H(z)) + \epsilon_0 \tau(X_0^{(n)}, Y_0^{(n)}; Z_0^{(n)}) \tag{3.4}$$

Let $x_1 = \frac{X_0^{(n)}(Ey)'Y_0}{Y_0'Y_0}$, $Y_0 = (n, n, \ldots, n)'$, $\beta_\epsilon = \frac{Y_0}{X_0^{(n)}}$ and $x_2 = x(G_{\epsilon_0}^{(n)}, H_{\epsilon_0}^{(n)})$. Set $x_t = x_2 + t(x_1 - x_2)$ for $t \in (0,1]$. Define

$$a(t) = \int \|z - x_t\beta_\epsilon\| dH_\epsilon \tag{3.5}$$

The derivative of $a(t)$ w.r.t. t becomes,

$$\frac{da(t)}{dt} = (1-\epsilon)\int \frac{(x_2 - x_1)\beta_\epsilon'(z - x_t\beta_\epsilon)}{\|z - x_t\beta_\epsilon\|} dH - \epsilon\|(x_2 - x_t)\beta_\epsilon\| \tag{3.6}$$

Choose a sequence of $\{X_0^{(n)}, Y_0^{(n)}; Z_0^{(n)}\}$ such that for n large enough,

$$|x_2 - x_1|\|\beta_\epsilon\| \le \frac{\epsilon}{2(1-\epsilon)} \frac{np}{X_0^{(n)}} \tag{3.7}$$

By Schwartz's inequality, (3.6) becomes,

$$\begin{aligned} \frac{da(t)}{dt} &\le (1-\epsilon)\int |x_2 - x_1|\|\beta_\epsilon\| dH - \epsilon\|(x_2 - x_t)\beta_\epsilon\| \\ &\le \frac{\epsilon}{2}\frac{(np)}{X_0^{(n)}} - \epsilon\frac{np}{X_0^{(n)}}(1 + o(1)) \\ &< 0 \end{aligned} \tag{3.8}$$

Since $a(t)$ is convex w.r.t. t, and has a minimum at $t = 0$, we have a contradiction. □

The above theorem indicates that on the global basis, the maximum portion of contaminated data in the median calibration becomes 0 and no part of the data is allowed to be contaminated. In view of infinitesimal basis, Theorem 3.1 results from non-robustness of the influence function w.r.t. the input variable.(See Theorem 2.1). In the next theorem, we will show that the median calibration retains its robust property when there is no outlier in the input variable.

Theorem 3.2. *If the marginal distribution of x in $G(x,y)$ is $F(x)$, the distribution of the design matrix in (2.1), then the break down point of the median calibration estimation is* 1/2.

Proof: We will prove for $0 \leq \epsilon < 1/2$. The case for $1/2 < \epsilon \leq 1$ can be proved similarly.
Suppose $G_\epsilon^{(n)}(x,y)$ be an ϵ-contaminated distribution around $G(x,y)$,

$$G_\epsilon^{(n)} = (1-\epsilon)G + \epsilon P_n, \quad P_n \in \Psi \quad \text{for } n = 1, 2, \ldots \tag{3.9}$$

where Ψ has $F(x)$ as a marginal distribution of x.
Let

$$x_0 = x(G; H), \quad x_1 = x(G_\epsilon^{(n)}; H_\epsilon) \tag{3.10}$$

$$x_t = x_0 + (x_1 - x_0)t, \quad 0 \leq t \leq 1 \tag{3.11}$$

Define

$$a(t) = \int \|z - x_t\beta_\epsilon\| dH_\epsilon \quad \text{with } \beta_\epsilon = \beta(G_\epsilon^{(n)}). \tag{3.12}$$

Since $a(t)$ is convex, it should have a minimum at $t = 1$. Hence $a(t) < 0$ on $t \in [0, 1)$.
By taking the derivative of $a(t)$,

$$\frac{da(t)}{dt} = (1-\epsilon)\int \frac{(x_1 - x_0)\beta_\epsilon'(z - x_t\beta_\epsilon)}{\|z - x_t\beta_\epsilon\|} dH + \epsilon \frac{(x_1 - x_0)\beta_\epsilon'(Z_0 - x_t\beta_\epsilon)}{\|Z_0 - x_t\beta_\epsilon\|} \tag{3.13}$$

Now assume $\|x_1\| \to \infty$ as n becomes large.
By the fact that the marginal distribution of x is $F(x)$ and by the robustness of β_ϵ,

$$\lim_{n\to\infty} \frac{(x_0 - x_1)\beta_\epsilon'(z - x_t\beta_\epsilon)}{\|(x_0 - x_1)\beta_\epsilon\| \|(z - x_t\beta_\epsilon)\|} = 1 \quad \text{as } n \to \infty \tag{3.14}$$

It follows that,

$$\begin{aligned} \frac{da(t)}{dt} &\geq (1-\epsilon)\textstyle\int \|(x_1 - x_0)\beta_\epsilon\| dH - \epsilon\|(x_0 - x_t)\beta_\epsilon\| \\ &\geq (1-2\epsilon)\|(x_0 - x_1)\beta_\epsilon\| + o(1) \\ &> 0 \quad \text{as } n \to \infty \end{aligned} \tag{3.15}$$

which leads a contradiction to the convexity of $a(t)$. □

4. SIMULATION RESULTS

In simulation for MSE (Mean Squared Errors), 20 data points are generated at input value of $x = -1, 1$ with model parameters $\beta_0 = 0, \beta_1 = 0.5$ and 100 repetitions. Similar results can be obtained for other values of parameters. In this section, we denote each calibration estimations of input value based on the inverse, classical and median calibration methods as inverse, classical and LD(Least Distance) estimators respectively. The performance of classical, inverse and LD estimators are compared by measuring their MSE. Since finding the LD calibration estimator is same as solving the corresponding Linear Programming problem (see, for example, R.D.Armstrong et al (1979)), we used SAS/OR and SAS Macro in the calculation of the LD estimator.

When there are no outliers in the data set, both LD and classical estimators show smaller MSE than the inverse estimator except near the centor of the input variable.(Figure 4.1). This result corresponds to the asymptotic MSE expression for classical and inverse estimators (Lwin and Maritz(1982)). On the other hand when there is an outlier in the data set, the robust feature of LD estimator can be seen. (Figure 4.2). We use the empirical expression of influence function: $IF_n(Y_0) = n(\hat{x}_{n+1} - \hat{x}_n)$ for plotting influence functions of three estimators (where Y_0 is the outlier in the output ranging from -30 to 30, $\hat{x}_n$ is the estimation of the input value at $x = 1$). Figure 4.3 shows simulated influence functions of each estimators. The LD estimator is plotted as a flat horizontal line indicating its robustness against the output outlier. But the classical and inverse estimators show linear and quadratic trend according to remark (2.3) in section 2. When the magnitude of the outlier becomes larger, the inverse estimator seems to be more sensitive due to its quadratic trend. Similar plots are obtained for the case of error distributions other than normal.

For the real data, the comparisons of the three estimators are made in table 4.1-2. The data used were the water content in soil specimens (Aitchison and Dunsmore(1975, p182)). Here the precise input measurements were made in laboratory and the outputs were observed by on cite direct measurements. For 16 observations of data, each accurate x_0 is estimated by various estimation methods with corresponding output readings and as a measure of the performance of each estimator, the average of the squared deviation are calculated.

When there are no outliers in the data (Table 4.1), the inverse estimator has smallest value but the differences between the three estimators are not significant. However if we increased the seventh output data by three times as an outlier, the LD estimator still shows good estimations while the other estimators show significant draw backs.(Table 4.2). The inverse estimator shows poor performance for observations away from the center of input variable (obs.1,2 13,14,15,16), while for the classical estimator, large deviations are observed around the center of the x variable (obs.4,5,6). The large deviation of the inverse estimator for the observations far from the center w.r.t. the input variables represents its quadratic trend.

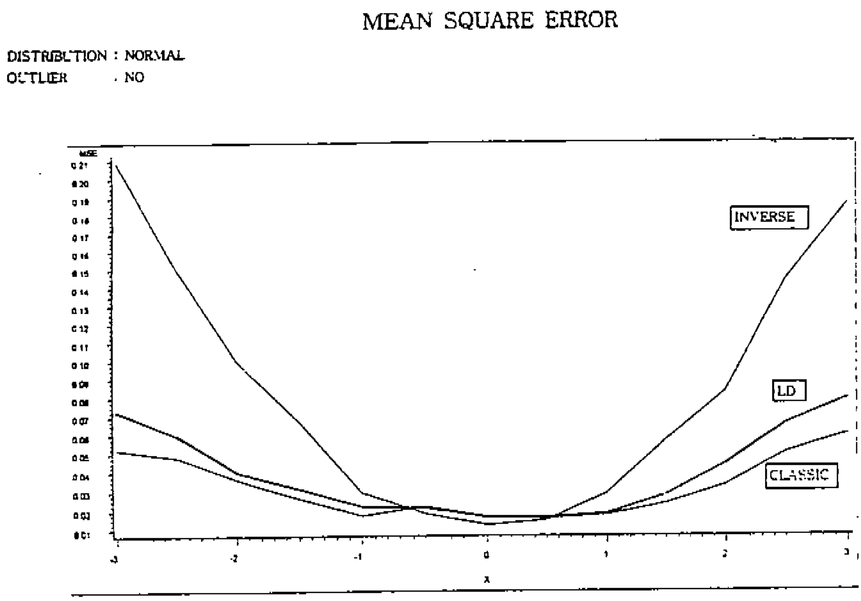

Figure 4.1: The comparison of MSE when there is no outlier in the output data.

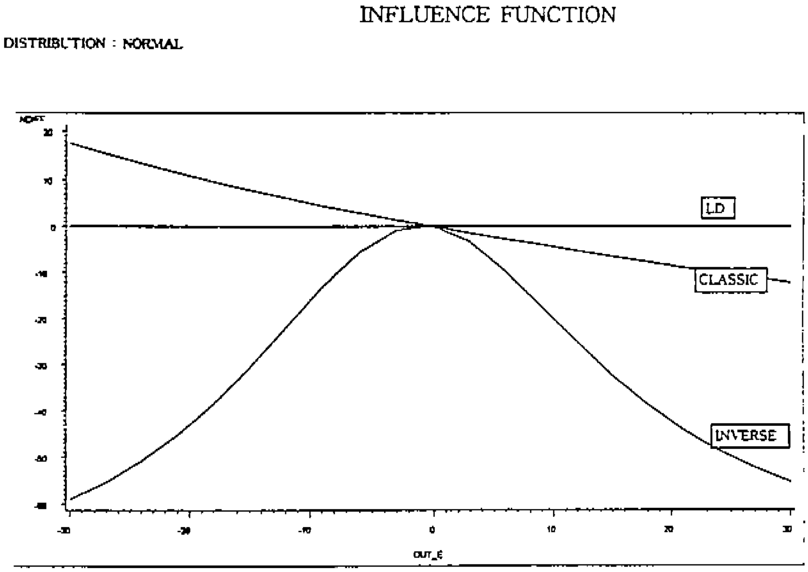

Figure 4.2: The comparison of MSE when there is an outlier in the output data.

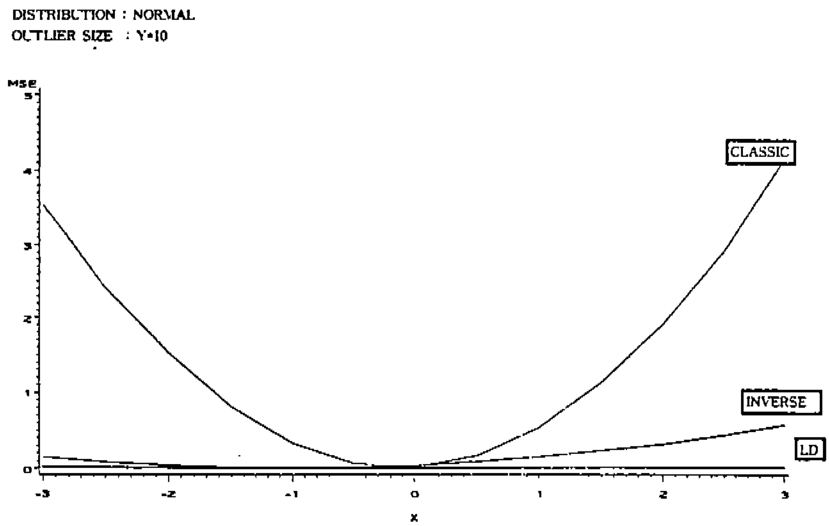

Figure 4.3: The comparison of influence functions under the normal error distribution.

Table 4.1: The estimation of the water content of soil by various predictors with no outliers in the output.

OUTLIER : NO

	Measurements on water content of soil specimens		Prediction by various predictors		
OBS	Y	X	X_{IN}	X_{CL}	X_{LD}
1	10.6	12.8	16.3027	15.8360	14.4911
2	10.1	16.1	15.6828	15.1870	13.8076
3	12.6	18.8	18.7827	18.4317	17.2253
4	12.7	19.3	18.9067	18.5615	17.3620
5	11.4	19.6	17.2947	16.8743	15.5848
6	15.8	21.6	22.7506	22.5849	21.6000
7	15.2	23.1	22.0066	21.8061	20.7797
8	17.8	24.1	25.2305	25.1806	24.3342
9	19.7	26.1	27.5865	27.6465	26.9316
10	19.0	27.5	26.7185	26.7380	25.9747
11	20.2	27.6	28.2065	28.2955	27.6152
12	24.3	33.1	33.2903	33.6167	33.2203
13	23.7	35.3	32.5464	32.8380	32.4000
14	24.5	36.2	33.5383	33.8763	33.4937
15	29.2	39.8	39.3662	39.9762	39.9190
16	31.8	39.8	42.5901	43.3507	43.4734

Table 4.2: The estimation of the water content of soil by various predictors with an outlier in the output.

OUTLIER : y * 3 (OBS = 7)

	Measurements on water content of soil specimens		Prediction by various predictors		
OBS	Y	X	X_{IN}	X_{CL}	X_{LD}
1	10.6	12.8	20.8264	11.6338	14.4832
2	10.1	16.1	20.5516	10.8978	13.8000
3	12.6	18.8	21.9252	14.5781	17.2158
4	12.7	19.3	21.9801	14.7253	17.3525
5	11.4	19.6	21.2659	12.8115	15.5762
6	15.8	21.6	23.6834	19.2889	21.5881
7	15.2	23.1	23.3537	18.4057	20.7683
8	17.8	24.1	24.7822	22.2332	24.3208
9	19.7	26.1	25.8261	25.0303	26.9168
10	19.0	27.5	25.4415	23.9998	25.9604
11	20.2	27.6	26.1008	25.7663	27.6000
12	24.3	33.1	28.3535	31.8021	33.2020
13	23.7	35.3	28.0238	30.9188	32.3822
14	24.5	36.2	28.4634	32.0965	33.4752
15	29.2	39.8	31.0457	39.0156	39.8970
16	31.8	39.8	32.4742	42.8432	43.4495

ACKNOWLEDGEMENTS

The authors thank the referees for their careful reading and helpful comments.

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