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A Method of Obtaning Least Squares Estimators of Estimable Functions in Classification Linear Models

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Abstract

In the problem of estimating estimable functions in classification linear models, we propose a method of obtaining least squares estimators of estimable functions. This method is based on the hierarchical Bayesian approach for estimating a vector of unknown parameters. Also, we verify that estimators obtained by our method are identical to least squares estimators of estimable functions obtained by using either generalized inverses or full rank reparametrization of the models. Some examples are given which illustrate our results.

Keywords: Column space; Full rank reparametrization; Generalized inverse; Hierarchical Bayes; Row space.

1. INTRODUCTION

Consider the linear model

$$\mathbf{y} = X\boldsymbol{\alpha} + \mathbf{e}, \tag{1.1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of observations y_i, X is an $n \times p$ $(p < n)$ known design matrix with $rank(X) = q$ $(q < p)$, $\boldsymbol{\alpha}$ is an $p \times 1$ vector of unknown parameters, and $\mathbf{e}$ is an $n \times 1$ vector of random errors. Classification linear models have this attribute. Then the normal equations corresponding to the model (1.1) derived by the method of least squares turn out to be

$$X'X\boldsymbol{\alpha} = X'\mathbf{y}. \tag{1.2}$$

It is well known that (1.2) is consistent and has an infinity of solutions with respect to $\boldsymbol{\alpha}$. To get any one of them we find any generalized inverse $(X'X)^-$ of $X'X$ and write the corresponding solution as $\boldsymbol{\alpha}_0 = (X'X)^- X'\mathbf{y}$, which depends

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entirely upon generalized inverses of $X'X$. But the least squares estimator (LSE) $\boldsymbol{\lambda}'(X'X)^{-}X'\mathbf{y}$ of an estimable function $\boldsymbol{\lambda}'\boldsymbol{\alpha}$ remains invariant for any generalized inverse $(X'X)^{-}$.

Now we consider the problem of estimating a vector of estimable functinons, $X\boldsymbol{\alpha}$. Generally, in the standard textbook of linear models, we use the g-inverses (see Searle (1970), Rao (1973), Graybill (1976)) or full rank reparametrization with suitable side conditions imposed on the parameters (see Scheffé (1959), Searle (1970), Graybill (1961, 1976)) to obtain the LSE $\widehat{X\boldsymbol{\alpha}}_{LS}$.

In this paper, we propose a method of obtaining the LSE of $X\boldsymbol{\alpha}$ which is based on hierarchical Bayesian approach for estimating $\boldsymbol{\alpha}$ under the sum of squared error losses. This method uses the mathematical inverse alone instead of using above methods. For the hierarchical Bayesian approach in linear models, see, for examples, Lindley and Smith(1972), Smith(1973), Goel and DeGroot(1981), Albert(1988), and Robert(1994).

We now consider the following hierarchical Bayesian model: (i) conditionally on α and r, $\mathbf{y} \sim N_n(X\alpha,\ r^{-1}I_n)$, where X is a $n \times p(p < n)$ design matrix with rank$(X) = q < p$; (ii) conditionally on r and $\mathbf{b}$, $\alpha \sim N_p(Z\mathbf{b}, (ar)^{-1}I_p)$, where a is a positive known constant, Z is a $p \times q$ known matrix with rank$(Z) = q$, and $\mathbf{b}$ is a $q \times 1$ known vector; (iii) $\mathbf{b}$ and r are marginally independently distributed with $\mathbf{b} \sim$ uniform$(\mathbf{R}^q)$ and $r \sim r^{\beta}$, $\beta \geq 0$, over $(0, \infty)$. Then after routine, but lengthy calculations, we obtain a hierarchical Bayes estimator of α, $\widehat{\alpha}_{HB} = [X'X + a(I_p - Z(Z'Z)^{-1}Z')]^{-1}X'\mathbf{y}$ under the sum of squared error losses. A detailed treatment for this subject can be found in Chang(1995).

In Section 2, we provide some results which show that the estimator obtained by our method is identical to the LSE expressed by g-inverses. In Section 3, we present some examples which illustrate the previous results.

2. A METHOD OF OBTAINING LSE

In this section, we propose a method of finding the LSE of $X\boldsymbol{\alpha}$. We first introduce a lemma necessary for our main results.

Lemma 2.1. *Let A be a $p \times p(p \geq 2)$ symmetric matrix with rank$(A) = k < p$ and B be a $p \times p$ symmetric matrix with rank$(B) = p - k$. Assume that $AB' = O$ where "B' " denotes the transpose of a matrix B, i.e., $\mathcal{C}(B) = \mathcal{R}^{\perp}(A)$ where $\mathcal{C}(B)$ is the column space of B, that is, the space spanned by columns of B and $\mathcal{R}^{\perp}(A)$ is the row null space of A which is the space of all vectors orthogonal to rows of*

A. Then $A+B$ is nonsingular and $(A+B)^{-1}$ exists.

Proof: This can be shown by contradiction as follows :
Suppose that $A+B$ is not nonsingular. Then, without loss of generality, we can assume that $rank(A+B) = p-1$. Let

$$A = \begin{pmatrix} \mathbf{a}_1' \\ \vdots \\ \mathbf{a}_p' \end{pmatrix} \text{ and } B = \begin{pmatrix} \mathbf{b}_1' \\ \vdots \\ \mathbf{b}_p' \end{pmatrix},$$

where $\mathbf{a}_i'$'s and $\mathbf{b}_j'$'s are $p \times 1$ row vectors. Then there exists a $(p-1)\times 1$ nonzero vector $(\lambda_1, \dots, \lambda_{p-1})'$ such that

$$\mathbf{a}_p' + \mathbf{b}_p' = \lambda_1(\mathbf{a}_1' + \mathbf{b}_1') + \cdots + \lambda_{p-1}(\mathbf{a}_{p-1}' + \mathbf{b}_{p-1}')$$

or, equivalently,

$$\mathbf{a}_p' - \lambda_1\mathbf{a}_1' - \cdots - \lambda_{p-1}\mathbf{a}_{p-1}' = \lambda_1\mathbf{b}_1' + \cdots + \lambda_{p-1}\mathbf{b}_{p-1}' - \mathbf{b}_p'. \tag{2.1}$$

But

$$-\lambda_1\mathbf{a}_1' - \cdots - \lambda_{p-1}\mathbf{a}_{p-1}' + \mathbf{a}_p' \in \mathcal{R}(A)$$

and

$$\lambda_1\mathbf{b}_1' + \cdots + \lambda_{p-1}\mathbf{b}_{p-1}' - \mathbf{b}_p' \in \mathcal{R}(B) = \mathcal{C}(B) = \mathcal{R}^{\perp}(A). \tag{2.2}$$

Hence, since $\mathcal{R}(A) \cap \mathcal{R}^{\perp}(A) = \{\mathbf{0}'\}$, we get, from (2.1) and (2.2),

$$-\lambda_1\mathbf{a}_1' - \cdots - \lambda_{p-1}\mathbf{a}_{p-1}' + \mathbf{a}_p' = \mathbf{0}' = \lambda_1\mathbf{b}_1' + \cdots + \lambda_{p-1}\mathbf{b}_{p-1}' - \mathbf{b}_p', \tag{2.3}$$

where $\mathbf{0}'$ is the $p \times 1$ vector in which all elements are zero.
Now

$$-\lambda_1\mathbf{a}_1' - \cdots - \lambda_{p-1}\mathbf{a}_{p-1}' + \mathbf{a}_p' = \mathbf{0}' \text{ implies } (\lambda_1, \dots, \lambda_{p-1}, -1) \in \mathcal{R}^{\perp}(A),$$

and

$$\lambda_1\mathbf{b}_1' + \cdots + \lambda_{p-1}\mathbf{b}_{p-1}' - \mathbf{b}_p' = \mathbf{0}' \text{ implies } (\lambda_1, \dots, \lambda_{p-1}, -1) \in \mathcal{R}(A). \tag{2.4}$$

Hence by (2.3) and (2.4) $(\lambda_1, \dots, \lambda_{p-1}, -1) = \mathbf{0}'$ which is impossible and therefore $rank(A+B) = p$. □

Now, we apply Lemma 2.1 to $[X'X+a\{I_p-Z(Z'Z)^{-1}Z'\}]^{-1}$, $a \neq 0$. Let $A = X'X$ and $B = a\{I_p - Z(Z'Z)^{-1}Z'\}$, $a \neq 0$. Take all linearly independent rows of X as columns of Z (i.e., $\mathcal{C}(Z) = \mathcal{R}(X)$). Then, since $\mathcal{C}(Z) = \mathcal{C}(Z(Z'Z)^{-1}Z')$ and $\mathcal{C}^{\perp}(Z) = \mathcal{C}(a\{I_p - Z(Z'Z)^{-1}Z'\})$,

$$\begin{aligned}&\mathcal{C}(a\{I_p - Z(Z'Z)^{-1}Z'\})\\&= \mathcal{R}(a\{I_p - Z(Z'Z)^{-1}Z'\})\\&= \mathcal{R}^{\perp}(X)\\&= \mathcal{R}^{\perp}(X'X).\end{aligned}$$

Hence by Lemma 2.1,

$$rank[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}] = p \quad \text{for all } a \neq 0$$

and hence $[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}$ exist for all $a \neq 0$.

In the following theorem we show that $[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}$, $a \neq 0$, correspond to generalized inverses of $X'X$ for some appropriate Z :

Theorem 2.1. *$X'X[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}X'X = X'X$ for all $a \neq 0$, where $\mathcal{C}(Z) = \mathcal{R}(X)$.*

Proof: It is clear that

$$[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}] = I_p.$$

Hence,

$$\begin{aligned}&[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}X'X\\&= I_p - [X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}[a\{I_p - Z(Z'Z)^{-1}Z'\}]. \quad (2.5)\end{aligned}$$

Therefore, premultiplying both sides of (2.5) by $X'X$ gives

$$\begin{aligned}&X'X[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}X'X\\&= X'X - X'X[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}[a\{I_p - Z(Z'Z)^{-1}Z'\}] \quad (2.6)\end{aligned}$$

Let $A = [X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}$ and $B = a\{I_p - Z(Z'Z)^{-1}Z'\}$. Then A is nonsingular and

$$\mathcal{C}(B) = \mathcal{R}(B) = \mathcal{C}^{\perp}(Z) = \mathcal{R}^{\perp}(X) = \mathcal{R}^{\perp}(X'X).$$

Also, $\mathcal{R}(AB) = \mathcal{R}(B)$ and $\mathcal{R}(X'X) = \mathcal{C}(X'X)$. Hence $X'X[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}$
$[a\{I_p - Z(Z'Z)^{-1}Z'\}] = O$ and the result now folows from (2.6). □

Remark 2.1. Theorem 2.1 implies that if $\mathcal{C}(Z) = \mathcal{R}(X)$, then for all $a \neq 0$, $X'X+a\{I_p-Z(Z'Z)^{-1}Z'\}$ are generalized inverses of $X'X$. This provides another method of obtaining generalized inverses of $X'X$.

Next we show that $X[X'X+a\{I_p-Z(Z'Z)^{-1}Z'\}]^{-1}X'$ remains invariant with respect to $a \neq 0$. To do this we need the following two lemmas :

Lemma 2.2. *For any matrix X, $tr(X'X) = 0 \Rightarrow X = O$.*

Proof: $tr(X'X)$ is the sum of the squares of the elements of X. Hence

$$tr(X'X) = 0 \Rightarrow X = O. \qquad \square$$

Lemma 2.3. *For any matrix X and for any matrices P and L of appropriate dimensions,*

$$PX'X = LX'X \Rightarrow PX' = LX'(\text{ or } XP' = XL').$$

Proof:

$$\begin{aligned} O &= PX'X - LX'X \\ \Longrightarrow\quad O &= (PX'X - LX'X)(P-L)' \\ &= (P-L)X'X(P-L)' \\ &= [X(P-L)']'[X(P-L)'] \\ \Longrightarrow\quad 0 &= tr\{[X(P-L)']'[X(P-L)']\} \\ \Longrightarrow\quad O &= X(P-L)' \quad \text{by Lemma 2.2} \\ \Longrightarrow\quad XP' &= XL'. \end{aligned} \qquad \square$$

Corollary 2.1. *Let X be any matrix and G_1 and G_2 be any two generalized inverses of $X'X$. Then*

$$XG_1X' = XG_2X'.$$

Theorem 2.2. *$X[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}X'$ remains invariant with respect to $a \neq 0$ where $\mathcal{C}(Z) = \mathcal{R}(X)$.*

Proof: By Theorem 2.1, $X'X[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}X'X = X'X$ for all $a \neq 0$. By Corollary 2.1, $X[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}X'$ is invariant for all $a \neq 0$. $\square$

We conclude from Theorems 2.1 and 2.2 that the estimator

$$\widehat{X\boldsymbol{\alpha}}_M = X[X'X + a\{I_p - Z(Z'Z)^{-1}Z'\}]^{-1}X'\mathbf{y}, \qquad a \neq 0 \tag{2.7}$$

of $X\boldsymbol{\alpha}$ is identical to the LSE $\widehat{X\boldsymbol{\alpha}}_{LS} = X(X'X)^{-}X'\mathbf{y}$ of $X\boldsymbol{\alpha}$. Note that for $a > 0$, $\widehat{X\boldsymbol{\alpha}}_M$ is identical to $X\widehat{\boldsymbol{\alpha}}_{HB}$ where $\widehat{\boldsymbol{\alpha}}_{HB}$ is the hierarchical Bayes estimator of $\boldsymbol{\alpha}$ given in Section 1. We also note that $\widehat{X\boldsymbol{\alpha}}_{LS}$ can be obtained by using full rank reparameterization.

3. EXAMPLES

We present simple examples which illustrate the previous results.

Example 3.1. Now we consider the two-way classification linear model with unbalanced data

$$y_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk}, \ i,j = 1,2, \ k = 1,\cdots, \ n_{ij} \text{ with } n_{11} = n_{12} = n_{22} = 2, n_{21} = 3$$

where y_{ijk} is the k-th observation on the i-th level of the α-factor and the j-th level of the β-factor, μ is the true mean, α_i is the effect due to the i-th level of the α-factor, β_j is the effect due to the j-th level of the β-factor, and e_{ijk} is the random error associated with y_{ijk}. Equivalently, we can write

$$\mathbf{y} = X\boldsymbol{\alpha} + \mathbf{e},$$

where

$$\mathbf{y} = \begin{pmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{213} \\ y_{221} \\ y_{222} \end{pmatrix}, X = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \boldsymbol{\alpha} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix}, \text{and } \mathbf{e} = \begin{pmatrix} e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{211} \\ e_{212} \\ e_{213} \\ e_{221} \\ e_{222} \end{pmatrix}.$$

Note that $rank(X) = 3$. Now,

$$X'X = \begin{pmatrix} 9 & 4 & 5 & 5 & 4 \\ 4 & 4 & 0 & 2 & 2 \\ 5 & 0 & 5 & 3 & 2 \\ 5 & 2 & 3 & 5 & 0 \\ 4 & 2 & 2 & 0 & 4 \end{pmatrix}, \text{and } (X'X)^- = \frac{1}{22}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & -6 & -5 \\ 0 & 0 & -6 & 8 & 3 \\ 0 & 0 & -5 & 3 & 8 \end{pmatrix}$$

is a g-inverse of $X'X$. Hence the least squares estimator of $X\boldsymbol{\alpha}$ is given by $\widehat{X\boldsymbol{\alpha}_{LS}} = X(X'X)^- X'\mathbf{y}$. Now we take $Z = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ so that $\mathcal{C}(Z) = \mathcal{R}(X)$.

Without loss of generality, we can take $a = 1$ in (2.7). Then we have

$$\widehat{X\boldsymbol{\alpha}_M} = X\left[X'X + \{I_5 - Z(Z'Z)^{-1}Z\}\right]^{-1} X'\mathbf{y}$$

$$= \frac{1}{22}\begin{pmatrix} 8 & 8 & 3 & 3 & 2 & 2 & 2 & -3 & -3 \\ 8 & 8 & 3 & 3 & 2 & 2 & 2 & -3 & -3 \\ 3 & 3 & 8 & 8 & -2 & -2 & -2 & 3 & 3 \\ 3 & 3 & 8 & 8 & -2 & -2 & -2 & 3 & 3 \\ 2 & 2 & -2 & -2 & 6 & 6 & 6 & 2 & 2 \\ 2 & 2 & -2 & -2 & 6 & 6 & 6 & 2 & 2 \\ 2 & 2 & -2 & -2 & 6 & 6 & 6 & 2 & 2 \\ -3 & -3 & 3 & 3 & 2 & 2 & 2 & 8 & 8 \\ -3 & -3 & 3 & 3 & 2 & 2 & 2 & 8 & 8 \end{pmatrix}\mathbf{y}.$$

Now, it can be shown that

$$\begin{aligned} \widehat{X\boldsymbol{\alpha}_M} &= X\left[X'X + \{I_5 - Z(Z'Z)^{-1}Z\}\right]^{-1} X'\mathbf{y} \\ &= X(X'X)^- X'\mathbf{y} \\ &= \widehat{X\boldsymbol{\alpha}_{LS}}. \end{aligned}$$

Also, it can be verified that $X'X\left[X'X + \{I_5 - Z(Z'Z)^{-1}Z\}\right]^{-1} X'X = X'X$.

Example 3.2. Consider the two-way nested linear model with balanced data

$$y_{ijk} = \mu + \alpha_i + \beta_{ij} + e_{ijk},\ i = 1,2\ ;\ j = 1,2\ ;\ k = 1,2,$$

where y_{ijk} is the k-th observation on the j-th level of the β-factor nested within the i-th level of the α-factor, μ is the true mean, α_i is the effect due to the i-th level of the α-factor, β_{ij} is the effect due to the j-th level of the β-factor nested within the i-th level of the α-factor, and e_{ijk} is the random error associated with y_{ijk}.

Equivalently, we can write

$$\mathbf{y} = X\boldsymbol{\alpha} + \mathbf{e}.$$

where

$$\mathbf{y} = \begin{pmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \end{pmatrix}, X = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \boldsymbol{\alpha} = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_{11} \\ \beta_{12} \\ \beta_{21} \\ \beta_{22} \end{pmatrix}, \text{and } \mathbf{e} = \begin{pmatrix} e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{211} \\ e_{212} \\ e_{221} \\ e_{222} \end{pmatrix}.$$

Note that $rank(X) = 4$. Now, take $Z = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ so that $\mathcal{C}(Z) = \mathcal{R}(X)$.

Without loss of generality we can take $a = 1$ in (2.7). Then we have

$$\widehat{X\boldsymbol{\alpha}_M} = X[X'X + \{I_7 - Z(Z'Z)^{-1}Z'\}]^{-1}X'\mathbf{y}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \mathbf{y}.$$

Also it can be verified that $X'X[X'X + \{I_7 - Z(Z'Z)^{-1}Z'\}]^{-1}X'X = X'X$. We next calculate the least squares estimator of $X\boldsymbol{\alpha}$ by using full rank reparametrization of the model. To do this, we use side conditions $\sum_{i=1}^{2} \alpha_i = 0$ and $\sum_{j=1}^{2} \beta_{ij} = 0$ for $i = 1, 2$. Then we get the following full rank model :

$$\mathbf{y} = X^* \boldsymbol{\alpha}^* + \mathrm{e},$$

where $X^* = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$ and $\boldsymbol{\alpha}^* = \begin{pmatrix} \mu \\ \alpha_1 \\ \beta_{11} \\ \beta_{21} \end{pmatrix}$. Now the least squares estimator of $\boldsymbol{\alpha}^*$ is given by

$$\begin{aligned} \widehat{\boldsymbol{\alpha}^*}_{LS} &= \begin{pmatrix} \widehat{\mu} \\ \widehat{\alpha_1} \\ \widehat{\beta_{11}} \\ \widehat{\beta_{21}} \end{pmatrix} \\ &= (X^{*'}X^*)^{-1}X^{*'}\mathbf{y} \\ &= \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} \mathbf{y}. \end{aligned}$$

Hence it can be shown by some calculations that the least squares estimator of

$X\boldsymbol{\alpha}$ is given by

$$\widehat{X\boldsymbol{\alpha}}_{LS} = \begin{pmatrix} \widehat{\mu} + \widehat{\alpha_1} + \widehat{\beta_{11}} \\ \widehat{\mu} + \widehat{\alpha_1} + \widehat{\beta_{11}} \\ \widehat{\mu} + \widehat{\alpha_1} - \widehat{\beta_{11}} \\ \widehat{\mu} + \widehat{\alpha_1} - \widehat{\beta_{11}} \\ \widehat{\mu} - \widehat{\alpha_1} + \widehat{\beta_{21}} \\ \widehat{\mu} - \widehat{\alpha_1} + \widehat{\beta_{21}} \\ \widehat{\mu} - \widehat{\alpha_1} - \widehat{\beta_{21}} \\ \widehat{\mu} - \widehat{\alpha_1} - \widehat{\beta_{21}} \end{pmatrix}$$
$$= \widehat{X\boldsymbol{\alpha}}_{M}.$$

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