일반역행렬을 이용한 복합하중을 받는 구조물의 안정경계에 관한 연구

A Study on the Stability Boundary for Multi-Loading System by Using Generalized Inverse

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요 지

본 연구는 복합하중을 받는 구조물에 있어서 구조물의 안정경계점을 계산하는 방법을 제시하고 있다. 여기에서는 우선 안정경계점에 놓여 있는 기지의 점에 대한 선형해를 일반역행열을 이용하여 선형 증분 평형방정식의 여해와 특이해의 선형결합으로 나타내었다. 다음으로 두개의 하중계수를 구속하는 선형조건을 도입하고, 그 구속조건하에서 하중계수 比가 일정하게 되도록 반복계산을 수행하므로써, 안정경계점위의 다음 목표점이 얻어진다. 얻어진 이 점을 초기점으로 이용한다. 평형경로를 추적할 때, 본래의 두 개의 하중계수 문제는 하중계수의 比가 일정하다는 조건을 도입하여 단일 하중계수의 문제로 된다. 두 개의 예를 들어 수치해석을 행하였으며, 얻어진 결과로부터 본 연구에서 채택된 방법은 구조물의 경계안정점을 찾는 문제에 적합하며 더욱 개발할 여지가 있음을 보여주고 있다.

핵심용어 : 복합하중, 안정경계, 일반역행렬, 구속조건

Abstract

A strategy for computing the stability boundary for two-parameter structural problems has been proposed. In this strategy, the incremental solution to a known point lying on the stability boundary is first expressed as the linear combination of homogeneous and particular solution of the linear incremental equilibrium equations by using generalized inverse. Next by imposing a linear constraint on the two loading parameters and then carrying out iteration at constant load, a point lying in the vicinity of the next target point on the stability boundary is obtained. Numerical analysis has been carried out on two examples. From the obtained results, it can be concluded that the proposed strategy is worthy of further development for multi-parameter problems.

Keywords : *multi-loading, stability boundary, generalized inverse, constraints*

1. Introduction

One of the main purpose of carrying out nonlinear static analysis on a given structural problem is to determine the critical deformed configuration at which dynamic failure might

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occur due to instability. Whether the instability problem is of snap-through or bifurcation buckling type will depend on numerous factors such as load type, loading intensity, loading direction, loading positions, imperfection of initial structural geometry, variation of material constant, boundary condition etc. All or some of these factors when considered in combination can reveal phenomena which may be overlooked if they are considered separately[13]. Futhermore for structural design purpose, evaluation of the critical combination of loading conditions is necessary in order to provide adequate safety against failure[2]. Therefore in order to obtain a more complete understanding of the behavior of a structure with respect to stability, it is necessary to perform analysis considering all or some of the above mentioned factors in combination. This type of problem is commonly called multi-parameter problem.

For a given problem with N degrees of freedom(DOF), we will have N number of equilibrium equations in hand. If the influence of $M(\geq 1)$ different parameters on structural stability is to be investigated, additional constraining equations with regards to the M parameters are necessary. Furthermore, since we are interested in only those solutions representing critical deformed configurations, additional condition representing the criteria of the occurrence of critical solutions must also be specified. The locus of critical solutions when projected on to the parameter space is commonly called stability boundary(Fig. 1 and Fig. 2). The computational works carried out in order to obtain the stability boundary is termed stability analysis. Huseyin[3] has extended the use of perturbation technique to compute the stability boundary for multi-parameter nonlinear problems. The vanishing of the determinant D of tangent stiffness matrix has been used in order to obtain the critical solutions. With Huseyin's approach, derivatives of the determinant of tangent stiffness matrix are required. Holzer et. al[5] have solved the multi-parameter problem by imposing additional relations among the different loading parameters leaving only one independent parameter.

By changing the relations imposed, stability boundary could then be obtained. As criteria for detecting the occurrence of critical solutions the vanishing of D is again used. The method of solving a multi-parameter system

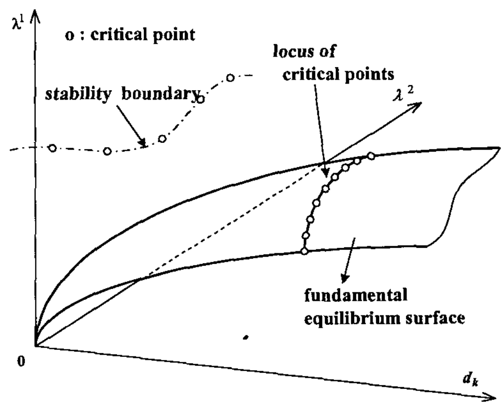

Fig. 1 Fundamental equilibrium surface

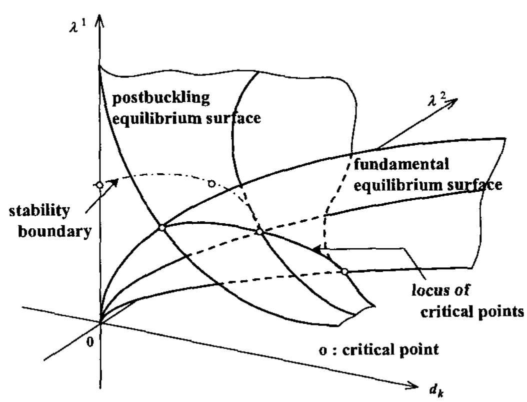

Fig. 2 Intersection of two equilibrium surfaces

by reducing it to a single-parameter system is the most simple method. Nevertheless, in order to obtain a fairly accurate representation of the stability boundary, considerable amount of computation must be carried out. In order to reduce the amount of computational effort required, Rheinboldt[6] has proposed three different approaches whereby the stability boundary could be computed directly once a point on it has been obtained. In two of Rheinboldt's proposal, solution of an extended system of equation is necessary. Futhermore, second derivatives of those quantities appearing in the criteria for critical condition are necessary. The third method involves a two step execution of standard continuation method with different starting tracing direction. Waszczyszyn and Cichon[7] have traced the stability boundary starting from a known point on it by constraining the different loading parameters by means of a specified 'loading process'. In this way, the analysis of a multi-parameter loading problem is being replaced by the analysis of a successive series of single parameter loading problem. In order to distinguish between limit and bifurcation point, D is used in combination with the determinant of the coefficient matrix of the linearized incremental equations used in the arch-length method. As a possible way of avoiding any numerical trouble due to near-critical condition of tangent stiffness matrix when computing the stability boundary, Hangai and Kawaguchi[4] have presented a method of stability analysis by using generalized inverse. By Specifying the unknown coefficients appearing in the expression of tangent vector to a known point on the stability boundary in a trial-and-error manner, subsequent point lying on the stability boundary could be determined.

In this paper, the possibility of using the generalized inverse in stability analysis has been further explored. A particular aspect investigated is the replacement of the trial-and-error approach in obtaining subsequent points on stability boundary with a set of procedures based on continuation method. The purpose of this study is to present an alternative strategy for stability analysis with which numerical instability could be avoided and standard continuation method could be advantageously utilized. Here, the continuation method is a numerical method used to trace nonlinear curves in a point-wise(point by point) matter. That is, after determining one point, we continue to determine the following point using the already determined point as starting point. After a brief summary of stability analysis using generalized inverse, the proposed procedures are described. This is then followed by the results of two numerical experiments carried out in order to test applicability of the proposed strategy. Remarks and conclusion with regards to the proposed procedures are then presented.

2. Stability analysis using generalized inverse[2]

The equilibrium equations and its linearized incremental form for a discretized N DOF problem subjected to the influence of M loading parameters could be written as follows;

$$\mathbf{r}=\mathbf{f}(\mathbf{d})-\lambda^{\alpha}\mathbf{p}^{\alpha}=0 \tag{1}$$

$$\mathbf{K}\Delta\mathbf{d} = \Delta\lambda^{\alpha}\mathbf{p}^{\alpha} \tag{2}$$

where,

$\mathbf{r}=\{r_i\}^T$: residual force vector

$\mathbf{f}=\{f_i\}^T$: internal resistant force vector

$\mathbf{d}=\{d_i\}^T$: generalized displacement vector

λ^α : loading parameter corresponding to the α^{th} constant loading vector $\mathbf{p}^\alpha$

$\mathbf{K}$: N×N tangent stiffness matrix

$\Delta\mathbf{d}$: vector of incremental $\mathbf{d}$

$\Delta\lambda^\alpha$: vector of increment of loading parameter λ^α ($i=1\sim N$, $\alpha=1\sim M$)

^{T} : notation for transposition

Solutions to eq.(1) will form an equilibrium surface in the N+M space of $\mathbf{d}-\lambda^1, \lambda^2, \cdot, \lambda^M$. Equilibrium surface that passes through the origin of $d=0$ and $\lambda^\alpha=0$ is called fundamental equilibrium surface. The purpose of stability analysis is to locate the locus of critical points, namely the stability boundary, which lies on this surface(Fig. 1 and 2). There are two types of critical points to be encountered for structures subjected to multi-parameter loadings : i) general points and ii) special points. The former points are limit point-type (Fig. 1) whilst the latter ones are bifurcation point-type critical points(Fig. 2). General points also include bifurcation points as special case which occur at certain combination of λ^α. Incremental solution to any known point on the equilibrium surface could be obtained by solving eq.(2). These solutions to eq.(2) are composed of linear combination of homogeneous and particular solution as follow[4];

$$\Delta\mathbf{d}=[\mathbf{I}-\mathbf{K}^-\mathbf{K}]\mathbf{c}+\mathbf{K}^-(\Delta\lambda^\alpha\mathbf{p}^\alpha) \tag{3}$$

where,

$\mathbf{c}$: arbitrary vector

$\mathbf{I}$: Identity matrix

$\mathbf{K}^-$: Moore-Penrose generalized inverse of $\mathbf{K}$

For the case of simple critical point where rank $[\mathbf{I}-\mathbf{K}^-\mathbf{K}]=1$, eq(3) will become

$$\Delta\mathbf{d} = \mathbf{c}\mathbf{h}+\mathbf{K}^-\Delta\lambda^\alpha\mathbf{p}^\alpha \tag{4}$$

where,

$\mathbf{h}$: linearly independent vector of $[\mathbf{I}-\mathbf{K}^-\mathbf{K}]$

For conservatives system, it could be proved that

$$\mathbf{h}=\beta\phi \tag{5}$$

where,

ϕ : eigenvector corresponding to the smallest eigenvalue of $\mathbf{K}$

β : arbitrary constant.

By specifying appropriate values to the 1+M unknownsrm $\mathbf{c}$ and $\Delta\lambda^\alpha$ solutions which satisfy eq.(1) and with rank $[\mathbf{I}-\mathbf{K}^-\mathbf{K}]=1$ could be obtained through trial-and-error method. Hence, if a point lying on the stability boundary is known stability boundary could be traced in a point-wise manner. Nevertheless, since there is no criteria upon which the selection of $\mathbf{c}$ and $\Delta\lambda^\alpha$ could be based, considerable computational effort is necessary in order to obtain the stability boundary. As a way to solve this problem, we have proposed an alternative strategy which makes use of both generalized inverse and continuation method. In this paper, only multi-parameter problem with M=2 has been considered.

3. Tracing of stability boundary with the use of generalized inverse and continuation method

Concept of the proposed strategy for stability analysis is illustrated in Fig. 3. We assume that a point $\mathbf{P}_c^{(k)}$(k=1,2,3..) could be computed by using any standard continuation method with the constraint $\lambda^2=m_1\lambda^1$ imposed on the

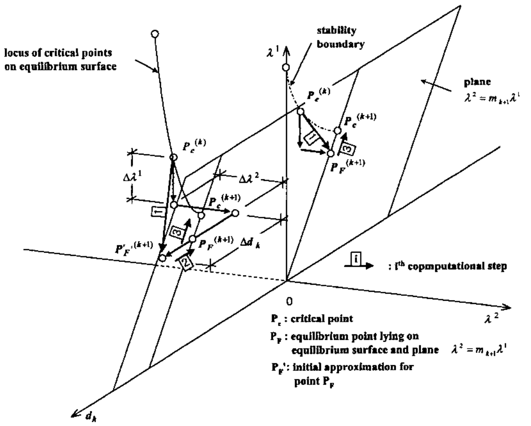

Fig. 3 Concept of the proposed strategy for the tracing of stability boundary

two loading parameters, where m_1 : arbitrary real constant. In order to obtain the next point $P_c^{(k+1)}$, firstly it is necessary to assign values to the three unknowns c, $\Delta\lambda^1$ and $\Delta\lambda^2$ in eq.(4). In this paper, the values of $\Delta\lambda^1$ and $\Delta\lambda^2$ are selected based on the following condition.

$$\frac{\lambda^{2,(k+1)}}{\lambda^{1,(k+1)}} = \frac{\lambda^{2,(k)}+\Delta\lambda^2}{\lambda^{1,(k)}+\Delta\lambda^1} = m_{k+1} \tag{6}$$

By specifying a chosen value to $\Delta\lambda^1$, $\Delta\lambda^2$ could be determined by using eq.(5) as follow

$$\Delta\lambda^2 = m_{k+1}(\lambda^{1,(k)}+\Delta\lambda^1)-\lambda^{2,(k)} \tag{7}$$

The value of $\mathbf{c}$ in eq.(4) could be chosen such that a particular critical component of $\mathbf{h}$ $(\beta\phi)$ is prevented from exceeding certain prescribed limiting value. Addition of the obtained $\Delta\mathbf{d}$[4] to point.

$P_c^{(k)}$ will yield $P_F^{(k+1)}$which in general will not be satisfied the equilibrium equations[1]. By loading the two sets of load constant at $\lambda^{1,(k+1)}$ and $\lambda^{2,(k+1)}$, iteration to obtain an equilibrium point lying in the vicinity of $P^{(k)}$ and on the loading plane $\lambda^2 = m_{k+1}\lambda^1$ is executed. This equilibrium point is denoted as $P_F^{(k+1)}$ in Fig. 3. After $P_F^{(k+1)}$ has been computed, $P_c^{(k+1)}$ is then obtained by tracing the equilibrium curve using continuation method with the following constant imposed on the two loading parameters

$$\lambda^2 = \dot{m}_{k+1}\,\lambda^1 \tag{8}$$

After $P_c^{(k+1)}$ is obtained, the above set of procedures can then be repeated in order to determined $P_c^{(k+2)}$. In this manner, stability boundary could be traced successively. Fig. 4 and Fig. 5 illustrate the proposed strategy in loading parameter space and loading parameter-displacement space, respectively. The use of eq.(7) will reduce the two-parameter problem to that of a single-parameter loading problem with λ^1 acting as the system parameter.

In order to identify the type of critical point $P_c^{(k)}$ encountered, the following criteria has been adopted in this paper. Let's denote the triangularization of $\mathbf{K}$ as

$$\mathbf{K}=\mathbf{L}\mathbf{D}\mathbf{L}^{T} \tag{9}$$

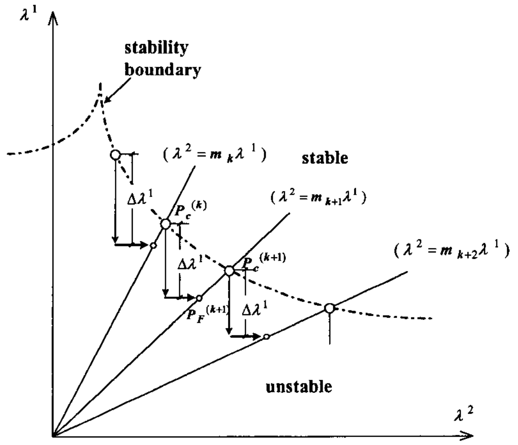

Fig. 4 Illustration of the strategy in loading parameter plane

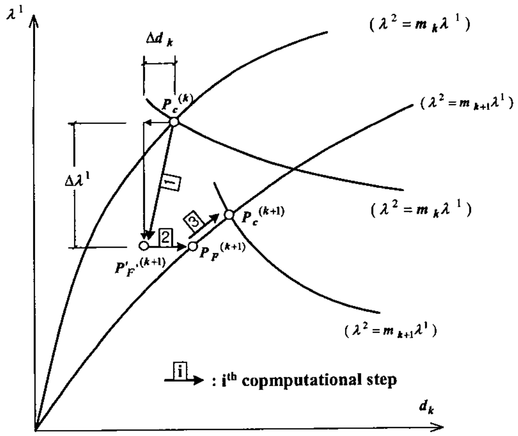

Fig. 5 Illustration of the strategy in loading parameter plane

where

D : diagonal matrix

L : lower triangular matrix.

Let's denote NP as the number of negative diagonal elements of matrix **D**. Whenever NP is observed to be changing from 0 to 1(or vice versa), $\Delta\lambda^1$ is examined and $\mathbf{P}_c^{(k)}$ is classified either as limit point or bifurcation point as follows;

$sign(\Delta\lambda^1_{j+1}) = sign(\Delta\lambda^1_j) \Rightarrow$ bifurcation point

$sign(\Delta\lambda^1_{j+1}) \neq sign(\Delta\lambda^1_j) \Rightarrow$ limit point

We note that criteria described here can be applied to only the case that the $\mathbf{P}_c^{(k)}$ encountered corresponds to the first critical point lying on the primary equilibrium curve.

4. Numerical examples

Stability analysis using the procedures described in section 3 has been carried out on two numerical models. Stability boundary which consists of general points and special points have been traced for the first and second numerical example, respectively. Modified Riks' arc-length method with normal plane constraint has been used as the continuation method.

4. 1 Bergan's truss

This 2-DOF model(Fig. 6) consists of three linear springs with spring constant k_1 and k_2. The horizontal spring is assumed to retain its initial inclination during deformation. Two point loads P^1 and P^2 act at node 2 in the directions of z-axis and y-axis, respectively. The loading parameters λ^1 and λ^2 are defined as rm P^1/P^1_0 and P^2/P^2_0, respectively, with both the reference load P^1_0 and P^2_0 assumed as unity. Two sets of stability analysis with the ratio H/L assumed as 0.35 and 0.45 respectively have been carried out. The values of **c** = 0.05 and $\Delta\lambda^1 = -5$ have been used during the analysis for both cases of H/L. Fig. 7 shows the stability boundaries traced for two cases of H/L. The values of λ^1 and λ^2 for points lying on two stability boundaries are listed in Table 1 and 2. All critical points lying on both stability boundaries are general points. For the case of H/L=0.45, bifurcation point

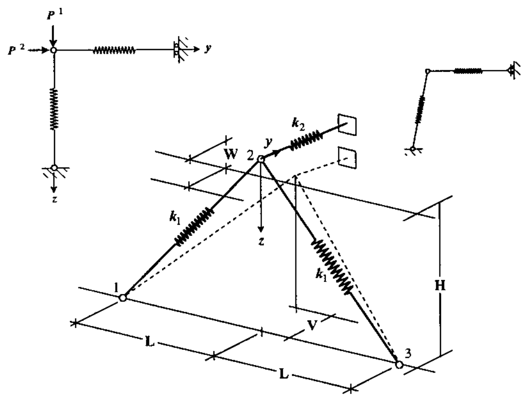

Fig. 6 Bergan's truss

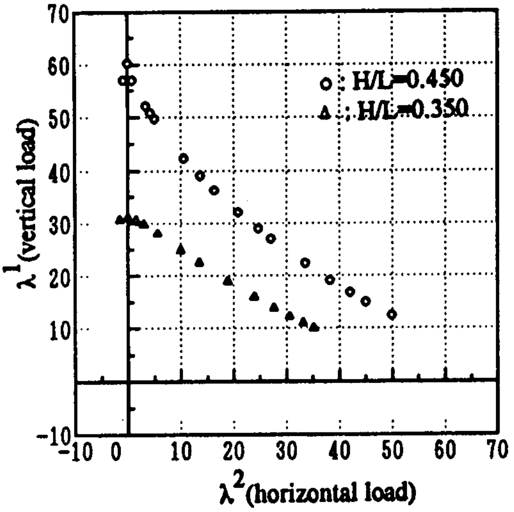

Fig. 7 The two stability boundaries of Bergan's truss

Table 1 Result of Stability analysis carried out on Bergan' s truss(H/L=0.350)

H/L=0.350		
λ^2	λ^1	$m\ (\lambda^2=m\lambda^1)$
-1.534	30.678	-0.050
0.000	31.135	0.000
1.534	30.678	0.050
2.984	29.845	0.100
5.606	28.030	0.200
9.961	24.901	0.400
13.471	22.452	0.600
18.910	18.910	1.000
23.873	15.915	1.500
27.625	13.812	2.000
30.535	12.214	2.500
33.121	11.040	3.000
35.153	10.044	3.500

has been detected when the constraint $P^2=0$ is imposed. The occurrence of such bifurcation point will appear in the form of a cusp on the stability boundary. For both cases, the

Table 2 Result of Stability analysis carried out on Bergan' s truss(H/L=0.450)

H/L=0.450		
λ^2	λ^1	$m\ (\lambda^2=m\lambda^1)$
-1.534	30.678	-0.050
0.000	31.135	0.000
1.534	30.678	0.050
2.984	29.845	0.100
5.606	28.030	0.200
9.961	24.901	0.400
13.471	22.452	0.600
18.910	18.910	1.000
23.873	15.915	1.500
27.625	13.812	2.000
30.535	12.214	2.500
33.121	11.040	3.000
35.153	10.044	3.500

first point on the stability boundary has been computed by setting $P^2=0$.

4. 2 Circular arch

This was taken from the paper by Huseyin[1]. The model(Fig. 8) has 2 DOF which correspond to the two generalized displacement Q_1 and Q_2 as appearing in the following expression for the radial displacement W :

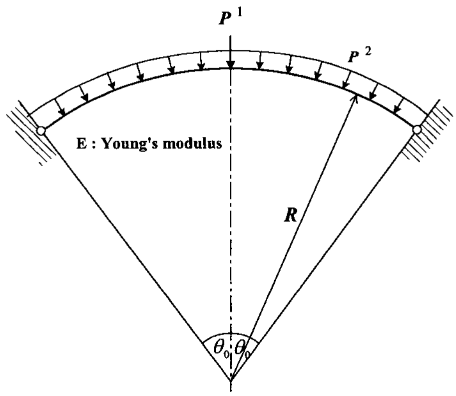

Fig. 8 Circular arch

$$W/R = Q_1 \cos[\pi\theta/Q2\theta_0] + Q_2 \sin[(\pi\theta)/\theta_0] \quad (8)$$

where R indicates the radius of circular arch. This model is subjected to P^1 and P^2 which correspond to concentrated load at the apex of arch and uniform distributed pressure, respectively. P^1 and P^2 are related to the loading parameter λ^1 and λ^2 as follows : $P^1 = \lambda P^1_0$ and $P^2 = \lambda^2 P^2_0$. Both reference loads P^1_0 and P^2_0 are again assumed to be unity. The width of the arch is taken as unity so that cross sectional area A will be equal to the thickness h. Stability analysis has been carried out using $\theta_0 = 30°$, $R/h = 0.025$, $h = 0.025$, $c = 0.001$, $\Delta\lambda^1 =$ -0.001 and -0.0001. Fig. 9 shows the stability boundary. For this model, all critical points lying on the stability boundary are special points. The computed values of λ^1 and λ^2 for points on the stability boundary are listed in Table 3. The first critical point $P^{(1)}_c$ has been obtained by imposing the constraintrm $P^2 = 0$.

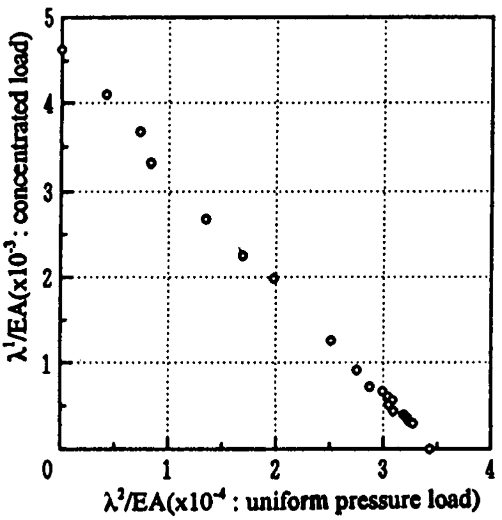

Fig. 9 Stability boundaries of circular arch

Table 3 Result of Stability analysis carried out on circular arch

H/L=0.450		
$\lambda^1/EA(\times 10^{-6})$	$m(\lambda^2 = m\lambda^1)$	$\lambda^2/EA(\times 10^{-4})$
46.622	0.000	0.000
41.010	0.410	0.010
36.660	0.733	0.020
33.130	0.828	0.025
26.830	1.342	0.050
22.510	1.688	0.075
19.810	1.981	0.100
12.580	2.516	0.200
9.180	2.754	0.300
7.180	2.872	0.400
6.660	2.997	0.450
6.090	3.045	0.500
5.610	3.085	0.550
5.090	3.054	0.600
4.420	3.094	0.700
3.990	3.192	0.800
3.580	3.222	0.900
3.240	3.240	1.000
2.980	3.278	1.100
0.000	4.430	∞

5. Remarks and Conclusion

We note that during the numerical analysis, the point $P^{(k+1)}_F$ could be obtained without difficulty starting from the point $P^{(k)}_c$ (Fig. 3). One problem in tracing direction was however faced during the execution of continuation process in order to obtain the next target point $P^{(k+1)}_c$ in the second numerical example. In this study, the tracing direction was specified in such a way that tracing will proceed in the direction of increasing loading parameter. For the case where critical points lying on the stability boundary are general points, such specification has worked well. On the other hand, such specification worked well in the case of stability boundary with special points only if the loading parameter at $P_F^{(k+1)}$ is less than the loading parameter at $P_c^{(k+1)}$ as can be seen from Fig. 10. Under the second situation as depicted in Fig. 10, tracing direction must be altered in order that the target point $P_c^{(k+1)}$ could be obtained. In this study, such alteration in tracing direction was carried out whenever it was monitored that the determinant of the tangent stiffness matrix tends to increase rather than decrease. This means that a more general criteria for the determination of tracing direction might be necessary.

A strategy for stability analysis by combining the use of generalized inverse and combination method has been proposed. From the numerical results, it can be concluded that the strategy is worthy of further development. Testing of the applicability of the proposed strategy for two-parameter and more general multi-parameter problem with larger DOF may be the subject of future research.

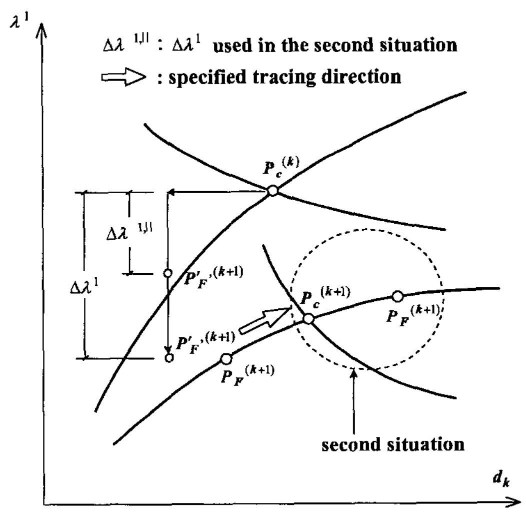

Fig. 10 Illustration of a situation where target critical point will not be obtained without alteration in tracing direction

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Reference

1. Huseyin K., "The multi-parameter perturbation technique for the analysis of nonlinear system", International Journal of Nonlinear Mechanics, Vol.8, 1973, pp.31～43
2. 鋼構造骨組の臨界擧動と限界狀態, 平成5年度 科學硏究費補助金(統合硏究(B))硏究成果報告書, 1994
3. 野中光彦, 三宅修平, 登坂宣好, "積分方程式による複合荷重を受ける偏平アーチの非線型解析",構造工學における數値解析法シンポジウム論文集제14卷(平成2年7月), 1990, pp.97～102
4. 半谷裕彦, 川口健一, "形態解析—一般逆行列とその應用", 培風館, 東京, 1991, p.198
5. Holzer S. M., Plaut R. H., Soners A.E. and White W. S. "Stability of lattice structures under combined loading", Journal of Engineering Mechanics Division, Proceedings of ASCE, Vol. 106, EM2, 1980, April, pp.289～305
6. Rheinboldt W. C. "Computation of critical boundaries on equilibrium manifolds", SIAM Journal of Numerical Analysis, Vol.19, No. 13, 1982, June, pp.653～669
7. Waszczyszyn Z. and Cichon C., "Nonlinear stability analysis of structures under multiple parameter loads", Engineering Computers, Vol.5, 1988, pp.10～14

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