Journal of the Optical Society of Korea, Vol. 3, No. 1, March 1999, pp. 1-9

The Phase-sensitivity of a Mach-Zehnder Interferometer for Coherent Light

Jongtae Shin, Yang Ha, Heonoh Kim, Goodong Park, and Taesoo Kim*

Department of Physics, University of Ulsan, Ulsan 680-749, KOREA

Dae-Yoon Park

Department of Physics, Inha University, Inchon 420-752, KOREA

(Received January 4, 1999)

We have studied the sensitivity of four different phase shift measurement schemes with a Mach-Zehnder interferometer. The input light is considered to be in a coherent state and the detectors are assumed to be ideal with the quantum efficiency of unity. It is shown by direct calculation of the operators corresponding to the measurement schemes that the uncertainty of the phase-shift measurement is limited to the classical one $\frac{1}{\sqrt{m}}$ (m is the average number of the photons in the input state) regardless of the phase-shift measurement schemes.

I. INTRODUCTION

Precise measurement of the phase shift (or phase difference) has been an important issue for both theoretical and experimental optics. One of the main practical reasons is its application to gravitational wave detection. In a conventional interferometry experiment using a Michelson interferometer, coherent light from a laser is fed into the input port. The beam splitter splits light beam into two beams which propagate along different paths. A phase shift, θ, is induced between the two paths when the two beams are combined and interfere with each other at the same beam splitter. A phase shift is determined by measuring the photon number at the output. In this case one has a phase sensitivity of the classical limit, $\frac{1}{\sqrt{N}}$, where N is the total number of photons during the measurement time [1].

Much effort has been focused on achieving the phase-sensitivity of the Heisenberg limit, 1/N. It has been believed that the fluctuation in the input light is the main cause of the classical limit of the sensitivity in the interferometer. An input light field prepared in special quantum states without (or with much less) fluctuations was studied as a potential source of the quantum limit in the literature [2-7]. Some nonclassical light such as a squeezed state, Fock-state light has been proposed to beat the limit [2,5]. A recent study showed that the sensitivity of a Mach-Zehnder interferometer depends on measurement scheme of the phase shift even for the Fock-state light input [8,9]. According to the study, the measurement of one output or half the difference of two outputs does not give any information about the phase shift, while the coincidence detection of two outputs results in the sensitivity at the Heisenberg limit [10].

In the following we report on the sensitivity of various measurement schemes for the coherent light inputs of a Mach-Zehnder interferometer. Suppose that there is an operator $\hat{M}$ representing a measurement scheme which can be realized by a certain physical apparatus. All measurements are performed on one or both of the two outputs of the Mach-Zehnder interferometer. We are only interested in a few measurement schemes which are most common or easily available. The detection schemes to be considered are the measurement of one output photon number, M_1 which is the detection for the 2nd-order interference; the measurement of the half the difference of two output photon numbers, M_2 ; the measurement of the square of half the difference of two photon numbers, M_3; and the measurement of the coincidence detection of the two output photons which represents the 4th-order interference, M_4.

The purpose of this study is to theoretically investigate the sensitivities of a Mach-Zehnder interferometer for the different measurement schemes with the ideal detectors when coherent light is assumed as an input. The basic theory of SU(2) Lie group for a Mach-Zehnder interferometer is presented in Section II. In Section III we calculate the mean square fluctuation of the phase-shift θ of each detection scheme, $(\Delta\theta)^2$, for

*electronic mail : tskim @ uou.ulsan.ac.kr.

the coherent light $|\alpha_1, \alpha_2>$. Discussion and conclusion are presented in Section IV.

II. THE PHASE-SENSITIVITY OF SU(2) INTERFEROMETERS

We consider an experimental setup shown in Fig.1, in which two lossless 50/50 beam splitters, BS1 and BS2, are used to form a Mach-Zehnder interferometer. Photon annihilation operators $\hat{a}_3$ and $\hat{a}_4$ for the modes after BS1 and BS2 can be expressed in terms of the input mode operators $\hat{a}_1$ and $\hat{a}_2$, which satisfy the commutation relations $[a_i, a_j^+] = \delta_{ij}$, for $i,j = 1,2$, as

$$\begin{aligned} \hat{a}_3 &= \frac{i}{\sqrt{2}}\hat{a}_1 + \frac{1}{\sqrt{2}}\hat{a}_2, \\ \hat{a}_4 &= \frac{1}{\sqrt{2}}\hat{a}_1 + \frac{i}{\sqrt{2}}\hat{a}_2. \end{aligned} \tag{1}$$

If we assign the overall phases θ_3 and θ_4 for two paths, two output annihilation operators $\hat{a}_5$ and $\hat{a}_6$ are given by

$$\begin{aligned} \hat{a}_5 &= \frac{i}{\sqrt{2}}\hat{a}_3 e^{i\theta_3} + \frac{1}{\sqrt{2}}\hat{a}_4 e^{i\theta_4} \\ &= \frac{1}{2}\left[(-\hat{a}_1 + i\hat{a}_2)e^{i\theta_3} + (\hat{a}_1 + i\hat{a}_2)e^{i\theta_4}\right], \\ \hat{a}_6 &= \frac{1}{\sqrt{2}}\hat{a}_3 e^{i\theta_3} + \frac{i}{\sqrt{2}}\hat{a}_4 e^{i\theta_4} \\ &= \frac{1}{2}\left[(i\hat{a}_1 + \hat{a}_2)e^{i\theta_3}\right] + \frac{1}{2}\left[(i\hat{a}_1 - \hat{a}_2)e^{i\theta_4}\right]. \end{aligned} \tag{2}$$

For the input states $|\phi_1>$ for mode 1 and $|\phi_2>$ for mode 2, we have expectation values of output photon numbers as

$$\begin{aligned} <\hat{n}_5> &= <\phi_1,\phi_2 \mid \hat{a}_5^+\hat{a}_5 \mid \phi_1,\phi_2>, \\ <\hat{n}_6> &= <\phi_1,\phi_2 \mid \hat{a}_6^+\hat{a}_6 \mid \phi_1,\phi_2> \end{aligned} \tag{3}$$

measured with detectors D_1 and D_2, respectively. When we define the phase shift, θ, between two paths as $\theta_4 - \theta_3$, we get

$$\begin{aligned} \hat{J}_{z,out} &= \frac{1}{2}(\hat{n}_6 - \hat{n}_5) \\ &= -(\sin\theta)\hat{J}_x + (\cos\theta)\hat{J}_z, \end{aligned} \tag{4}$$

where J_x, J_z are defined as $\frac{1}{2}(a_1^+a_2 + a_1a_2^+)$ and $\frac{1}{2}(a_1^+a_1 - a_2^+a_2)$, respectively.

$$\begin{aligned} <\hat{J}_{z,out}> = &-\sin\theta <\phi_1,\phi_2 \mid \hat{J}_x \mid \phi_1,\phi_2> \\ &+\cos\theta <\phi_1,\phi_2 \mid \hat{J}_z \mid \phi_1,\phi_2>, \end{aligned} \tag{5}$$

in the Heisenberg picture.

Let a Hermitian operator $\hat{M}$ represent an operator corresponding to a specific detection scheme. The expectation value and variance of the operator $\hat{M}$ is given by

$$\begin{aligned} <\hat{M}> &= <\phi_1,\phi_2|\hat{M}_{out}|\phi_1,\phi_2>, \\ (\Delta M)^2 &= <\phi_1,\phi_2 \mid \hat{M}_{out}^2 \mid \phi_1,\phi_2> \\ &\quad - <\phi_1,\phi_2 \mid \hat{M}_{out} \mid \phi_1,\phi_2>^2. \end{aligned} \tag{6}$$

Then the mean-square noise of θ is given by

$$(\Delta\theta)_M^2 = \frac{(\Delta M)^2}{\left[\frac{\partial<\hat{M}>}{\partial\theta}\right]^2}. \tag{7}$$

The detection $\hat{M}$ determines the relative phase shift θ with a phase sensitivity of $\Delta\theta$, which is the root-mean-square fluctuation of θ.

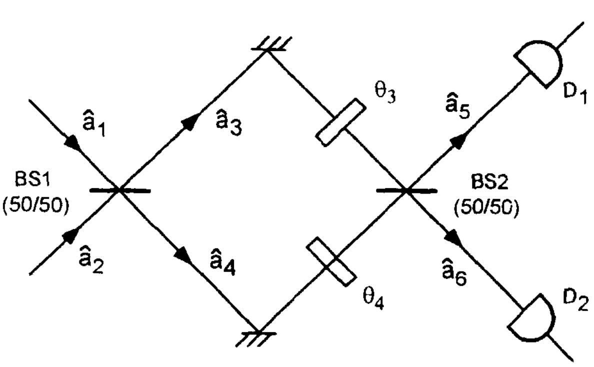

FIG. 1. The schematic diagram of a Mach-Zehnder interferometer.

III. MEASUREMENTS WITH COHERENT LIGHT INPUTS

A. Measurement of the photon number of one output, M_1.

In this section we assume the inputs of the Mach-Zehnder interferometer, $|\phi_1,\phi_2>$, are in pure coherent states $|\alpha_1,\alpha_2>$. In case of the measurement of the photon number of one output [Fig.2(a)], the number operator $\hat{n}_5$ is given by Eq.(2) as

$$\begin{aligned} \hat{n}_5 = \hat{a}_5^+\hat{a}_5 = &\frac{1}{2}[\hat{n}_1(1-\cos\theta) + \hat{n}_2(1+\cos\theta) \\ &+ (\hat{a}_1^+\hat{a}_2 + \hat{a}_2^+\hat{a}_1)\sin\theta], \end{aligned} \tag{8}$$

where the photon number operators $\hat{n}_1$ and $\hat{n}_2$ represent $\hat{a}_1^+\hat{a}_1$ and $\hat{a}_2^+\hat{a}_2$, respectively.

From the relationship of the operators $\hat{a}$, $\hat{a}^+$ and the coherent state $|\alpha>$

$$\begin{aligned} \hat{a}\mid\alpha> &= \alpha\mid\alpha>, \\ <\alpha\mid\hat{a}^+ &= \alpha^* <\alpha\mid, \\ <\alpha\mid\alpha> &= 1, \end{aligned} \tag{9}$$

the expectation value of $\hat{n}_5$, can be obtained as

$$< \alpha_1, \alpha_2 \mid \hat{n}_5 \mid \alpha_1, \alpha_2 >= \frac{1}{2}\left[|\alpha_1|^2(1-\cos\theta) + |\alpha_2|^2(1+\cos\theta) + (\alpha_1^*\alpha_2 + \alpha_2^*\alpha_1)\sin\theta\right]. \tag{10}$$

The expectation value of the $\hat{n}_5^2$ is given by

$$\begin{aligned} < \alpha_1,\ \alpha_2 \mid \hat{n}_5\hat{n}_5 \mid \alpha_1,\ \alpha_2 > \ = & \frac{1}{4}\{[(1-\cos\theta)^2(|\alpha_1|^4 + |\alpha_1|^2) + 2(1-\cos\theta^2)|\alpha_1|^2|\alpha_2|^2 \\ & + (1+\cos\theta)^2(|\alpha_2|^4 + |\alpha_2|^2)\ ] \\ & + \sin\theta(1-\cos\theta)\left[\alpha_1^*\alpha_2 + 2|\alpha_1|^2(\alpha_1^*\alpha_2 + \alpha_1\alpha_2^*) + \alpha_1\alpha_2^*\right] \\ & + \sin\theta(1+\cos\theta)\left[2|\alpha_2|^2(\alpha_1\alpha_2^*) + \alpha_1\alpha_2^* + \alpha_1^*\alpha_2\right] \\ & + \sin^2\theta\left[2|\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2 + |\alpha_2|^2 + \alpha_1^2\alpha_2^{*2} + \alpha_1^{*2}\alpha_2^2\right]\}. \end{aligned} \tag{11}$$

Then the variance of operator $\hat{n}_5$ becomes

$$(\Delta n_5)^2 = \frac{1}{2}\left[|\alpha_1|^2 + |\alpha_2|^2 + \cos\theta(|\alpha_2|^2 - |\alpha_1|^2) + \sin\theta(\alpha_1^*\alpha_2 + \alpha_1\alpha_2^*)\right] \tag{12}$$

and the square of the phase shift uncertainty $(\Delta\theta)^2$ with the measurement of $\hat{n}_5$ becomes

$$(\Delta\theta)^2_{M_1} = \frac{\frac{1}{2}\left[|\alpha_1|^2 + |\alpha_2|^2 + \cos\theta(|\alpha_2|^2 - |\alpha_1|^2) + \sin\theta(\alpha_1^*\alpha_2 + \alpha_1\alpha_2^*)\right]}{\frac{1}{4}\left[\sin^2\theta(|\alpha_1|^2 - |\alpha_2|^2)^2 + \cos^2\theta(\alpha_1^*\alpha_2 + \alpha_2^*\alpha_1)^2 + 2\sin\theta\cos\theta(|\alpha_1|^2 - |\alpha_2|^2)(\alpha_1^*\alpha_2 + \alpha_2^*\alpha_1)\right]}, \tag{13}$$

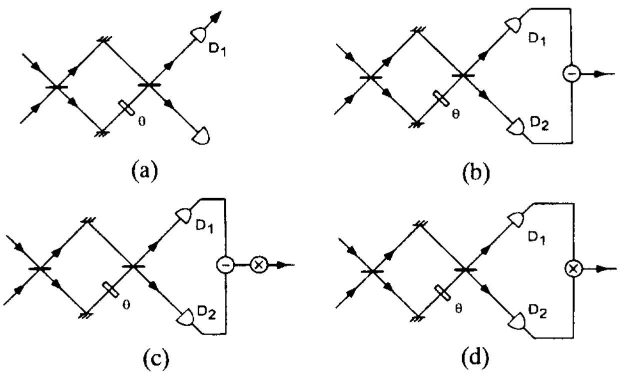

FIG. 2. The schematic diagram of four different measurement schemes. (a) M_1 scheme-measurement of one output, (b) M_2 scheme-measurement of half the difference between two outputs, (c) M_3 scheme-measurement the square of half the difference between two outputs, (d) M_4 scheme-the coincidence measurement of two outputs.

using Eq.(12). If one of two inputs is in the vacuum state, i.e. $\alpha_2 = 0$ and $\alpha_1 = \alpha(|\alpha|^2 >> 1)$, the expectation value and the square of the phase-shift uncertainty $(\Delta\theta)^2$ are given by

$$< \alpha, 0|\hat{n}_5|\alpha, 0 >= \frac{1}{2}|\alpha|^2(1-\cos\theta) \tag{14}$$

$$(\Delta\theta)^2_{M_1} = \frac{2}{|\alpha|^2(1+\cos\theta)}. \tag{15}$$

Then the phase-sensitivity of the measurement with $\hat{n}_5$ is

$$\Delta\theta_{M_1} = \frac{1}{|\alpha|} = \frac{1}{\sqrt{m}} \tag{16}$$

for $\theta = 0$, where m is the average photon number of the coherent input. As θ approaches π, $\Delta\theta$ goes to infinity. It means that we cannot obtain any information about the relative phase-shift if it is 180°.

In case of two intense, identical coherent inputs $\alpha_1 = \alpha_2 = \alpha(|\alpha|^2 >> 1)$, the expectation value and variance are given by

$$< \alpha, \alpha|\hat{n}_5|\alpha, \alpha >= |\alpha|^2(1+\sin\theta) \tag{17}$$

$$(\Delta\theta)^2_{M_1} = \frac{1}{|\alpha|^2(1-\sin\theta)}. \tag{18}$$

The uncertainty of θ becomes

$$\Delta\theta_{M_1} = \frac{1}{|\alpha|} = \frac{1}{\sqrt{m}} \tag{19}$$

for $\theta = 0$. It increases to infinity at $\theta = \frac{\pi}{2}$. Similar to the case of one coherent light input, $\Delta\theta$ depends on the phase shift θ of the interferometer and its minimum value is given by the average photon number of the inputs.

B. Measurement of $\hat{J}_z$, M_2

In this measurement scheme [Fig.2(b)], we measure the phase shift θ by the detection of $\hat{J}_z$, which is half the difference between the photon numbers of two outputs. $< \hat{J}_{z,out} >$ can be easily calculated by Eq.(5) for the two mode coherent inputs, $|\alpha_1, \alpha_2 >$, as

$$< \hat{J}_{z,out} >= -\frac{1}{2}(\alpha_1^*\alpha_2 + \alpha_2^*\alpha_1)\sin\theta + \frac{1}{2}(|\alpha_1|^2 - |\alpha_2|^2)\cos\theta, \quad (20)$$

while

$$< \hat{J}^2_{z,out} > = \frac{1}{4}(2|\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2 + |\alpha_2|^2 + \alpha_1^{*2}\alpha_2^2 + \alpha_2^{*2}\alpha_1^2)\sin^2\theta - \frac{1}{2}(\alpha_1^*\alpha_2 + \alpha_1\alpha_2^*)(|\alpha_1|^2 - |\alpha_2|^2)\sin\theta\cos\theta + \frac{1}{4}(|\alpha_1|^4 + |\alpha_2|^4 + |\alpha_1|^2 + |\alpha_2|^2 - 2|\alpha_1|^2|\alpha_2|^2)\cos^2\theta. \quad (21)$$

Then the variance of $\hat{J}_z$ becomes

$$(\Delta J_z)^2 = \frac{1}{4}(|\alpha_1|^2 + |\alpha_2|^2). \quad (22)$$

Therefore one easily find the phase sensitivity as

$$(\Delta\theta)^2_{M_2} = (|\alpha_1|^2 + |\alpha_2|^2) \times[(\alpha_1^*\alpha_2 + \alpha_2^*\alpha_1)^2\cos^2\theta + (|\alpha_1|^2 - |\alpha_2|^2)^2\sin^2\theta + 2(|\alpha_1|^2 - |\alpha_2|^2)(\alpha_1^*\alpha_2 + \alpha_2^*\alpha_1)\sin\theta\cos\theta]^{-1}. \quad (23)$$

If one input is in the vacuum state, i.e., $\alpha_1 = \alpha$, $\alpha_2 = 0$ we have

$$(\Delta\theta)^2_{M_2} = \frac{1}{|\alpha|^2\sin^2\theta}. \quad (24)$$

It means that the uncertainty of the phase shift is

$$\Delta\theta_{M_2} = \frac{1}{|\alpha||\sin\theta|}. \quad (25)$$

$\Delta\theta$ has the minimum value $\frac{1}{|\alpha|}$ at $\theta = \frac{\pi}{2}$, but it increases as θ decreases from $\frac{\pi}{2}$ to 0. On the other hand, if we have two identical coherent inputs ($\alpha_1 = \alpha_2 = \alpha$), the square of the phase shift uncertainty is given by

$$(\Delta\theta)^2_{M_2} = \frac{1}{2|\alpha|^2\cos^2\theta}. \quad (26)$$

The minimum phase-shift uncertainty is

$$\Delta\theta_{M_2} = \frac{1}{\sqrt{2}}\frac{1}{|\alpha|} \quad (27)$$

at $\theta = 0$. As before, $\Delta\theta$ also depends on θ but it differs by the factor of $1/\sqrt{2}$ from the M_1 case for the same input.

For both kinds of input states, this measurement scheme can have phase sensitivity corresponding to the classical limit $\frac{1}{\sqrt{m}}$.

C. Measurement of $\hat{J}_z^2$, M_3

In this scheme [Fig.2(c)], the relative phase shift θ is measured by the square of $\hat{J}_z$. $< \hat{J}^2_{z,out} >$ is already given by Eq.(21) as

$$< \hat{J}^2_{z,out} >= A_1\sin^2\theta + A_2\sin\theta\cos\theta + A_3\cos^2\theta, \quad (28)$$

where coefficient A's are defined as follows.

$$A_1 \equiv < \alpha_1, \alpha_2|\hat{J}_z^2|\alpha_1, \alpha_2 >,$$
$$A_2 \equiv < \alpha_1, \alpha_2|(\hat{J}_z\hat{J}_x + \hat{J}_x\hat{J}_z)|\alpha_1, \alpha_2 >,$$
$$A_3 \equiv < \alpha_1, \alpha_2|\hat{J}_x^2|\alpha_1, \alpha_2 > . \quad (29)$$

The expectation value of $\hat{J}^4_{z,out}$ for two coherent inputs $|\alpha_1, \alpha_2 >$ is given by

$$< \hat{J}^4_{z,out} >= B_1\sin^4\theta + B_2\cos^4\theta + B_3\sin^2\theta\cos^2\theta + B_4\sin^3\theta\cos\theta + B_5\sin\theta\cos^3\theta, \quad (30)$$

where coefficient B's are defined as follows.

$$B_1 \equiv < \alpha_1, \alpha_2|\hat{J}_x^4|\alpha_1, \alpha_2 >,$$
$$B_2 \equiv < \alpha_1, \alpha_2|\hat{J}_z^4|\alpha_1, \alpha_2 >,$$
$$B_3 \equiv < \alpha_1, \alpha_2|\left[(\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x)^2 + \hat{J}_x^2\hat{J}_z^2 + \hat{J}_z^2\hat{J}_x^2\right]|\alpha_1, \alpha_2 >,$$

$$B_4 \equiv -<\alpha_1,\alpha_2|\left[\hat{J}_x^2(\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x)+(\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x)\hat{J}_x^2\right]|\alpha_1,\alpha_2>,$$
$$B_5 \equiv -<\alpha_1,\alpha_2|\left[\hat{J}_z^2(\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x)+(\hat{J}_x\hat{J}_z+\hat{J}_z\hat{J}_x)\hat{J}_z^2\right]|\alpha_1,\alpha_2>. \quad (31)$$

The variance of $\hat{J}^2_{z,out}$ is given by

$$\begin{aligned}(\Delta J_z^2)^2 &= <\alpha_1,\alpha_2|\hat{J}^4_{z,out}|\alpha_1,\alpha_2> - <\alpha_1,\alpha_2|\hat{J}^2_{z,out}|\alpha_1,\alpha_2>^2\\ &= (B_1-A_1^2)\sin^4\theta+(B_2-A_3^2)\cos^4\theta+(B_3-2A_1A_3-A_2^2)\sin^2\theta\cos^2\theta\\ &\quad +(B_4-2A_1A_2)\sin^3\theta\cos\theta+(B_5-2A_2A_3)\sin\theta\cos^3\theta,\end{aligned} \quad (32)$$

this lead us to the expression of $(\Delta\theta)^2_{M_3}$ as

$$\begin{aligned}(\Delta\theta)^2_{M_3} &= \frac{(\Delta J_z^2)^2}{\left[\frac{\partial<\hat{J}^2_{z,out}>}{\partial\theta}\right]^2}\\ &= [(B_1-A_1^2)\sin^4\theta+(B_2-A_3^2)\cos^4\theta+(B_3-2A_1A_3-A_2^2)\sin^2\theta\cos^2\theta\\ &\quad +(B_4-2A_1A_2)\sin^4\theta\cos\theta+(B_5-2A_2A_3)\sin\theta\cos^3\theta]\\ &\quad [4(A_1-A_3)^2\sin^2\theta\cos^2\theta+4A_2^2\cos^4\theta+A_2^2-4A_2^2\cos^2\theta\\ &\quad -4A_2(A_1-A_3)\sin\theta\cos\theta-8A_2(A_1-A_3)\sin\theta\cos^3\theta]^{-1}.\end{aligned} \quad (33)$$

Let us consider two special input cases. When we have only one coherent input ($\alpha_1=\alpha,\ \alpha_2=0,\ |\alpha|^2>>1$), the coefficient B_2 is zero. Then the denominator in Eq.(33) is

$$\left[\frac{\partial<\hat{J}^2_{z,out}>}{\partial\theta}\right]^2 = \frac{1}{4}|\alpha|^8\sin^2\theta\cos^2\theta. \quad (34)$$

From these the phase sensitivity can be obtained as

$$\begin{aligned}(\Delta\theta)^2_{M_3} &= \frac{\frac{1}{4}|\alpha|^6\cos^4\theta+\frac{1}{4}|\alpha|^6\sin^2\theta\cos^2\theta}{\frac{1}{4}|\alpha|^8\sin^2\theta\cos^2\theta}\\ &= \frac{1}{|\alpha|^2}\frac{1}{\sin^2\theta}.\end{aligned} \quad (35)$$

For the case of two identical coherent inputs ($\alpha_1=\alpha_2=\alpha,\ |\alpha|^2>>1$), A_2 is also zero and the denominator in Eq.(33) becomes

$$\left[\frac{\partial<\hat{J}^2_{z,out}>}{\partial\theta}\right]^2 = 4|\alpha|^8\sin^2\theta\cos^2\theta. \quad (36)$$

Therefore $\Delta\theta$ is obtained as

$$\begin{aligned}(\Delta\theta)^2_{M_3} &= \frac{2|\alpha|^6\sin^4\theta+2|\alpha|^6\sin^2\theta\cos^2\theta}{4|\alpha|^8\sin^2\theta\cos^2\theta}\\ &= \frac{1}{2|\alpha|^2}\frac{1}{\cos^2\theta}.\end{aligned} \quad (37)$$

For both cases the phase sensitivity of the phase-shift is bounded by the classical limit $\frac{1}{\sqrt{m}}$.

D. The coincidence detection, M_4

The measurement of the phase-shift θ is based on the coincidence photon detection at the two outputs of the interferometer in this scheme [Fig.2(d)]. The operator $\hat{n}$, which is the average of two input photon numbers, is defined as

$$\hat{n} = \frac{1}{2}(\hat{a}_1^+\hat{a}_1+\hat{a}_2^+\hat{a}_2) \quad (38)$$

$\hat{n}_1$ and $\hat{n}_2$ can be expressed in terms of $\hat{n}$ and $\hat{J}_z$ as

$$\begin{aligned}\hat{n_1} &= \hat{a_1}^+\hat{a_1} = \hat{n}+\hat{J}_z,\\ \hat{n_2} &= \hat{a_2}^+\hat{a_2} = \hat{n}-\hat{J}_z.\end{aligned} \quad (39)$$

The coincidence detection $\hat{N}_c$ of the photons at two output ports is given by

$$\hat{N}_c = \hat{n}_5\hat{n}_6 = \hat{n}^2-\hat{J}^2_{z,out}, \quad (40)$$

where the commutation relation

$$[\hat{n},\ \hat{J}_z] = 0 \quad (41)$$

is used. Then the expectation of the coincidence measurement for the coherent inputs, $|\alpha_1,\alpha_2>$, is obtained as

$$\begin{aligned}<\alpha_1,\alpha_2|\hat{N}_c|\alpha_1,\alpha_2> &= C_1-C_2\sin^2\theta-C_3\cos^2\theta\\ &\quad +C_4\sin\theta\cos\theta,\end{aligned} \quad (42)$$

where the coefficient C's are defined as

$$C_1 \equiv <\alpha_1,\alpha_2|\hat{n}^2|\alpha_1,\alpha_2>,$$
$$C_2 \equiv <\alpha_1,\alpha_2|\hat{J}_x^2|\alpha_1,\alpha_2>,$$
$$C_3 \equiv <\alpha_1,\alpha_2|\hat{J}_z^2|\alpha_1,\alpha_2>,$$

$$C_4 \equiv < \alpha_1, \alpha_2 | (\hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x) | \alpha_1, \alpha_2 > . \quad (43)$$

The expectation value of $\hat{N}_c^2$ is given by

$$< \alpha_1, \alpha_2 | \hat{N}_c^2 | \alpha_1, \alpha_2 > = D_1 - 2D_2 \sin^2\theta - 2D_5 \cos^2\theta + 2(D_3 + D_4)\sin\theta\cos\theta + B_1 \sin^4\theta + B_2\cos^4\theta + B_3 \sin^2\theta\cos^2\theta + B_4 \sin^3\theta\cos\theta + B_5 \sin\theta\cos^3\theta, \quad (44)$$

where the coefficient D's are defined as

$$D_1 \equiv < \alpha_1, \alpha_2 | \hat{n}^4 | \alpha_1, \alpha_2 >,$$
$$D_2 \equiv < \alpha_1, \alpha_2 | \hat{n}^2 \hat{J}_x^2 | \alpha_1, \alpha_2 >,$$
$$D_3 \equiv < \alpha_1, \alpha_2 | \hat{n}^2 \hat{J}_x \hat{J}_z | \alpha_1, \alpha_2 >,$$
$$D_4 \equiv < \alpha_1, \alpha_2 | \hat{n}^2 \hat{J}_z \hat{J}_x | \alpha_1, \alpha_2 >,$$
$$D_5 \equiv < \alpha_1, \alpha_2 | \hat{n}^2 \hat{J}_z^2 | \alpha_1, \alpha_2 > . \quad (45)$$

Then the variance of the coincidence measurement is given by

$$(\Delta N_c)^2 = (D_1 - C_1^2) + (-2D_2 + 2C_1C_2)\sin^2\theta + (2C_1C_3 - 2D_5)\cos^2\theta + 2\left[(D_3 + D_4) - C_1C_4\right]\sin\theta\cos\theta + (B_1 - C_2^2)\sin^4\theta + (B_2 - C_3^2)\cos^4\theta + (B_3 - 2C_2C_3 - C_4^2)\sin^2\theta\cos^2\theta + (B_4 + 2C_2C_4)\sin^3\theta\cos\theta + (B_5 + 2C_3C_4)\sin\theta\cos^3\theta \quad (46)$$

and the square of the phase-shift uncertainty, $\Delta\theta$, becomes

$$(\Delta\theta)^2_{M_4} = [(D_1 - C_1^2) + (-2D_2 + C_1C_2)\sin^2\theta + (2C_1C_3 - 2D_5)\cos^2\theta + 2[(D_3 + D_4) - C_1C_4]\sin\theta\cos\theta + (B_1 - C_2^2)\sin^4\theta + (B_2 - C_3^2)\cos^4\theta + (B_3 - 2C_2C_3 - C_4^2)\sin^2\theta\cos^2\theta] [4(C_3 - C_2)^2\sin^2\theta\cos^2\theta + C_4^2\cos^4\theta + C_4^2\sin^4\theta + 4C_4(C_3 - C_2)\sin\theta\cos^3\theta - 2C_4^2\cos^2\theta\sin\theta - 4C_4(C_3 - C_2)\sin^3\theta\cos\theta]^{-1}. \quad (47)$$

We now consider the two special input cases. If we have only one intense coherent input ($\alpha_1 = \alpha$, $\alpha_2 = 0$, $|\alpha|^2 >> 1$) for the coincidence measurement. The denominator in Eq.(49) is reduced to

$$\left[\frac{\partial < \hat{N}_c >}{\partial\theta}\right]^2 = \frac{1}{4}|\alpha|^8 \sin^2\theta\cos^2\theta. \quad (48)$$

Those generate phase sensitivity as

$$(\Delta\theta)^2_{M_4} = \frac{1}{|\alpha|^2\cos^2\theta}. \quad (49)$$

For the case of two intense, identical coherent light inputs ($\alpha_1 = \alpha_2 = \alpha$, $|\alpha|^2 >> 1$), the variance of the coincidence measurement and the square of the derivative of the coincidence measurement with respect to the phase-shift θ can be approximated as

$$(\Delta N_c)^2 = 2|\alpha|^6 - 4|\alpha|^6\sin^2\theta + 2|\alpha|^6\sin^4\theta + 2|\alpha|^6\sin^2\theta\cos^2\theta \quad (50)$$

and

$$\left[\frac{\partial < \hat{N}_c >}{\partial\theta}\right]^2 = 4|\alpha|^8\sin^2\theta\cos^2\theta, \quad (51)$$

respectively.

The mean-square fluctuation of θ then becomes

$$(\Delta\theta)^2_{M_4} = \frac{1}{2|\alpha|^2\sin^2\theta}. \quad (52)$$

The phase shift measurement by coincidence detection, therefore, shows θ-dependent resolutions that is also proportional to $\frac{1}{\sqrt{m}}$ corresponding to the classical limit. The case of two mode coherent input has the best phase shift resolution of $\frac{1}{\sqrt{2}|\alpha|}$ at the $\frac{\pi}{2}$ phase shift, while the best resolutions $\frac{1}{|\alpha|}$ is achieved at zero phase shift for the case of one coherent light input.

The results of this section show that the phase sensitivity in a Mach-Zehnder interferometer is limited by the classical limit $\frac{1}{\sqrt{m}}$, irrespective of the measurement schemes for the coherent light inputs. But it depends on θ, the value of the phase shift, in all cases of the measurements.

IV. DISCUSSION AND CONCLUSION

We have examined the sensitivity of four different measurement schemes of a Mach-Zehnder interferometer by direct calculation of operators representing some

TABLE 1. $(\Delta\theta)^2$ in a Mach-Zehnder interferometer for the special coherent state inputs.

Scheme	Coherent state light input $\|\alpha_1, \alpha_2\rangle$	
	$\alpha_1 = \alpha, \alpha_2 = 0$, $\|\alpha\|^2 \gg 1$	$\alpha_1 = \alpha_2 = \alpha$, $\|\alpha\|^2 \gg 1$
M_1	$\frac{2}{\|\alpha\|^2(1+\cos\theta)}$	$\frac{1}{\|\alpha\|^2(1-\sin\theta)}$
M_2	$\frac{1}{\|\alpha\|^2\sin^2\theta}$	$\frac{1}{2\|\alpha\|^2\cos^2\theta}$
M_3	$\frac{1}{\|\alpha\|^2\sin^2\theta}$	$\frac{1}{2\|\alpha\|^2\cos^2\theta}$
M_4	$\frac{1}{\|\alpha\|^2\cos^2\theta}$	$\frac{1}{2\|\alpha\|^2\sin^2\theta}$

measurement schemes for the coherent light as an input light in an ideal situation with the quantum efficiency of 1. The phase sensitivity for some special cases are presented in Table 1. The accuracy of interferometric measurements for conventional coherent light input is bounded by the classical limit for any kind of measurement scheme. When we consider the nonideal efficiency of detectors, the phase sensitivity of the interferometer can be worse than $\frac{1}{\sqrt{m}}$. This calculation shows an agreement with Paris's analysis of the sensitivity for coherent light input [1]. On the other hand,the study of the Mach-Zehnder interferometer for the Fock-state light input shows that the phase sensitivity depends on the measurement schemes, in which the accuracy of the M_1 and M_2 schemes have the classical limit,while those of M_3 and M_4 schemes can reach the Heisenberg limit [10].

The phase sensitivity is also dependent on the choice of the phase shift, θ, at which we set up the interferometer, as shown in Table 1. The dependence of the phase sensitivity on the phase shift is different from those of Fock-state cases, in which the phase sensitivity has the term of $\tan^2\theta$ in M_3 and M_4 schemes. It means that the best accuracy can be achieved in zero phase shift. For coherent light input,on the other hand, the minimum uncertainties are at different phase shifts θ according to the measurement scheme.

The calculation of the phase sensitivity for coherent light input shows that we can measure the signals in these 4 measurement schemes. This is somewhat different from Mandel's experiment [11,12], which had signals measured at one output port, M_1 in our case, but didn't have the coincidence measurement, M_4, with mutually coherent light from a laser. Here we assumed pure coherent light as an input with long coherence time and the resolution of the detection system is very short compared with the coherence time of the input light.

ACKNOWLEDGMENTS

One of us (T. Kim) would like to express gratitude to Professor John L. Hall at JILA. This work was supported by the research fund of the University of Ulsan.

APPENDIX A: THE CALCULATION OF THE COEFFICIENTS A'S, B'S, C'S, D'S.

Using the commutation relation $[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}$ for $i, j = 1, 2$, we can calculate all the coefficients A's, B's, C's, D's for the coherent state input $|\alpha_1, \alpha_2>$.

$$\begin{aligned}
A_1 &\equiv <\alpha_1,\alpha_2|\hat{J}_z^2|\alpha_1,\alpha_2> \\
&= <\alpha_1,\alpha_2|\frac{1}{4}(\hat{a}_1^+\hat{a}_1 - \hat{a}_2^+\hat{a}_2)(\hat{a}_1^+\hat{a}_1 - \hat{a}_2^+\hat{a}_2)|\alpha_1,\alpha_2> \\
&= \frac{1}{4} <\alpha_1,\alpha_2|(\hat{a}_1^+\hat{a}_1\hat{a}_1^+\hat{a}_1 - \hat{a}_1^+\hat{a}_1\hat{a}_2^+\hat{a}_2 - \hat{a}_2^+\hat{a}_2\hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2\hat{a}_2^+\hat{a}_2)|\alpha_1,\alpha_2> \\
&= \frac{1}{4} <\alpha_1,\alpha_2|(\hat{a}_1^+\hat{a}_1^+\hat{a}_1\hat{a}_1 + \hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2^+\hat{a}_2\hat{a}_2 + \hat{a}_2^+\hat{a}_2 - 2\hat{a}_1^+\hat{a}_1\hat{a}_2^+\hat{a}_2)|\alpha_1,\alpha_2> \\
&= \frac{1}{4}(|\alpha_1|^4 + |\alpha_2|^4 - 2|\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2 + |\alpha_2|^2), \qquad \text{(A1)}
\end{aligned}$$

$$A_2 \equiv <\alpha_1,\alpha_2|(\hat{J}_z\hat{J}_x + \hat{J}_x\hat{J}_z)|\alpha_1,\alpha_2> = \frac{1}{2}(|\alpha_1|^2 - |\alpha_2|^2)(\alpha_1^*\alpha_2 + \alpha_1\alpha_2^*), \qquad \text{(A2)}$$

$$A_3 \equiv <\alpha_1,\alpha_2|\hat{J}_x^2|\alpha_1,\alpha_2> = \frac{1}{4}(2|\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2 + |\alpha_2|^2 + \alpha_1^{*2}\alpha_2^2 + \alpha_1^2\alpha_2^{*2}). \qquad \text{(A3)}$$

$$\begin{aligned} B_1 &\equiv < \alpha_1, \alpha_2 | \hat{J}_x^4 | \alpha_1, \alpha_2 > \\ &= \frac{1}{16}(|\alpha_1|^2 + |\alpha_2|^2 + 4\alpha_1^{*2}\alpha_2^2 + 3|\alpha_1|^4 + 14|\alpha_1|^2|\alpha_2|^2 \\ &+ 3|\alpha_2|^4 + 4\alpha_1^2\alpha_2^{*2} + 2|\alpha_2|^2\alpha_1^*\alpha_2 + 6|\alpha_1|^2\alpha_1^{*2}\alpha_2^2 + 4|\alpha_2|^2\alpha_1^{*2}\alpha_2^2 \\ &+ 12|\alpha_1|^2|\alpha_2|^4 + 13|\alpha_1|^4|\alpha_2|^2 + 6|\alpha_2|^2\alpha_1^2\alpha_2^{*2} + 3|\alpha_1|^2\alpha_1^2\alpha_2^{*2} \\ &+ 4|\alpha_1|^2|\alpha_2|^2\alpha_1^*\alpha_2 + \alpha_1^{*4}\alpha_2^4 + 6|\alpha_1|^4|\alpha_2|^4 + 3|\alpha_1|^2|\alpha_2|^2\alpha_1^{*2}\alpha_2^2 \\ &+ |\alpha_1|^4|\alpha_2|^2\alpha_1^*\alpha_2 + 4|\alpha_1|^2|\alpha_2|^2\alpha_1^2\alpha_2^{*2} + \alpha_1^4\alpha_2^{*4}), \end{aligned} \quad \text{(A4)}$$

$$\begin{aligned} B_2 &\equiv < \alpha_1, \alpha_2 | \hat{J}_z^4 | \alpha_1, \alpha_2 > \\ &= \frac{1}{16}(|\alpha_1|^2 + |\alpha_2|^2 - 2|\alpha_1|^2|\alpha_2|^2 + 7|\alpha_1|^4 + 7|\alpha_2|^4 - 6|\alpha_1|^4|\alpha_2|^2 - 6|\alpha_1|^2|\alpha_2|^4 \\ &+ 6|\alpha_1|^6 + 6|\alpha_2|^6 - 3|\alpha_1|^6|\alpha_2|^2 + 6|\alpha_1|^4|\alpha_2|^4 - |\alpha_1|^6|\alpha_2|^2 - 4|\alpha_1|^2|\alpha_2|^6 + |\alpha_1|^8 + |\alpha_2|^8), \end{aligned} \quad \text{(A5)}$$

$$\begin{aligned} B_3 &\equiv < \alpha_1, \alpha_2 | \left[(\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x)^2 + \hat{J}_x^2\hat{J}_z^2 + \hat{J}_z^2\hat{J}_x^2 \right] | \alpha_1, \alpha_2 >, \\ &= \frac{1}{16}(2|\alpha_1|^2 + 2|\alpha_2|^2 + 8\alpha_1^{*2}\alpha_2^2 + 6|\alpha_1|^4 - 2|\alpha_1|^2|\alpha_2|^2 - |\alpha_2|^2\alpha_1^{*2} \\ &+ 6|\alpha_2|^4 + 8\alpha_1^2\alpha_2^{*2} + 2|\alpha_1|^2\alpha_1^{*2}\alpha_2^2 + 2|\alpha_2|^2\alpha_1^{*2}\alpha_2^2 + 3|\alpha_1|^4|\alpha_2|^2 \\ &+ 4|\alpha_1|^6 + |\alpha_1|^4|\alpha_2|^2 - 4|\alpha_1|^2\alpha_1^2\alpha_2^{*2} - 3|\alpha_2|^2\alpha_1^{*2}\alpha_2^2 - 3|\alpha_1|^2|\alpha_2|^4 + |\alpha_2|^6 + 5|\alpha_1|^2\alpha_1^2\alpha_2^{*2}), \end{aligned} \quad \text{(A6)}$$

$$\begin{aligned} B_4 &\equiv - < \alpha_1, \alpha_2 | \left[\hat{J}_x^2(\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x) + (\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x)\hat{J}_x^2 \right] | \alpha_1, \alpha_2 >, \\ &= -\frac{1}{8}(|\alpha_1|^4 + \alpha_1^{*2}\alpha_2^2 - 4|\alpha_1|^2|\alpha_2|^2 - 2|\alpha_2|^2\alpha_1^*\alpha_2 - 5|\alpha_2|^2\alpha_1^{*2}\alpha_2^2 \\ &+ 3|\alpha_1|^2\alpha_1^{*2}\alpha_2^2 + 3|\alpha_1|^4\alpha_1^*\alpha_2 + 3|\alpha_1|^4\alpha_1\alpha_2^* - 7|\alpha_1|^2|\alpha_2|^4 + 4|\alpha_1|^2|\alpha_2|^2\alpha_1^*\alpha_2 \\ &+ 2|\alpha_1|^2|\alpha_2|^2\alpha_1\alpha_2^* - 4|\alpha_2|^4\alpha_1^*\alpha_2 - |\alpha_2|^6 - 5|\alpha_2|^4\alpha_1\alpha_2^* + 3|\alpha_1|^4|\alpha_2|^2 \\ &+ 2|\alpha_1|^2\alpha_1^{*3}\alpha_2^3 - 2|\alpha_1|^2|\alpha_2|^4\alpha_1^*\alpha_2 + 2|\alpha_1|^4|\alpha_2|^2\alpha_1^*\alpha_2 - 2|\alpha_2|^2\alpha_1^{*3}\alpha_2^3 \\ &+ 2|\alpha_1|^4|\alpha_2|^2\alpha_1\alpha_2^* - 2|\alpha_2|^2\alpha_1^3\alpha_2^{*3} + |\alpha_1|^2\alpha_1^3\alpha_2^{*3} - 2|\alpha_1|^2|\alpha_2|^4\alpha_1\alpha_2^* \\ &+ 2|\alpha_1|^2|\alpha_2|^2\alpha_1^{*2}\alpha_2^2 - 2|\alpha_1|^2|\alpha_2|^6 + |\alpha_1|^4|\alpha_2|^4 - 2|\alpha_2|^4\alpha_1^{*2}\alpha_2^2 \\ &+ 2|\alpha_1|^4|\alpha_2|^2\alpha_1^*\alpha_2 - |\alpha_1|^2|\alpha_2|^4\alpha_1\alpha_2^* + 2|\alpha_1|^4|\alpha_2|^2\alpha_1\alpha_2^* + |\alpha_1|^2|\alpha_2|^2\alpha_1^2\alpha_2^{*2}). \end{aligned} \quad \text{(A7)}$$

$$C_1 \equiv < \alpha_1, \alpha_2 | \hat{n}^2 | \alpha_1, \alpha_2 > = \frac{1}{4}(|\alpha_1|^2 + |\alpha_2|^2)(|\alpha_1|^2 + |\alpha_2|^2 + 1), \quad \text{(A8)}$$

$$C_2 \equiv < \alpha_1, \alpha_2 | \hat{J}_x^2 | \alpha_1, \alpha_2 > = \frac{1}{4}(2|\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2 + |\alpha_2|^2 + \alpha_1^{*2}\alpha_2^2 + \alpha_1^2\alpha_2^{*2}), \quad \text{(A9)}$$

$$C_3 \equiv < \alpha_1, \alpha_2 | \hat{J}_z^2 | \alpha_1, \alpha_2 > = \frac{1}{4}(|\alpha_1|^4 + |\alpha_2|^4 - 2|\alpha_1|^2|\alpha_2|^2 - |\alpha_1|^2 + |\alpha_2|^2), \quad \text{(A10)}$$

$$C_4 \equiv < \alpha_1, \alpha_2 | \hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x | \alpha_1, \alpha_2 > = \frac{1}{2}(|\alpha_1|^2 - |\alpha_2|^2)(\alpha_1^*\alpha_2 + \alpha_2^*\alpha_1). \quad \text{(A11)}$$

$$\begin{aligned} D_1 &\equiv < \alpha_1, \alpha_2 | \hat{n}^2 | \alpha_1, \alpha_2 > \\ &= \frac{1}{16}(|\alpha_1|^2 + |\alpha_2|^2 + 7|\alpha_1|^4 + 7|\alpha_2|^4 + 16|\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^6 + 6|\alpha_2|^6 \\ &+ 18|\alpha_1|^2|\alpha_2|^4 + 18|\alpha_1|^4|\alpha_2|^2 + 4|\alpha_1|^4|\alpha_2|^4 + 4|\alpha_1|^6|\alpha_2|^2 + 4|\alpha_1|^2|\alpha_2|^6 + |\alpha_1|^8 + |\alpha_2|^8, \end{aligned} \quad \text{(A12)}$$

$$D_2 = < \alpha_1, \alpha_2 | \hat{n}^2 \hat{J}_x^2 | \alpha_1, \alpha_2 >$$

$$= \frac{1}{16}(|\alpha_1|^2 + |\alpha_2|^2 + 3|\alpha_1|^4 + 18\alpha_1^2\alpha_2^{*2} + 4\alpha_1^{*2}\alpha_2^2 + 3|\alpha_2|^4 + 13|\alpha_1|^4|\alpha_2|^2$$
$$+ |\alpha_1|^2\alpha_1^{*2}\alpha_2^2 + |\alpha_1|^2\alpha_1^2\alpha_2^{*2} + 5|\alpha_2|^2\alpha_1^{*2}\alpha_2^2 + 5|\alpha_2|^4\alpha_1^2 + 4|\alpha_1|^4|\alpha_2|^4$$
$$+ 4|\alpha_1|^2\alpha_1^2\alpha_2^{*2} + 13|\alpha_1|^2|\alpha_2|^4 + 2|\alpha_1|^4|\alpha_2|^2\alpha_1^*\alpha_2 + 2|\alpha_1|^4|\alpha_2|^2\alpha_1\alpha_2^*$$
$$+ |\alpha_1|^6 + 2|\alpha_1|^6|\alpha_2|^2 + 4|\alpha_1|^2\alpha_1^{*2}\alpha_2^2 + |\alpha_1|^4\alpha_1^{*2}\alpha_2^2 + |\alpha_1|^4\alpha_1^2\alpha_2^{*2}$$
$$+ 2|\alpha_1|^2|\alpha_2|^2\alpha_1^{*2}\alpha_2^2 + |\alpha_2|^6 + 2|\alpha_1|^2|\alpha_2|^6 + |\alpha_2|^4\alpha_1^{*2}\alpha_2^2 + |\alpha_2|^4\alpha_1^2\alpha_2^{*2}), \quad \text{(A13)}$$

$$D_3 = < \alpha_1, \alpha_2|\hat{n}^2\hat{J}_x\hat{J}_z|\alpha_1, \alpha_2 >$$
$$= \frac{1}{16}(\alpha_1\alpha_2^* - \alpha_1^*\alpha_2 + 5|\alpha_1^*|^2\alpha_1\alpha_2^* + |\alpha_1|^2\alpha_1^*\alpha_2 - 7|\alpha_2|^2\alpha_1^*\alpha_2$$
$$- |\alpha_1|^2\alpha_1\alpha_2^* + 2|\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2\alpha_1^{*2}\alpha_2^2 + 6|\alpha_1|^4\alpha_1\alpha_2^* - 6|\alpha_2|^4\alpha_1^*\alpha_2$$
$$- 4|\alpha_2|^4\alpha_1\alpha_2^* + 2|\alpha_1|^4|\alpha_2|^2 + 3|\alpha_1|^2\alpha_1^{*2}\alpha_2^2 - 2|\alpha_1|^2|\alpha_2|^2\alpha_1^*\alpha_2$$
$$- |\alpha_1|^2|\alpha_2|^4\alpha_1^*\alpha_2 + |\alpha_1|^4|\alpha_2|^2\alpha_1\alpha_2^* - |\alpha_1|^2|\alpha_2|^4\alpha_1\alpha_2^*$$
$$+ |\alpha_1|^6\alpha_1^*\alpha_2 + |\alpha_1|^6\alpha_1\alpha_2^* - |\alpha_2|^6\alpha_1\alpha_2^* + |\alpha_1|^4|\alpha_2|^2\alpha_1^*\alpha_2), \quad \text{(A14)}$$

$$D_4 = < \alpha_1, \alpha_2|\hat{n}^2\hat{J}_z\hat{J}_x|\alpha_1, \alpha_2 >$$
$$= \frac{1}{16}(\alpha_1^*\alpha_2 - \alpha_1\alpha_2^* + 7|\alpha_1|^2\alpha_1^*\alpha_2 - 7|\alpha_2|^2\alpha_1\alpha_2^* + |\alpha_1|^2\alpha_1\alpha_2^* - |\alpha_2|^2\alpha_1^*\alpha_2$$
$$+ 6|\alpha_1|^4\alpha_1^*\alpha_2 + 2|\alpha_1|^2|\alpha_2|^2\alpha_1^*\alpha_2 + 4|\alpha_1|^4\alpha_1\alpha_2^* - 4|\alpha_2|^4\alpha_1^*\alpha_2$$
$$- 2|\alpha_1|^2|\alpha_2|^2\alpha_1\alpha_2^* - 6|\alpha_2|^4\alpha_1\alpha_2^* + |\alpha_1|^4|\alpha_2|^2\alpha_1^*\alpha_2 - |\alpha_1|^2|\alpha_2|^4\alpha_1^*\alpha_2$$
$$+ |\alpha_1|^4|\alpha_2|^2\alpha_1\alpha_2^* - |\alpha_1|^2|\alpha_2|^4\alpha_1\alpha_2^* + |\alpha_1|^6\alpha_1^*\alpha_2 + |\alpha_1|^6\alpha_1\alpha_2^*$$
$$- |\alpha_2|^6\alpha_1^*\alpha_2 - |\alpha_2|^6\alpha_1\alpha_2^*), \quad \text{(A15)}$$

$$D_5 = < \alpha_1, \alpha_2|\hat{n}^2\hat{J}_z^2|\alpha_1, \alpha_2 >$$
$$= \frac{1}{16}(|\alpha_1|^2 + |\alpha_2|^2 + 7|\alpha_1|^4 + 7|\alpha_2|^4 - 2|\alpha_1|^2|\alpha_2|^2 + 6|\alpha_1|^6 + 6|\alpha_2|^6$$
$$- 2|\alpha_1|^2|\alpha_2|^4 - 2|\alpha_1|^4|\alpha_2|^2 + |\alpha_1|^8 + |\alpha_2|^8 - 2|\alpha_1|^4|\alpha_2|^4). \quad \text{(A16)}$$

REFERENCES

[1] M. G. A. Paris, Phy. Lett. A 225, 23 (1997).
[2] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[3] M. O. Scully, Phys. Rev. Lett. 55, 2802 (1985).
[4] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).
[5] M. Holland and K. Burnett, Phys. Rev. Lett. **71**, 1355 (1993).
[6] B. C. Sanders and G. J. Milburn, Phys. Rev. Lett. **75**, 2944 (1995).
[7] C. Brif and A. Mann, Phys. Rev. A **54**, 4505 (1996).
[8] T. Kim, O. Pfister, J. Noh, M. Holland, and J. Hall, Phys. Rev. A **57**, 4004 (1998).
[9] J. Noh, K. Kim, and T. Kim, J. Kor. Phy. Soc. **31**, 599 (1997).
[10] Taesoo Kim, Jongtae Shin, Yang Ha, Heonoh Kim, Goodong Park, Tae Gon Noh, and Chung Ki Hong, Opt. Comm. **156**, 37 (1998)
[11] Z. Y. Ou, X. Y. Zou, L. J. Wang, and L. Mandel, Phys. Rev. A **42**, 2957 (1990).
[12] L. Mandel, Phys. Rev. A **28**, 929 (1983).