Suboptimal Robust Generalized H_2 Filtering using Linear Matrix Inequalities

Won-Sang Ra, Seung-Hee Jin, Tae-Sung Yoon, and Jin-Bae Park

Abstract : The robust generalized H_2 filtering problem for a class of discrete time uncertain linear systems satisfying the sum quadratic constraints(SQCs) is considered. The objective of this paper is to develop robust stability condition using SQCs and design a robust generalized H_2 filter to take place of the existing robust Kalman filter. The robust generalized H_2 filter is designed based on newly derived robust stability condition. The robust generalized H_2 filter bounds the energy to peak gain from the energy bounded exogenous disturbances to the estimation errors under the given positive scalar γ. Unlike the robust Kalman filter, it does not require any spectral assumptions about the exogenous disturbances. Therefore the robust generalized H_2 filter can be considered as a deterministic formulation of the robust Kalman filter. Moreover, the variance of the estimation error obtained by the proposed filter is lower than that by the existing robust Kalman filter. The robustness of the robust generalized H_2 filter against the uncertainty and the exogenous signal is illustrated by a simple numerical example.

Keywords : robust generalized H_2 filter, sum quadratic constraint, linear matrix inequality, S-procedure, robust Kalman filter

I. Introduction

During the last 40 years, the celebrated Kalman filter has been used widely in many engineering areas. But it is now very well known that the Kalman filter has some drawbacks, that is, the Kalman filter requires the following assumptions which are not practical in realistic applications [1]:

· system models are perfect, i.e., there must be no parameter uncertainties

· spectral properties about the exogenous disturbances are known, i.e., they must be white Gaussian

Unfortunately, in the cases where the above conditions do not hold, the Kalman filter shows degradation in estimation performance and, in the worst case, it may even diverge. Recently, various robust filtering schemes, which have acceptable performance in the presence of modeling errors, have been developed. The robust Kalman filter (or robust H_2 filter) and the robust H_∞ filter are representatives of robust filtering schemes.

The robust Kalman filter has been under consideration, which are to remove the first assumption mentioned above, and some results are already available [11,14]. However, it still requires *a priori* statistical information about the exogenous disturbances. And, it is also noted that, in nominal cases, its estimation performance is not as good as the standard Kalman filter.

On the other hand, the robust H_∞ filter does not require statistical knowledge about the exogenous disturbances, except that they are energy-bounded signals. Of course it was designed for the cases that there exist parameter uncertainties. The robust H_∞ filter, which minimizes the energy-to-energy gain from the exogenous disturbances to the estimation errors, is able to guarantee the robustness against worst-case disturbances and modeling errors. But it has been indicated that the robust H_∞ filter was overly conservative by considering the whole uncertain situations. Namely, the robust H_∞ filter may not have better performance in H_2 sense than the robust Kalman filter in [8]. Moreover, if *a priori* statistical information about the exogenous disturbances is partially available, the robust H_∞ filter can not come up with the robust Kalman filter.

Therefore the development of a new robust filter, which performs well and guarantees robustness against the uncertain disturbances and parameter uncertainties in system models, is necessary. This motivates our robust generalized H_2 filter. The generalized H_2 norm, so-called $l_2 - l_\infty$ convolution operator norm, was introduced by Wilson in the late 80's [13]. Using this operator norm, the generalized H_2 filtering pro-blem was already considered in [6] for *nominal* systems. This filter was designed to minimize the energy-to-peak gain from the exogenous noises to the estimation errors. Therefore, the nominal generalized H_2 filter does not require any statistical assumptions about the exogenous noises. Furthermore, if a bound is known for the energy of the disturbance, the generalized H_2 filter can be used to bound the state estima-tion error amplitude against excessive dynamic excursions. Note that, in the single-input single-output(SISO) system, the H_2 norm of the error dynamics is exactly the same as the generalized H_2 norm [13]. It is the reason why we can interpret the generalized H_2 filtering problem as a deterministic formulation of the Kalman filtering problem [6]. But the robust generalized H_2 filtering problem to guarantee the robustness in the presence of parameter uncertainties was not considered in [6].

The objective of this paper is to design the robust generalized H_2 filter to take place of the robust Kalman filter for the emphasis on the performance. By considering generalized H_2 norm, the proposed robust generalized H_2 filter can relax the second assumption on the knowledge of the spectral properties as well as the first assumption mentioned above. To describe a class of discrete time uncertain linear systems to be estimated, we use the sum quadratic constraints

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(SQCs) which can cover the various uncertainties [10]. This type of uncertain system modeling was used in [7] for the robust H_∞ filtering problem. Generally, the standard norm bounded uncertainty description has been used to deal with the various filtering problems. However, in this paper, we use the SQCs to design a more practical robust filter for general uncertain systems. To develop the robust filter, first we derive new robust stability condition using SQCs and S-procedure. We can not find this robust stability condition in other literature to date. This is one of the main contributions of this paper. And then, the robust generalized H_2 filtering problem is formulated by two QMIs. Solving these QMIs, we can obtain the generalized H_2 filter which has better performance than the existing robust Kalman filter.

This paper is organized as follows : In next section, the preliminaries to solve the robust generalized H_2 filtering problem are introduced. Then, in Section 3, the robust generalized H_2 filtering problem is formulated as several quadratic matrix inequalities(QMIs) which can be converted into LMIs. Then, this convex optimization problem is solved by using the solvability conditions of the QMIs. In Section 4, the simulation results are depicted. Finally, in Section 5, we give conclusions.

II. Preliminaries

1. Sum quadratic constraints and S-procedure

The SQCs are discrete version of the integral quadratic constraints(IQCs). Therefore they have same properties as IQCs. In the 1960-70's, the SQCs were used to treat the stability problems in advanced and complex nonlinear systems. Also, it is very well known that the SQCs can be used as general tools to specify uncertain linear time-invariant dynamics, unmodeled dynamics, constant or time varying signals, delay, and nonlinearities, etc. [9,10]. Furthermore, by using S-procedure, we can convert the SQCs to the combined LMIs. Because efficient algorithms are already available for the convex optimization problem, it seems that we can solve many robust filtering problems via SQCs and S-procedure. The SQCs in the state-space can be classified into the following two large groups.

Given the state-space realization :

$$x(k+1) = Ax(k) + B_w w(k) + B_v v(k)$$

Soft SQCs

$$\sum_{k=0}^{\infty} \sigma(x(k), w(k), v(k)) \geq 0$$

Hard SQCs

$$\sum_{k=0}^{T} \sigma(x(k), w(k), v(k)) \geq 0, \quad T \geq 0$$

where $\sigma(\cdot)$ is a quadratic form and $w(k), v(k) \in l_2[0,\infty)$. In contrast to the soft SQCs, the hard SQCs will be nonnegative at any moments. Therefore, the soft SQCs are more general. S-procedure was introduced by Yakubovich to treat multiple quadratic constraints. Many researchers pointed out that S-procedure may be conservative. Nevertheless, it is very useful tool. By using S-procedure we can easily handle the SQCs and combine them with other LMI constraints [2,7,9]. The following lemma can be found in a lot of related literature, e.g. [2].

Lemma 1 (S-procedure : Non-Strict Inequality Case) : *Let F_0, Λ, F_p be quadratic functions of the variable $\zeta \in \Re^m$:*

$$F_i(\zeta) \triangleq \zeta^T T_i \zeta + 2u_i^T \zeta + v_i, \quad i = 0, \Lambda, p$$

where $T_i = T_i^T$. The following condition on F_0, Λ, F_p :

$$F_0(\zeta) \geq 0 \ \ for\ all\ \zeta\ such\ that\ F_i(\zeta) \geq 0,\ i = 1, \Lambda, p$$

holds if there exist $\tau_1 \geq 0, \Lambda, \tau_p \geq 0$ such that

$$\begin{bmatrix} T_0 & u_0 \\ u_0^T & v_0 \end{bmatrix} - \sum_{i=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix} \geq 0.$$

When $p = 1$ or functionals F_i are affine, the converse holds.

2. Generalized H_2 norm

Consider the following systems.

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

where the exogenous noise $u(k)$ is zero mean white Gaussian.

In H_2 theory, the performance criterion is originally to minimize the mean square error and other important signals which were to be protected against excessive dynamic excursions. From standard results in [3],

$$\text{minimize } \|G(z)\|_2 = \text{minimize } trace^{1/2}\left(CQC^T + DD^T\right)$$

where $G(z)$ is the transfer function from input signal $u(k)$ to output signal $y(k)$ and $Q \geq 0$ satisfies the following Lyapunov equation.

$$AQA^T + BB^T = Q$$

If we assume that the input signal is an unknown l_2 signal and the controlled output is a bounded l_∞ signal, to minimize the mean square error is the same as to minimize the maximum deviation of the controlled output in SISO discrete time systems [13]. But the above statement is not established in multi-input multi-output(MIMO) systems. In MIMO systems, if we want to minimize the maximum deviation of the controlled output, performance measure is defined by minimizing the following generalized H_2 norm [12,13].

$$\begin{aligned} &\text{minimize } \sup_{u \in l_2 - \{0\}} \frac{\|y\|_{l_\infty}}{\|u\|_{l_2}} \\ &= \text{minimize } \lim_{k \to \infty} \left(trace^{1/2}\left[(CQC^T + DD^T)^k\right]\right)^{1/k} \\ &= \text{minimize } f\left(CQC^T + DD^T\right) \end{aligned}$$

where the nonlinear functional $f(\cdot) = \lambda_{\max}^{1/2}(\cdot)$ or $d_{\max}^{1/2}(\cdot)$. $\lambda_{\max}$ denotes the maximum eigenvalue and $d_{\max}$ denotes the maximum diagonal element. If we are to deal with a specific application the peak of whose state or controlled (filtered) output must be bounded under the desired level, we can design an appropriate controller or filter by using the above result.

III. Robust generalized H_2 filtering problem

1. Problem statement

Consider the following uncertain discrete time linear systems [7].

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + \sum_{i=1}^{p} B_{2i}\xi_i(k) \\ y(k) &= Cx(k) + D_1 w(k) + \sum_{i=1}^{p} D_{2i}\xi_i(k) \\ z(k) &= Lx(k) \end{aligned} \tag{1}$$

where $x(k) \in \Re^n$ is state, $w(k) \in \Re^q$ is assumed to be energy bounded noise, $y(k) \in \Re^m$ is measurement output, $z(k) \in \Re^l$ is an arbitrary linear combination of the states, and $\xi_i(k) \in \Re^r$ is uncertain input satisfying the following SQCs. The given matrices have appropriate dimensions.

$$\sum_{k=0}^{T} \|\xi_i(k)\|^2 \le \sum_{k=0}^{T} \|E_{1i}x(k) + E_{2i}(k)w(k) + E_{3i}\xi(k)\|^2 \tag{2}$$

$$\text{for all } i = 1, \Lambda, p$$

where T is the final time and $\xi(k) = [\xi_1^T(k), \Lambda, \xi_p^T(k)]^T$.

The robust generalized H_2 filtering problem associated with the system (1) and the SQCs (2) is to find a stable full-order filter of the form

$$\begin{aligned} \hat{x}(k+1) &= G\hat{x}(k) + Hy(k) \\ \hat{z}(k) &= J\hat{x}(k) + Ky(k) \end{aligned} \tag{3}$$

such that

$$\sup_{w \in l_2 - \{0\}} \frac{\|e\|_{l_\infty}}{\|w\|_{l_2}} < \gamma \tag{4}$$

where γ is a given positive scalar and $e = z - \hat{z}$. To solve the suboptimal robust generalized H_2 filtering problem, we convert the uncertain system (1) to the following general form,

$$\begin{bmatrix} x(k+1) \\ \zeta(k) \\ y(k) \\ z(k) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ E_1 & E_2 & E_3 \\ C & D_1 & D_2 \\ L & 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ \xi(k) \end{bmatrix} \tag{5}$$

where

$$B_2 = [B_{21} \;\; \Lambda \;\; B_{2p}], \quad E_1 = [E_{11}^T \;\; \Lambda \;\; E_{1p}^T]^T,$$

$$E_2 = [E_{21}^T \;\; \Lambda \;\; E_{2p}^T]^T, \quad E_3 = [E_{31}^T \;\; \Lambda \;\; E_{3p}^T]^T,$$

$$D_2 = [D_{21} \;\; \Lambda \;\; D_{2p}],$$

and the uncertainty output $\zeta(k) = [\zeta_1^T(k) \;\; \Lambda \;\; \zeta_p^T]^T$.

2. Robust generalized H_2 filter design

Robust generalized H_2 filter design is based on the robust stability condition for the given uncertain error dynamics. That is, $e(\infty) = L(x(\infty) - \hat{x}(\infty)) \to 0$ for all admissible uncertainties. So far, the necessary and sufficient condition for robust stability is difficult or almost impossible to find. Therefore, we derive only the sufficient condition for robust stability in robust generalized H_2 filtering problem using SQCs and well-known S-procedure.

Theorem 1 (Robust Stability) : *The uncertain system (5) is robustly stable if there exist a matrix $\tilde{P} > 0$ and a diagonal matrix $\tilde{Q} > 0$ which satisfy the following quadratic inequality.*

$$\begin{bmatrix} A & B_2 \\ E1 & E_3 \end{bmatrix} \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{Q} \end{bmatrix} \begin{bmatrix} A & B_2 \\ E_1 & E3 \end{bmatrix}^T < \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{Q} \end{bmatrix} \tag{6}$$

Proof : To check the stability using Lyapunov theory, consider the uncertain system (5) and SQCs (2) without $w(k)$. Let the Lyapunov function candidate for uncertain system (5) be

$$V = x^T(k)\tilde{P}^{-1}x(k) + \sum_{i=1}^{p}\sum_{t=0}^{k-1} \tau_i \left(\|E_{1i}x(t) + E_{3i}\xi(t)\|^2 - \|\xi_i(t)\|^2 \right)$$

where the arbitrary scalars $\tau_i > 0$ should be optimized. According to the Lyapunov theory, the uncertain linear system is asymptotically stable if the following inequality holds [4].

$$\Delta V = x^T(k+1)\tilde{P}^{-1}x(k+1) - x^T(k)\tilde{P}^{-1}x(k)$$

$$+ \sum_{i=1}^{p} \tau_i \left(\|E_{1i}x(t) + E_{3i}\xi(t)\|^2 - \|\xi_i(k)\|^2 \right) < 0$$

(We can also derive the above inequality by S-procedure, in Lemma 1 : if $\|\xi_i(k)\|^2 \le \|E_{1i}x(k) + E_{3i}\xi(k)\|^2$ [2].)

Rearranging the above inequality, yields

$$\begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix}^T \left(\begin{bmatrix} \tilde{P}^{-1} & 0 \\ 0 & \tilde{Q}^{-1} \end{bmatrix} - \begin{bmatrix} A^T & E_1^T \\ B_2^T & E_3^T \end{bmatrix} \begin{bmatrix} \tilde{P}^{-1} & 0 \\ 0 & \tilde{Q}^{-1} \end{bmatrix} \begin{bmatrix} A & B_2 \\ E_1 & E_3 \end{bmatrix} \right) \begin{bmatrix} x(k) \\ \xi(k) \end{bmatrix} > 0 \tag{7}$$

where a diagonal matrix $\tilde{Q}^{-1} > 0$ is defined as

$$\begin{bmatrix} I^{d_2} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I^{\tau_2} & 0 \\ 0 & \cdots & 0 & I^{1\,2} \end{bmatrix} \tag{8}$$

Now, taking the Schur complement of (7) we can obtain (6).

From Theorem 1, we can derive the following inequality related to generalized H_2 performance measures.

Corollary 1 (Robust stability) : *If there exist a matrix $\tilde{P} > 0$ and a diagonal matrix $\tilde{Q} > 0$ which satisfy (6), there exist a matrix $P > 0$ and a diagonal matrix $Q > 0$ satisfying the following inequality.*

$$\begin{bmatrix} A & B_2 & B_1 \\ E_1 & E_3 & E_2 \end{bmatrix} \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A & B_2 & B_1 \\ E_1 & E_3 & E_2 \end{bmatrix}^T < \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \tag{9}$$

Proof : Equation (6) implies the following inequality for sufficiently small $\varepsilon > 0$.

$$\begin{bmatrix} A & B_2 \\ E_1 & E_3 \end{bmatrix} \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{Q} \end{bmatrix} \begin{bmatrix} A & B_2 \\ E_1 & E_3 \end{bmatrix}^T + \varepsilon \begin{bmatrix} B_1 \\ E_2 \end{bmatrix} \begin{bmatrix} B_1 \\ E_2 \end{bmatrix}^T < \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{Q} \end{bmatrix}$$

Dividing both sides by ε and replacing $\frac{1}{\varepsilon}\tilde{P}, \frac{1}{\varepsilon}\tilde{Q}$ by P, Q, respectively, we have (9).

Given the system (5), the robust generalized H_2 performance is defined by the following Lemma 2.

Lemma 2 : (Robust generalized H_2 performance) : *The robust generalized H_2 gain of the system (5) is under the given positive scalar γ if the matrix $P > 0$ satisfies the following matrix inequality and $D_2 = 0$.*

$$CPC^T + D_1 D_1^T < \gamma^2 I \tag{10}$$

where, the matrix $P > 0$ satisfies robust stability condition (9).

Proof : See [6], [15] and references therein for more details.

The robust generalized H_2 filtering problem is formulated by some QMIs. To solve these QMIs, it is necessary to check the solvability conditions for a general QMI as follows :

Lemma 3. : (Solvability conditions of general QMI : Elimination Lemma) : *Let matrices Θ, Γ, Λ and $R > 0$, $S > 0$ be given, and the solution to the following QMI be F.*

$$(\Theta + \Gamma F \Lambda)^T R (\Theta + \Gamma F \Lambda) < S \tag{11}$$

Then there exists a feasible solution F to the QMI (11) *if the following conditions are satisfied.*

$$\Lambda^{T\perp}(S - \Theta^T R \Theta)\Lambda^{T\perp T} > 0 \tag{12}$$

$$\Gamma^{\perp}(R^{-1} - \Theta S^{-1}\Theta^T)\Gamma^{\perp T} > 0 \tag{13}$$

where, $A^{\perp}$ denotes the left annihilator of A, i.e., a matrix with the following properties; the null-space of $A^{\perp}$ is equal to the range space of A, $N(A^{\perp}) = R(A)$, and $A^{\perp}A^{\perp T} > 0$. Note that, for a given A, $A^{\perp}$ is not unique, but throughout this paper, any choice is acceptable.

Proof : See [6] and references therein for more details. ■

Now, the robust generalized H_2 filtering problem can be solved by considering the robust stability condition and the generalized H_2 performance condition of the error dynamics. The error dynamics (14) can be obtained by combining the uncertain system (5) and the filter equation (3).

$$\begin{bmatrix} \begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} \\ \zeta(k) \\ e(k) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A & 0 \\ HC & G \end{bmatrix} & \begin{bmatrix} B_1 \\ HD_1 \end{bmatrix} & \begin{bmatrix} B_2 \\ HD_2 \end{bmatrix} \\ \begin{bmatrix} E_1 & 0 \end{bmatrix} & E_2 & E_3 \\ \begin{bmatrix} L-KC & -J \end{bmatrix} & -KD_1 & -KD_2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} \\ w(k) \\ \xi(k) \end{bmatrix} \tag{14}$$

From Corollary 1 and Lemma 2, we can obtain two QMIs (15), (16).

Corollary 2 (Two QMIs in robust generalized H_2 filtering problem) : *The robust generalized H_2 filtering problem is to find the filter matrices G, H, J and K which satisfy the following two QMIs.*

$$\begin{bmatrix} \begin{bmatrix} A & 0 \\ HC & G \end{bmatrix} & \begin{bmatrix} B_2 \\ HD_2 \end{bmatrix} & \begin{bmatrix} B_1 \\ HD_1 \end{bmatrix} \\ \begin{bmatrix} E_1 & 0 \end{bmatrix} & E_3 & E_2 \end{bmatrix} \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} A & 0 \\ HC & G \end{bmatrix} & \begin{bmatrix} B_2 \\ HD_2 \end{bmatrix} & \begin{bmatrix} B_1 \\ HD_1 \end{bmatrix} \\ \begin{bmatrix} E_1 & 0 \end{bmatrix} & E_3 & E_2 \end{bmatrix}^T < \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \tag{15}$$

$$[[L-KC \ \ -J] \ \ KD_1]\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}[[L-KC \ \ -J] \ \ KD_1]^T < \gamma^2 I$$

$$[[L-KC \ \ -J] \ \ KD_1]\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}[[L-KC \ \ -J] \ \ KD_1]^T < \gamma^2 I \tag{16}$$

Proof : This corollary can be easily verified by using (9), (10), (14). From the robust stability condition (9) of the error dynamics (14), we have (15). And we need (16) for the error dynamics (14) to satisfy the generalized H_2 performance condition (10).

Remark : *In the case $D_2 \neq 0$, we can obtain the robust generalized H_2 filter by letting K equal to zero. For the generalized H_2 performance measure of the given error dynamics to exist, KD_2 must be zero by Lemma 2. However, this choice limits the feasible set for the filter to exist, it is reasonable to assume $D_2 = 0$. Henceforth, we use $D_2 = 0$.*

We can obtain the following results from the solvability conditions of QMIs (15), (16).

Theorem 2 (Solvability conditions of the robust generalized H_2 filtering problem) : *There exist γ-suboptimal robust generalized H_2 filter matrices F_1, F_2 in Corollary 2 if there exist matrices X, $Z > 0$, and a diagonal matrix $Q^{-1} > 0$ satisfying the QMIs* (17), (18), (19).

$$\begin{bmatrix} X - A^T Y^{-1} A - E_1^T Q^{-1} E_1 & -A^T X B_2 - E_1^T Q^{-1} E_3 & -A^T X B_1 - E_1^T Q^{-1} E_2 \\ -B_2^T X A - E_3^T Q^{-1} E_1 & Q^{-1} - B_2^T X B_2 - E_3^T Q^{-1} E_3 & -B_2^T X B_1 - E_3^T Q^{-1} E_2 \\ -B_1^T X A - E_2^T Q^{-1} E_1 & -B_1^T X B_2 - E_2^T Q^{-1} E_3 & I - B_1^T X B_1 - E_2^T Q^{-1} E_2 \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} C^T \\ 0 \\ D_1^T \end{bmatrix}^{\perp} \begin{bmatrix} Z - A^T Z A - E_1^T Q^{-1} E_1 & -A^T Z B_2 - E_1^T Q^{-1} E_3 & -A^T Z B_1 - E_1^T Q^{-1} E_2 \\ -B_2^T Z A - E_3^T Q^{-1} E_1 & Q^{-1} - B_2^T Z B_2 - E_3^T Q^{-1} E_3 & -B_2^T Z B_1 - E_3^T Q^{-1} E_2 \\ -B_1^T Z A - E_2^T Q^{-1} E_1 & -B_1^T Z B_2 - E_2^T Q^{-1} E_3 & I - B_1^T Z B_1 - E_2^T Q^{-1} E_2 \end{bmatrix} \begin{bmatrix} C^T \\ 0 \\ D_1^T \end{bmatrix}^{\perp T} > 0 \tag{18}$$

$$\begin{bmatrix} C^T \\ D_1^T \end{bmatrix}^{\perp} \begin{bmatrix} Z - \frac{1}{\gamma^2} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C^T \\ D_1^T \end{bmatrix}^{\perp T} > 0 \tag{19}$$

Proof : QMIs (15), (16) can be converted to the following general QMIs.

$$(\Theta_1 + \Gamma_1 F_1 \Lambda_1)^T R_1 (\Theta_1 + \Gamma_1 F_1 \Lambda_1) < S_1 \tag{20}$$

$$(\Theta_2 + \Gamma_2 F_2 \Lambda_2)^T R_2 (\Theta_2 + \Gamma_2 F_2 \Lambda_2) < S_2 \tag{21}$$

where

$$\Theta_1 = \begin{bmatrix} A^T & 0 & E_1^T \\ 0 & 0 & 0 \\ B_2^T & 0 & E_3^T \\ B_1^T & 0 & E_2^T \end{bmatrix} \quad \Gamma_1 = \begin{bmatrix} 0 & C^T \\ I & 0 \\ 0 & 0 \\ 0 & D_1^T \end{bmatrix} \quad \Lambda_1 = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}^T$$

$$R_1 = \begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{bmatrix} \quad S_1 = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \quad F_1 = \begin{bmatrix} G^T \\ H^T \end{bmatrix}$$

$$\Theta_2 = \begin{bmatrix} L^T \\ 0 \\ 0 \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} 0 & -C^T \\ -I & 0 \\ 0 & D_1^T \end{bmatrix} \quad \Lambda_2 = I \tag{22}$$

$$R_2 = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \quad S_2 = \gamma^2 I \quad F_2 = \begin{bmatrix} J^T \\ K^T \end{bmatrix}$$

$$P = \begin{bmatrix} Y & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \quad P^{-1} = \begin{bmatrix} Z & Z_{12} \\ Z_{12}^T & Z_{22} \end{bmatrix} \quad Y^{-1} = X$$

Using the Lemma 3, we can obtain the following solvability conditions for QMI (21).

$$\Lambda_2^{T\perp}(S_2 - \Theta_2^T R_2 \Theta_2)\Lambda_2^{T\perp T} > 0 \qquad (23)$$

$$\Gamma_2^{\perp}(R_2^{-1} - \Theta_2 S_2^{-1}\Theta_2^T)\Gamma_2^{\perp T} > 0 \qquad (24)$$

Noting that $\Lambda_2^{T\perp} = 0$, QMI (23) can be ignored. Then it is easy to verify that (24) is equivalent to (19). Also, from the solvability conditions of QMI (20), the following matrix inequalities must hold.

$$\Lambda_1^{T\perp}(S_1 - \Theta_1^T R_1 \Theta_1)\Lambda_1^{T\perp T} > 0 \qquad (25)$$

$$\Gamma_1^{\perp}(R_1^{-1} - \Theta_1 S_1^{-1}\Theta_1^T)\Gamma_1^{\perp T} > 0 \qquad (26)$$

We can obtain (18) by substituting the matrices in (22) for (26). In the same way, we have (27) from (25) using (22).

At this point, we need to check whether the QMIs (18), (19), and (27) are jointly linear in the matrix variables, respectively. The QMIs (18) and (19) are jointly linear in the matrices Z and Q^{-1} whereas the QMI (27) is jointly linear in the matrices Y and Q. Therefore, it is difficult to find the feasible solution satisfying the QMI (18), (19), and (27) because these QMIs are not jointly linear in Q. To make the suboptimal robust general- ized H_2 filtering problem tractable, QMI (27) must be changed to appropriate form. To this end, let the outer factor of QMI (27) be

$$\begin{bmatrix}0\\I\\0\end{bmatrix}^{\perp}\begin{bmatrix} Y-AYA^T & Y_{12} & -AYE_1^T - B_2QE_3^T - B_1E_2^T \\ Y_{12}^T & Y_{22} & 0 \\ -E_1YA^T - E_3QB_2^T - E_2B_1^T & 0 & Q - E_1YE_1^T - E_3QE_3^T - E_2E_2^T \end{bmatrix}\begin{bmatrix}0\\I\\0\end{bmatrix}^{\perp T} > 0 \qquad (27)$$

$$\begin{bmatrix}0\\I\\0\end{bmatrix}^{\perp} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (28)$$

Then, we can convert (27) to

$$\begin{bmatrix} Y & 0 \\ 0 & Q \end{bmatrix} - \begin{bmatrix} A & B_2 & B_1 \\ E_1 & E_3 & E_2 \end{bmatrix}\begin{bmatrix} Y & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} A & B_2 & B_1 \\ E_1 & E_3 & E_2 \end{bmatrix}^T > 0 \qquad (29)$$

Taking Schur complement, (29) is transformed to

$$\begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} A & B_2 & B_1 \\ E_1 & E_3 & E_2 \end{bmatrix}^T \begin{bmatrix} Y^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix}\begin{bmatrix} A & B_2 & B_1 \\ E_1 & E_3 & E_2 \end{bmatrix} > 0 \qquad (30)$$

Finally, replacing Y^{-1} by X, we have (17). ■

Now, we can find the feasible solution to the QMIs (17), (18), (19) using the interior point algorithm [5] because those QMIs are jointly linear in X, Z, and Q^{-1}. The only to be left for the design of suboptimal robust generalized H_2 filter is to solve the two QMIs (15),(16) in Corollary 2. Substituting X, Z, and Q^{-1} for (15),(16), matrix variables of QMIs (15), (16) reduce to F_1, F which will result in the robust generalized H_2 filter.

IV. Numerical example

A simple example is presented to analyze the performance of the robust generalized H_2 filter. The (SISO) plant model is borrowed from [14].

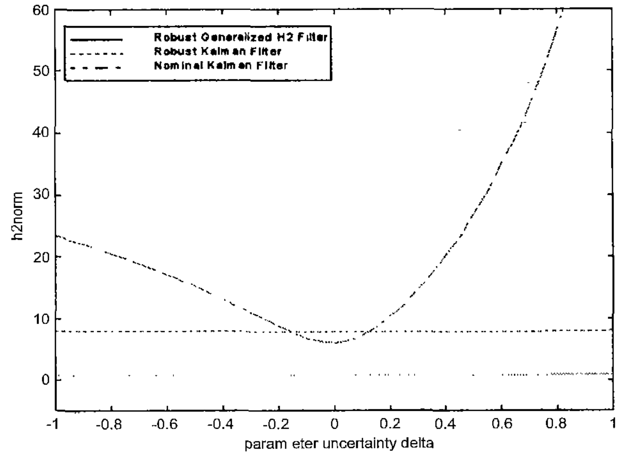

Fig. 1. Estimation error bound – case 1.

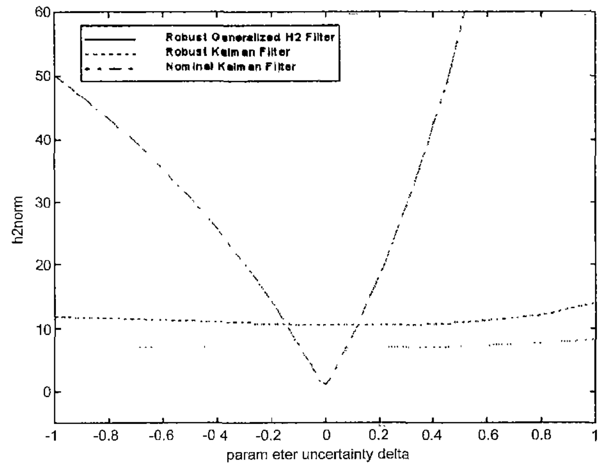

Fig. 2. Estimation error bound – case 2.

$$x(k+1) = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 10 \end{bmatrix}\Delta\begin{bmatrix} 0 & 0.03 \end{bmatrix} x(k) + \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix} w(k)$$
$$y(k) = \begin{bmatrix} -100 & 10 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(k)$$
$$z(k) = Lx(k)$$

where the parameter uncertainty Δ satisfies $\|\Delta\| \le 1$.

We consider two cases; $L = [1\ 0]$ and $L = [0\ 1]$. Given $\gamma = 10.6581$, we can obtain the following robust generalized H_2 filter by (15), (16) and their solvability conditions in Theorem 2.

Case 1 : $L = [1\ 0]$

$$\hat{x}(k+1) = \begin{bmatrix} 0.0502 & 0.1125 \\ 0.1283 & 0.2876 \end{bmatrix}\hat{x}(k) + \begin{bmatrix} -0.0001 \\ 0.0000 \end{bmatrix} y(k)$$
$$\hat{z}(k) = [5.9780\ 13.3969]\hat{x}(k) - 0.0088y(k)$$

Case 2 : $L = [0\ 1]$

$$\hat{x}(k+1) = \begin{bmatrix} 0.1373 & 0.2769 \\ 0.1547 & 0.3119 \end{bmatrix}\hat{x}(k) + \begin{bmatrix} -0.0001 \\ 0.0000 \end{bmatrix} y(k)$$
$$\hat{z}(k) = [50.7148\ 102.2785]\hat{x}(k) + 0.0117y(k)$$

The performance of the robust generalized H_2 filter, robust Kalman filter by L. Xie, *et. al.* [14], and nominal Kalman filter are compared by the H_2 norm of the error dynamics as the function of parameter uncertainties. The results are depicted in Fig. 1-2. The solid lines indicate the H_2 norm of the robust generalized H_2 filter error dynamics. The dotted lines and dashed lines show the H_2 norm of the robust and nominal Kalman filter error dynamics, respectively.

In Fig. 1, we can see that the proposed filter exhibits better estimation performance for all admissible parameter uncertainties than the other filters even in the nominal case ($\Delta = 0$). Generally, if there are no parameter uncertainties, the estimation performance of the robust filter is not better than the nominal filter because its design objective is to maintain a certain estimation performance. Therefore, the result in Fig. 1 is not common. This result may come from the dynamic characteristics of the given error dynamics. Figure 2 supports this assertion.

Note that in the simulation results, the estimation error bound of the nominal Kalman filter increases abruptly, while that of the proposed filter maintains a certain level less than that of the robust Kalman filter in [14]. In addition, the proposed filter is designed to bound the peak of the estimation errors in worst cases without any spectral assumptions on the exogenous noises. Therefore, it can relax the spectral assumptions which have been pointed out as the drawback of the nominal and the robust Kalman filter.

V. Conclusions

So far, many researchers have studied the robust Kalman filtering problem to achieve the robust performance against parameter uncertainties, but these filters have a disadvantage inherently; the robust Kalman filter assumes that the exogenous noises are white Gaussian.

In this paper, we have developed a suboptimal robust generalized H_2 filter using linear matrix inequalities, which can overcome the disadvantage of the robust Kalman filter by considering the generalized H_2 performance measure. The proposed filter bounds energy to peak gain from the exogenous noises to the estimation errors. Therefore, the robust generalized H_2 filter achieves robust performance against parameter uncertainties without *a priori* statistical information about the exogenous noises. The robust generalized H_2 filter is designed based on the newly derived robust stability condition by SQCs and S-procedure. And then, We have shown that, given $\gamma > 0$, the robust generalized H_2 filtering problem can be formulated by two QMIs and its solvability conditions. Numerical results have shown that the variance of the estimation error obtained by the proposed filter is lower than that by the existing Kalman filter in the presence of parameter uncertainty.

The proposed filter can be applied to many applications. Since the proposed filter bounds the peak of the estimation errors in worst cases, it is useful for systems whose performance is affected by the abrupt increase of the estimation errors such as target tracking systems, navigation systems, etc.. As well, it can be applied to many realistic applications because it is designed for the uncertain linear systems described by SQCs which can cover various uncertainties.

References

[1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Prentice-Hall, Inc., 1979.

[2] S. P. Boyd L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

[3] S. P. Boyd and C. H. Barrat, *Linear Control Design - Limits of Performance*, Prentice-Hall, Inc., 1996.

[4] W. L. Brogan, *Modern Control Theory*, Prentice-Hall, Inc., 1991.

[5] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*, Mathworks, Inc., 1995.

[6] K. M. Grigoriadis and J. T. Watson JR., "Reduced Order H_∞ and $l_2 - l_\infty$ Filtering via Linear Matrix Inequa-lities", *IEEE Trans. on Aerospace and Electronic System*, vol. 33, no. 4, pp.1326-1338, 1997.

[7] H. Li, and M. Fu, "A Linear Matrix Inequality Approach to Robust H_∞ Filtering", *IEEE Trans. on Signal Processing*, vol. 45, no. 9, pp. 2338-2350, 1997.

[8] L. Li, Z. Q. Luo, K. M. Wong, and E. Bosse, "Semidefinite Programming Solutions to the Robust State Estimation Problem", *Manuscript, Communications Research Lab., McMaster Univ.*, July 1998.

[9] A. Megretski and A. Rantzer, "System Analysis via Integral Quadratic Constraints", *IEEE Trans. on Automatic Control*, vol. 42, no. 6, pp. 819-830, 1997.

[10] M. Fu, N. E. Barabanov and H. Li, "Robust Analysis and Control of Linear Systems with Integral Quadratic Constraints", *European Control Conferences*, 1995.

[11] I. R. Peterson and D. C. McFarlane, "Optimal Guaranteed Cost Filtering for Uncertain Discrete-Time Linear Systems", *Int. J. of Robust and Nonlinear Control*, vol. 6, pp. 267-280, 1996.

[12] M. A. Rotea, "The Generalized H_2 Control Problem", *Automatica*, vol. 29, no. 2, pp. 373-385, 1993.

[13] D. A. Wilson, "Convolution and Hankel Operator Norms for Linear Systems", *IEEE Trans. on Automatic Control*, vol. 34, pp. 94-98, 1989.

[14] L. Xie, Y. C. Soh, and C. E. de Souza, "Robust Kalman Filtering for Uncertain Discrete Time Systems", *IEEE Trans. on Automatic Control*, vol. 39, no. 6, pp. 1310-1314, 1994.

[15] R. Zhu and R.E. Skelton, "Mixed L_2 and L_∞ Problems by Weight Selection in Quadratic Optimal Control", *Int. J. of Control*, vol. 53, no. 3, pp. 1161-1176, 1991.

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