Robust Stabilization of Uncertain Linear Systems with Time-delay

Young Soo Moon, PooGyeon Park, and Wook Hyun Kwon

Abstract : This paper presents a new delay-dependent robust stabilization condition for uncertain time-delay systems. An algorithm involving convex optimization is proposed to compute a suboptimal upper bound of the delay such that the system can be stabilized by the controller for all admissible uncertainties. It is illustrated by numerical examples that the proposed delay-dependent controller can be less conservative than previous results. It is also shown that the proposed delay-dependent controller can even capture the delay-independent stabilizability of the system, which is not possible with existing delay-dependent results.

Keywords : time-delay systems, robust stabilization, linear matrix inequalities

I. Introduction

Since time-delay is often a source of instability in many engineering systems, there has been considerable research on the control problem of time-delay systems [1]. Recently, for systems with uncertainty as well as time-delay, a number of robust stabilization methods have been proposed[2].

In general, the stabilization methods for time-delay systems can be classified into two types: delayindependent stabilization[3]-[6] and delay-dependent stabilization[7]-[10]. The delay-independent stabilization provides a controller which can stabilize the system irrespective of the size of the delay. On the other hand, the delay-dependent stabilizing controller is concerned with the size of the delay and usually gives the upper bound of the delay such that the system can be stabilized by the given controller. In general, the delay-dependent results are considered less conservative than the delay-independent ones. However, existing delay-dependent stabilization results are still too conservative in some cases. Especially, when applied to the system which is delayindependent stabilizable, existing delay-dependent controllers often guarantee the stability for only a small size of the delay, far from providing infinity as the upper bound of the delay.

In this paper, we present a new delay-dependent robust controller for uncertain time-delay systems which can stabilize the system for all admissible uncertainties. We also propose an algorithm involving convex optimization to compute a suboptimal upper bound of the delay such that the system can be stabilized by the controller. It is shown by numerical examples that the proposed controller can be less conservative than the existing results and it is even possible to capture the delay-independent stabilizability of the system, which is not the case for the previous results.

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This paper is organized as follows. In Section II, nominal time-delay systems without uncertainties are considered first and stability analysis and stabilization conditions are presented. In Section III, robust stability and stabilization results are developed for uncertain time-delay systems and the algorithm to construct the controller is proposed. Numerical examples are given in Section IV and finally, Section V makes conclusions.

II. Stability and stabilization of nominal systems

Let us consider a system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t-h) + Bu(t), \\ x(t) &= \phi(t), \quad t \in [-h, 0] \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control, $h > 0$ is the delay of the system, $\phi(\cdot)$ is the initial condition and A, A_1 and B are real constant matrices with appropriate dimensions. We assume that the pair $(A+A_1, B)$ is stabilizable. We are interested in designing a memoryless linear state-feedback controller

$$u(t) = Gx(t) \tag{2}$$

where $G \in R^{m \times n}$ is a constant gain matrix. Our aim is to develop a delay-dependent stabilization method which provides the controller gain G as well as the upper bound $\bar{h}$ of the delay such that the closed-loop system is stable for any h satisfying $0 \le h \le \bar{h}$.

The following lemma which was introduced by the authors in [11, 12] for stability analysis of time-delay systems plays an important role in obtaining the main results of this paper.

Lemma 1 [11, 12] : Assume that $a(\cdot) \in R^{n_a}$ and $b(\cdot) \in R^{n_b}$ are defined on the interval Ω. Then, for any positive definite matrix $X \in R^{n_a \times n_a}$ and any matrix $M \in R^{n_b \times n_b}$, the following holds:

$$-2\int_\Omega a^T(\theta)b(\theta)d\theta \le \int_\Omega \begin{bmatrix} a(\theta) \\ b(\theta) \end{bmatrix}^T \begin{bmatrix} X & XM \\ M^TX & (M^TX+I)X^{-1}(XM+I) \end{bmatrix} \begin{bmatrix} a(\theta) \\ b(\theta) \end{bmatrix} d\theta. \tag{3}$$

In the following theorem, we first consider the stability analysis of the unforced nominal system (1) with $u(t)=0$.

Theorem 1 : (Stability) The unforced system (1) with $u(t)=0$ is asymptotically stable for any time-delay h satisfying $0\le h\le \bar{h}$, if there exist $Z>0$, $V>0$, $q>0$ and M such that

$$\begin{bmatrix} Y_{11} & qZ+A_1M^T & Y_{13} & Y_{14} \\ MA_1^T+qZ & -\bar{h}qV & -\bar{h}VA_1^T & 0 \\ Y_{13}^T & -\bar{h}A_1V & -V & 0 \\ Y_{14}^T & 0 & 0 & -V \end{bmatrix} < 0 \qquad (4)$$

where

$$Y_{11} := Z(A+A_1)^T+(A+A_1)Z,$$

$$Y_{13} := Z(A+A_1)^T, \quad Y_{14} := A_1(M^T+\bar{h}V).$$

proof : Choose a Lyapunov function as

$$W(x(t-a),\ a\in[0,\bar{h}]) = W_1+W_2+W_3 \qquad (5)$$

where

$$W_1 := x^T(t)Z^{-1}x(t),$$

$$W_2 := \int_{-h}^{0}\int_{t+s}^{t}\{Ax(\theta)+A_1x(\theta-h)\}^TY^{-1}\{Ax(\theta)+A_1x(\theta-h)\}d\theta ds,$$

$$W_3 := q\int_{t-h}^{t}x^T(\theta)Y^{-1}x(\theta)d\theta$$

and $Y>0$.

Since it holds that

$$x(t)-x(t-h) \equiv \int_{t-h}^{t}\dot{x}(\theta)d\theta = \int_{t-h}^{t}\{Ax(\theta)+A_1x(\theta-h)\}d\theta, \qquad (6)$$

the system (1) can be written as

$$\dot{x}(t) = (A+A_1)x(t) - A_1\int_{t-h}^{t}\{A(\theta)+A_1x(\theta-h)\}d\theta \qquad (7)$$

and thus the derivative of W_1 satisfies the relation

$$\dot{W}_1 = 2x^T(t)Z^{-1}(A+A_1)x(t) - 2x^T(t)Z^{-1}A_1\int_{t-h}^{t}\{Ax(a)+A_1x(a-h)\}da.$$

Defining $a(\cdot)$ and $b(\cdot)$ in (3) as, for all $\theta\in[t-h,t]$,

$$a(\theta) := Ax(\theta)+A_1x(\theta-h),$$

$$b(\theta) := A_1^TZ^{-1}x(t),$$

and using Lemma 1 will supply

$$\begin{aligned}\dot{W}_1 \le\ & x^T(t)\{(A+A_1)^TZ^{-1}+Z^{-1}(A+A_1) \\ & +hZ^{-1}A_1(M^TX+I)X^{-1}(XM+I)A_1^TZ^{-1}\}x(t) \\ & +2x^T(t)Z^{-1}A_1M^TX\int_{t-h}^{t}\{Ax(\theta)+A_1x(\theta-h)\}d\theta \\ & +\int_{t-h}^{t}\{Ax(\theta)+A_1x(\theta-h)\}^TX\{Ax(\theta)+A_1x(\theta-h)\}d\theta.\end{aligned}$$

Since $\dot{W}_2$ and $\dot{W}_3$ yield the relation

$$\begin{aligned}\dot{W}_2 =\ & h\{Ax(t)+A_1x(t-h)\}^TY^{-1}\{Ax(t)+A_1x(t-h)\} \\ & -\int_{t-h}^{t}\{Ax(\theta)+A_1x(\theta-h)\}^TY^{-1}\{Ax(\theta)+A_1x(\theta-h)\}d\theta, \\ \dot{W}_3 =\ & qx^T(t)Y^{-1}x(t)-qx^T(t-h)Y^{-1}x(t-h),\end{aligned}$$

choosing $X := Y^{-1}$ and $V := Y/\bar{h}$ yield

$$\begin{aligned}\dot{W} =\ & \dot{W}_1+\dot{W}_2+\dot{W}_3 \\ \le\ & x^T(t)\{(A+A_1)^TZ^{-1}+Z^{-1}(A+A_1) \\ & +Z^{-1}A_1(M^T+\bar{h}V)V^{-1}(M+\bar{h}V)A_1^TZ^{-1}\}x(t) \\ & +\{Ax(t)+A_1x(t-h)\}^TV^{-1}\{Ax(t)+A_1x(t-h)\} \\ & +2x^T(t)Z^{-1}A_1M^TY^{-1}\{x(t)-x(t-h)\} \\ & +qx^T(t)Y^{-1}x(t)-qx^T(t-h)Y^{-1}x(t-h) \\ =\ & \begin{bmatrix} Z^{-1}x(t) \\ x(t)-x(t-h)\end{bmatrix}^T \begin{bmatrix} (1,1) & (1,2) \\ (1,2)^T & (2,2)\end{bmatrix}\begin{bmatrix} Z^{-1}x(t) \\ x(t)-x(t-h)\end{bmatrix},\end{aligned}$$

where

$$\begin{aligned}(1,1) :=\ & Z(A+A_1)^T+(A+A_1)Z+ \\ & A_1(M^T+\bar{h}V)V^{-1}(M+\bar{h}V)A_1^T \\ & +Z(A+A_1)^TV^{-1}(A+A_1)Z, \\ (1,2) :=\ & A_1M^TY^{-1}+qZY^{-1}-Z(A+A_1)^TV^{-1}A_1, \\ (2,2) :=\ & -qY^{-1}+A_1^TV^{-1}A_1.\end{aligned}$$

This quantity will be negative if it holds that

$$\begin{bmatrix} Y_{11} & Y_{12} & Z(A+A_1)^T \\ Y_{12}^T & -qY^{-1} & -A_1^T \\ (A+A_1)Z & -A_1 & -V \end{bmatrix} < 0,$$

where

$$\begin{aligned}Y_{11} :=\ & Z(A+A_1)^T+(A+A_1)Z \\ & +A_1(M^T+\bar{h}V)V^{-1}(M+\bar{h}V)A_1^T, \\ Y_{12} :=\ & (A_1M^T+qZ)Y^{-1}.\end{aligned}$$

Then, using the Lyapunov-Krasovskii stability theorem [13] and Schur complement[14], we can conclude that the unforced system (1) is asymptotically stable if the condition (4) is satisfied. This completes the proof. ■

Now, we extend Theorem 1 to a synthesis problem to design a stabilizing state-feedback controller (2) for the system (1).

Theorem 2 (Controller Design) : If there exist $Z>0$, $V>0$, $q>0$, M and K such that

$$\begin{bmatrix} Y_{11} & qZ+A_1M^T & Y_{13} & Y_{14} \\ MA_1^T+qZ & -\bar{h}qV & -\bar{h}VA_1^T & 0 \\ Y_{13}^T & -\bar{h}A_1V & -V & 0 \\ Y_{14}^T & 0 & 0 & -V \end{bmatrix} < 0 \qquad (8)$$

where

$$Y_{11} := Z(A+A_1)^T+(A+A_1)Z+K^TB^T+BK,$$

$$Y_{13} := Z(A+A_1)^T+K^TB^T,$$

$$Y_{14} := A_1(M^T+\bar{h}V),$$

then the system (1) with the control $u(t)=KZ^{-1}x(t)$ is asymptotically stable for any time-delay h satisfying $0 \le h \le \bar{h}$.

proof : With the control (2), the closed-loop system matrix becomes $A_c = A + BG$. Hence, by applying Theorem 1 to the closed-loop system, it is easy to derive (8) with the change of a variable $K = GZ$. ■

In the next section, we extend the obtained stability and stabilization results to the systems with norm-bounded uncertainties.

III. Robust stability and stabilization of uncertain systems

Consider the following uncertain time-delay systems

$$\dot{x}(t) = (A + DF(t)E)x(t) + (A_1 + D_1F_1(t)E_1) x(t-h) + (B + DF(t)E_b)u(t), \quad (9)$$

$$x(t) = \phi(t), \quad t \in [-h, 0],$$

where $D \in R^{n\times j}$, $D_1 \in R^{n\times j_1}$, $E \in R^{k\times n}$, $E_1 \in R^{k_1\times n}$, $E_b \in R^{k\times m}$ are real constant matrices with appropriate dimensions and $F(t) \in R^{j\times k}$ and $F_1(t) \in R^{j_1\times k_1}$ are uncertainties satisfying

$$\|F(t)\| \le 1, \qquad \|F_1(t)\| \le 1. \quad (10)$$

First, the following theorem gives robust stability analysis of the unforced system (9) with $u(t)=0$.

Theorem 3 (Robust Stability) : The unforced system (9) with $u(t)=0$ is asymptotically stable for any time-delay h satisfying $0 \le h \le \bar{h}$, if there exist $Z>0$, $V>0$, $q>0$, M and scalars $e_1, e_2, \ldots, e_6$ such that

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & ZE^T & ZE^T & ZE_1^T & ZE_1^T \\ Y_{12}^T & Y_{22} & Y_{23} & 0 & 0 & 0 & ME_1^T & -\bar{h}VE_1^T \\ Y_{13}^T & Y_{23}^T & Y_{33} & 0 & 0 & 0 & 0 & 0 \\ Y_{14}^T & 0 & 0 & -V & 0 & 0 & Y_{47} & 0 \\ EZ & 0 & 0 & 0 & -e_1I & -e_3I & 0 & 0 \\ EZ & 0 & 0 & 0 & -e_3I & -e_2I & 0 & 0 \\ E_1Z & E_1M^T & 0 & Y_{47}^T & 0 & 0 & -e_4I & -e_6I \\ E_1Z & -\bar{h}E_1V & 0 & 0 & 0 & 0 & -e_6I & -e_5I \end{bmatrix} < 0, \quad (11)$$

where

$$Y_{11} := Z(A+A_1)^T + (A+A_1)Z + e_1DD^T + e_4D_1D_1^T,$$

$$Y_{13} := Z(A+A_1)^T + e_3DD^T + e_6D_1D_1^T,$$

$$Y_{12} := qZ + A_1M^T, \quad Y_{14} := A_1(M^T + \bar{h}V),$$

$$Y_{22} := -\bar{h}qV, \qquad Y_{33} := -V + e_2DD^T + e_5D_1D_1^T,$$

$$Y_{23} := -\bar{h}VA_1^T, \quad Y_{47} := (M + \bar{h}V)E_1^T.$$

proof : Replace A, A_1, and B in (4) with $A + DF(t)E$, $A_1 + D_1F_1(t)E_1$ and $B + DF(t)E_b$, respectively and multiply both sides of (4) by vectors x_i for $i = 1, \cdots, 4$. Next, define

$$p_1 := F^T(t)D^Tx_1, \qquad p_2 := F^T(t)D^Tx_3,$$

$$q_1 := F_1^T(t)D_1^Tx_1, \qquad q_2 := F_1^T(t)D_1^Tx_3.$$

Then we have the following condition

$$v^T [X_{ij}]_{(8\times 8)} v < 0 \quad (12)$$

for all admissible $F(t)$ and $F_1(t)$, where

$$v := [x_1 \ x_2 \ x_3 \ x_4 \ p_1 \ p_2 \ q_1 \ q_2]^T$$

and the upper triangular entries of the symmetric matrix X are

$$X_{11} := Z(A+A_1)^T + (A+A_1)Z, \ X_{12} := qZ + A_1M^T,$$

$$X_{13} := Z(A+A_1)^T, \qquad X_{14} := A_1(M^T + \bar{h}V),$$

$$X_{15} := ZE^T, \quad X_{16} := ZE^T, \quad X_{17} := ZE_1^T,$$

$$X_{18} := ZE_1^T, \quad X_{22} := -\bar{h}qV, \quad X_{23} := -\bar{h}VA_1^T,$$

$$X_{27} := ME_1^T, \qquad X_{28} := -\bar{h}VE_1^T,$$

$$X_{33} := X_{44} := -V, \qquad X_{47} := (M + \bar{h}V)E_1^T$$

and all the other upper triangular entries are zero. We shall now claim that the condition $\|F(t)\| < 1$ can be replaced with the condition that there exist e_1, e_2, e_3 such that

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \begin{bmatrix} e_1I & e_3I \\ e_3I & e_2I \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} < \begin{bmatrix} D^Tx_1 \\ D^Tx_3 \end{bmatrix}^T \begin{bmatrix} e_1I & e_3I \\ e_3I & e_2I \end{bmatrix} \begin{bmatrix} D^Tx_1 \\ D^Tx_3 \end{bmatrix}, \quad (13)$$

$$\begin{bmatrix} e_1I & e_3I \\ e_3I & e_2I \end{bmatrix} > 0. \quad (14)$$

To prove this claim, we first UDL-decompose the left side of (14) into

$$\begin{bmatrix} e_1I & e_3I \\ e_3I & e_2I \end{bmatrix} = \begin{bmatrix} I & f_1I \\ 0 & I \end{bmatrix} \begin{bmatrix} g_1I & 0 \\ 0 & g_2I \end{bmatrix} \begin{bmatrix} I & f_1I \\ 0 & I \end{bmatrix}^T,$$

where g_1 and g_2 are positive because the UDL-decomposition preserves matrix inertia. Now consider the left side of (13).

$$\begin{aligned} &\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \begin{bmatrix} e_1I & e_3I \\ e_3I & e_2I \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ &= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \begin{bmatrix} I & f_1I \\ 0 & I \end{bmatrix} \begin{bmatrix} g_1I & 0 \\ 0 & g_2I \end{bmatrix} \begin{bmatrix} I & f_1I \\ 0 & I \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ &= \begin{bmatrix} D^Tx_1 \\ D^Tx_3 \end{bmatrix}^T \begin{bmatrix} I & f_1I \\ 0 & I \end{bmatrix} \begin{bmatrix} g_1F^T(t)F(t) & 0 \\ 0 & g_2F^T(t)F(t) \end{bmatrix} \cdot \begin{bmatrix} I & f_1I \\ 0 & I \end{bmatrix}^T \begin{bmatrix} D^Tx_1 \\ D^Tx_3 \end{bmatrix} \\ &< \begin{bmatrix} D^Tx_1 \\ D^Tx_3 \end{bmatrix}^T \begin{bmatrix} I & f_1I \\ 0 & I \end{bmatrix} \begin{bmatrix} g_1I & 0 \\ 0 & g_2I \end{bmatrix} \begin{bmatrix} I & f_1I \\ 0 & I \end{bmatrix}^T \begin{bmatrix} D^Tx_1 \\ D^Tx_3 \end{bmatrix} \\ &= \begin{bmatrix} D^Tx_1 \\ D^Tx_3 \end{bmatrix}^T \begin{bmatrix} e_1I & e_3I \\ e_3I & e_2I \end{bmatrix} \begin{bmatrix} D^Tx_1 \\ D^Tx_3 \end{bmatrix}. \end{aligned}$$

Similarly, $\|F_1(t)\| < 1$ can be replaced with the condition that there exist e_4, e_5, e_6 such that

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}^T \begin{bmatrix} e_4I & e_6I \\ e_6I & e_5I \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} < \begin{bmatrix} D_1^Tx_1 \\ D_1^Tx_3 \end{bmatrix}^T \begin{bmatrix} e_4I & e_6I \\ e_6I & e_5I \end{bmatrix} \begin{bmatrix} D_1^Tx_1 \\ D_1^Tx_3 \end{bmatrix}, \quad (15)$$

$$\begin{bmatrix} e_4 I & e_6 I \\ e_6 I & e_5 I \end{bmatrix} > 0. \quad (16)$$

Now applying the S-procedure[14] to (12), (13), (14), (15) and (16), we can obtain (11). ■

Next, we extend the above result to the robust controller synthesis problem in the following theorem.

Theorem 4 (Robust Controller Design) : If there exist $Z>0$, $V>0$, $q>0$, M, K and scalars $e_1, e_2, \cdots, e_6$ such that

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & ZE_1^T & ZE_1^T \\ Y_{12}^T & Y_{22} & Y_{23} & 0 & 0 & 0 & ME_1^T & -\bar{h}VE_1^T \\ Y_{13}^T & Y_{23}^T & Y_{33} & 0 & 0 & 0 & 0 & 0 \\ Y_{14}^T & 0 & 0 & -V & 0 & 0 & Y_{47} & 0 \\ Y_{15}^T & 0 & 0 & 0 & -e_1 I & -e_3 I & 0 & 0 \\ Y_{16}^T & 0 & 0 & 0 & -e_3 I & -e_2 I & 0 & 0 \\ E_1 Z & E_1 M^T & 0 & Y_{47}^T & 0 & 0 & -e_4 I & -e_6 I \\ E_1 Z & -\bar{h}E_1 V & 0 & 0 & 0 & 0 & -e_6 I & -e_5 I \end{bmatrix} < 0, \quad (17)$$

where

$$Y_{11} := Z(A+A_1)^T + (A+A_1)Z + K^T B^T + BK + e_1 DD^T + e_4 D_1 D_1^T,$$

$$Y_{13} := Z(A+A_1)^T + K^T B^T + e_3 DD^T + e_6 D_1 D_1^T,$$

$$Y_{12} := qZ + A_1 M^T, \quad Y_{14} := A_1(M^T + \bar{h}V),$$

$$Y_{15} := ZE^T + K^T E_b^T, \quad Y_{16} := ZE^T + K^T E_b^T,$$

$$Y_{22} := -\bar{h}qV, \quad Y_{33} := -V + e_2 DD^T + e_5 D_1 D_1^T,$$

$$Y_{23} := -\bar{h}VA_1^T, \quad Y_{47} := (M + \bar{h}V)E_1^T.$$

then the system (9) with the control

$$u(t) = KZ^{-1}x(t)$$

is asymptotically stable for any time-delay h satisfying $0 \le h \le \bar{h}$.

proof : Applying the control (2) to the system (9), the resulting closed-loop system matrix is given by $A_c = A + BG + DF(t)(E + E_b G)$. Then, following the similar procedures as in the proof of Theorem 3 will provide (17). ■

Now, we consider the problem of maximizing the upper bound $\bar{h}$ guaranteed by the stabilizing controller in Theorem 4. Since (17) is not a convex function of the variables concerned, we cannot find in general the global maximum of the problem. However, using the similar procedure in [10], we can obtain the robust stabilizing controller with suboptimal maximum upper bound of the delay $\bar{h}$ as follows.

First, note that if we fix q, (17) has a form of a generalized eigenvalue problem which can be efficiently solved with recently developed numerical algorithms[14]. Also, if we fix Z and V in (17), then it becomes again a quasi-convex problem. Therefore,

we can propose the following algorithm to find the controller with suboptimal maximal delay $\bar{h}$:

Algorithm 1 :

1. Choose an initial $q>0$ arbitrary, for example, $q=1$. Also, select a sufficiently small initial $\bar{h}>0$ so that the condition (17) with the initial q and $\bar{h}$ is feasible.

2. For a given q, find the maximum $\bar{h}$ with incremental search such that (17) is feasible.

3. With fixed Z and V obtained in Step 2, search incrementally the maximum $\bar{h}$ such that (17) is feasible.

4. Exit if the convergence of $\bar{h}$ is attained with a prescribed precision. Otherwise, return to Step 2 with q obtained in Step 3.

Note that the maximum $\bar{h}$ obtained in each step is not smaller than that found in the previous step by the above-mentioned quasi-convexity. The next section presents numerical examples which compare the proposed controller with the previous results.

IV. Numerical examples

Example 1 : Consider the uncertain time-delay system (9) with system matrices

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and uncertainties

$$D = D_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad E = E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_b = 0.$$

The above system is actually delay-independent robust stabilizable with the control (2), that is, the closed-loop system is robust stabilizable for any h satisfying $0 \le h < \infty$. With the existing delaydependent stabilization results[7]-[9], however, one can obtain very conservative results. In fact, the largest upper bound of the delay guaranteeing the closed-loop stability is only $\bar{h} = 0.5557$ in [8]. On the other hand, our delay-dependent stabilization condition (17) using Algorithm 1 gives $\bar{h} = \infty$ from one iteration as shown in Table 1.

In this case, the values of the concerned variables are

$$Z = \begin{bmatrix} 1.7192 & -0.0181 \\ -0.0181 & 2.4056 \end{bmatrix}, \quad V = \begin{bmatrix} 5.5338 & -1.6032 \\ -1.6032 & 20.7008 \end{bmatrix},$$

$$M = 10^7 \times \begin{bmatrix} -0.5534 & 0.1603 \\ 0.1603 & -2.0701 \end{bmatrix}, \quad q = 4.3 \times 10^6,$$

$$K = [-1.2494 \ \ -0.6379], \quad e_1 = 40.3884,$$

$$e_2 = 37.5939, \quad e_3 = 29.9079, \quad e_4 = 41.8153,$$

$$e_5 = 39.1049, \quad e_6 = 31.3899,$$

and a stabilizing controller is given by

$$u(t) = KZ^{-1}x(t) = [-0.7296 \ \ -0.2707]x(t).$$

Table 1. Values of $\bar{h}$ in Algorithm 1 (Example 1).

Iteration number	Step 2	Step 3
1	0.4500	∞

Table 2. Values of $\bar{h}$ in Algorithm 1 (Example 2).

Iteration number	Step 2	Step 3
1	0.2000	0.3350
2	0.3600	0.3800
3	0.3850	0.3880
4	0.3902	0.3902

The next example is concerned with a system which is not delay-independent stabilizable.

Example 2 : Let us consider the uncertain time-delay system (9) with system matrices

$$A=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_1=\begin{bmatrix} -2 & -0.5 \\ 0 & -1 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the same uncertainties as in the above example. In this case, the delay-independent stabilizing controllers [3]-[6] cannot be applied since the pair (A, B) is not stabilizable. The largest time-delay attainable from the known delay-dependent robust stabilization methods in the literature is $\bar{h}$=0.3015 of [8]. As shown in Table. 2, our stabilization condition (17) using Algorithm 1 provides $\bar{h}=0.3902$ after four iterations. Hence, we can see that the robust stabilizing controller of this paper can be less conservative than the existing results.

When $\bar{h}=0.3902$, the values of the concerned variables are as follows.

$$\begin{aligned} Z &= \begin{bmatrix} 3531.7 & 276.1 \\ 276.1 & 325.6 \end{bmatrix},\ V=\begin{bmatrix} 10890.1 & 239.5 \\ 239.5 & 1953.3 \end{bmatrix}, \\ M &= \begin{bmatrix} 215.2 & 28.5 \\ 57.8 & -71.1 \end{bmatrix},\ K=[-236.9\ \ -888.5], \\ q &= 2.0335,\quad e_1=13762.3,\quad e_2=13785.8, \\ e_3 &= 13729.5,\quad e_4=20433.8,\quad e_5=21856.9, \\ e_6 &= -11628.4. \end{aligned}$$

In this case, a stabilizing state-feedback controller is given by

$$u(t) = KZ^{-1}x(t) = [0.1566\ \ -2.8612]x(t).$$

V. Conclusions

This paper addressed the problem of robust stabilization of uncertain time-delay systems. We also proposed an algorithm involving convex optimization to construct a controller with a suboptimal upper bound of the delay such that the system can be stabilized by the controller for all admissible uncertainties. It was shown by numerical examples that the proposed delay-dependent stabilization condition can be less conservative than previous results and even capture the delay-independent stabilizability of the system, which is not possible for the existing results. Extensions are expected to input-delay systems and time-varying delay cases.

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