H^{∞} Control for Linear Systems with Time-Varying Delayed States, Control Inputs, and Measurement Outputs

Eun Tae Jeung, Sung-Ha Kwon, Jong Hae Kim, and Hong Bae Park

Abstract : This paper presents an H^{∞} controller design method for linear systems with time-varying delayed states, inputs, and measurement outputs. Using a Lyapunov functional, the stability for delay systems is discussed independently of time delays. And a sufficient condition for the existence of H^{∞} controllers of n-th order is given in terms of three matrix inequalities. Based on the positive-definite solutions of their matrix inequalities, we briefly explain how to construct H^{∞} controller, which stabilizes time-delay systems independently of delays and guarantees an H^{∞} norm bound.

Keywords : Time-varying delay, H^{∞} control, LMI, output feedback

I. Introduction

Since 1980's, the H^{∞} control problem has been extensively studied. It is well known that the state-space result of Doyle et al. [6] is an efficient and numerically good method for the standard H^{∞} control problem. The existence conditions for an H^{∞} controller were described by two Riccati equations and a spectral radius condition. Gahinet and Apkarian [7] and Iwasaki and Skelton [8] extended to the general H^{∞} control problem using the bounded real lemma (BRL) and linear matrix inequalities(LMIs). Necessary and sufficient conditions for the existence of an H^{∞} controller of any order were given in terms of three LMIs. On the other hand, the study of time-delay systems has received considerable attention over the last few decades because time delay is frequently a source of instability and encountered in various engineering systems such as chemical process, hydraulic, and rolling mill systems, etc. [16]. Recently, many researcher have proposed many results for robust and/or H^{∞} control of time-delay systems, see, e. g., Cheres *et al.* [2], Choi and Chung [3]-[5], Jeung *et al.* [9]-[11], Kim *et al.* [12], Lee *et al.* [13], Li and de Souza [14], Mahmoud and Al-Muthairi [15], [16], Niculescu [18], Shaked and Yaesh [21], and the references therein.

The problem of robust control for linear time-delay systems with parameter uncertainties is considered by Choi and Chung [4] and Kim *et al.* [12] (memoryless state-feedback control via an algebraic Riccati inequality (ARI) approach, and Jeung *et al.* [9] (dynamic output-feedback control via an LMI approach proposed by Gahinet and Apkarian [7] and Iwasaki and Skelton [8]). Li and de Souza [14] tackled the problem of delay-dependent robust stability analysis and control design for a class of uncertain linear systems with delayed state and parameter uncertainty. Also, Lee *et al.* [13] and Choi and Chung [3] extended the memoryless H^{∞} controller design method proposed by Petersen [20] to state delay systems and both state and input delay systems, respectively. Jeung *et al.* [10] and Choi and Chung [5] presented the design method of H^{∞} output feedback controller for state delay systems via an LMI approach. The problem of static H^{∞} output feedback control of linear systems with measurement delay has been considered by Shaked and Yaesh [21]. And the design of memoryless H^{∞} state feedback controllers satisfying some α-stability constraints on the closed-loop poles for linear systems with delayed state has been proposed by Niculescu [18]. However, the problem of H^{∞} control for time-delay systems has not been yet fully investigated, although Jeung *et al.* [11] considered linear systems with constant delayed states, control inputs, and measurement outputs.

The objective of this paper is to present a design method of strictly proper H^{∞} output feedback controllers for linear systems with time-varying delayed states, control inputs, and measurement outputs. After developing a sufficient condition for asymptotic stability independently of delays, we obtain a sufficient condition which stabilizes the closed-loop time-delay system and guarantees an H^{∞} norm bound. And we present a sufficient condition for the existence of an H^{∞} output feedback controller using three matrix inequalities. Their matrix inequalities are LMIs for some variables (X, Y, γ), but not some variables(R_1, R_2, R_3). A simple example to verify our results is illustrated.

Manuscript received : April. 30, 1999., Accepted : Aug. 25, 1999.
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※This work was supported by the Korea Science and Engineering Foundation(KOSEF) through the Machine Tool Research Center at Changwon National University.

II. Problem formulation

Consider the delay system described by the state-space equations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t)+A_{d_1}x(t-d_1(t)) \\ &\quad + B_1w(t)+B_2u(t)+B_{d_2}u(t-d_2(t)) \\ z(t) &= C_1x(t)+D_{11}w(t)+D_{12}u(t) \\ y(t) &= C_2x(t)+C_{d_3}x(t-d_3(t))+D_{21}w(t) \\ x(t) &= 0, \quad t\le 0 \end{aligned} \tag{1}$$

where $x(t)\in \mathbf{R}^n$ is the state, $w(t)\in \mathbf{R}^l$ is the square-integrable disturbance input vector, $u(t)\in \mathbf{R}^m$ is the control, $z(t)\in \mathbf{R}^p$ is the controlled output, $y(t)\in \mathbf{R}^q$ is the measurement output, $d_1(t)$, $d_2(t)$, and $d_3(t)$ are time-varying delays with the following assumption:

$$0\le d_i(t) < \infty, \quad \dot{d}_i(t)\le m_i < 1, \quad i=1,2,3 \tag{2}$$

and A, A_{d_1}, B_1, B_2, B_{d_2}, C_1, C_2, C_{d_3}, D_{11}, D_{12}, and D_{21} are constant matrices with appropriate dimensions.

Also we assume that (A, B_2, C_2) is stabilizable and detectable. As an H^∞ controller of the delay system (1), we consider a strictly proper linear time-invariant dynamic controller with same order of the given plant as follows:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_K\hat{x}(t)+B_Ky(t) \\ u(t) &= C_K\hat{x}(t) \end{aligned} \tag{3}$$

where $\hat{x}(t)\in \mathbf{R}^n$ is the state of the controller and all matrices are constant with proper dimensions. When we apply the control (3) to the delay system (1), the closed-loop system from $w(t)$ to $z(t)$ is given by

$$\begin{aligned} \dot{\xi}(t) &= A_{cl}\xi(t)+A_{cl1}\xi(t-d_1(t)) \\ &\quad + A_{cl2}\xi(t-d_2(t)) \\ &\quad + A_{cl3}\xi(t-d_3(t))+B_{cl}w(t) \\ z(t) &= C_{cl}\xi(t)+D_{cl}w(t) \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\xi(t)=\begin{bmatrix} x(t) \\ \hat{x}(t)\end{bmatrix}, \quad A_{cl}=\begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K\end{bmatrix}, \\ &A_{cl1}=\begin{bmatrix} A_{d_1} & 0 \\ 0 & 0_n\end{bmatrix}, \quad A_{cl2}=\begin{bmatrix} 0_n & B_{d_2}C_K \\ 0 & 0_n\end{bmatrix}, \\ &A_{cl3}=\begin{bmatrix} 0_n & 0 \\ B_KC_{d_3} & 0_n\end{bmatrix}, \quad B_{cl}=\begin{bmatrix} B_1 \\ B_KD_{21}\end{bmatrix}, \\ &C_{cl}=[C_1 \quad D_{12}C_K], \quad D_{cl}=D_{11}. \end{aligned} \tag{5}$$

Here, we introduce the shorthands as follows:

$$K:=\begin{bmatrix} 0_{m\times q} & C_K \\ B_K & A_K\end{bmatrix}, \tag{6}$$

$$\begin{aligned} &A_{00}=\begin{bmatrix} A & 0 \\ 0 & 0_n\end{bmatrix}, \quad A_{10}=\begin{bmatrix} A_{d_1} \\ 0_n\end{bmatrix}, \\ &B_{00}=\begin{bmatrix} B_2 & 0 \\ 0 & I_n\end{bmatrix}, \quad B_{10}=\begin{bmatrix} B_1 \\ 0_{n\times l}\end{bmatrix}, \\ &B_{20}=\begin{bmatrix} B_{d_2} \\ 0_{n\times m}\end{bmatrix}, \quad C_{00}=\begin{bmatrix} C_2 & 0 \\ 0 & I_n\end{bmatrix}, \\ &C_{10}=[C_1 \quad 0_{p\times n}], \quad C_{30}=[C_{d_3} \quad 0_{q\times n}], \\ &D_{10}=[D_{12} \quad 0_{p\times n}], \quad D_{20}=\begin{bmatrix} D_{21} \\ 0_{n\times l}\end{bmatrix}, \\ &E_{10}=[I_n \quad 0_n], \quad E_{20}=[I_m \quad 0_{m\times n}], \\ &E_{30}=[I_q \quad 0_{q\times n}]^T, \end{aligned} \tag{7}$$

then

$$\begin{aligned} &A_{cl}=A_{00}+B_{00}KC_{00}, \quad A_{cl1}=A_{10}E_{10}, \\ &A_{cl2}=B_{20}E_{20}KC_{00}, \quad A_{cl3}=B_{00}KE_{30}C_{30}, \\ &B_{cl}=B_{10}+B_{00}KD_{20}, \quad C_{cl}=C_{10}+D_{10}KC_{00}, \\ &D_{cl}=D_{11}. \end{aligned} \tag{8}$$

Note that (7) involves only plant data and that all matrices of (8) are affine form of the controller data K. We consider the design of a stabilizing controller data K which yields the closed-loop system with H^∞ norm bounded above by a specified number. To help our results, we need to review well-known results.

Lemma 1 : ([1], [17]) : For any symmetric matrix $L=\begin{bmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22}\end{bmatrix}$, the following are equivalent.

i) $L<0$

ii) $L_{11}<0, \quad L_{22}-L_{12}^TL_{11}^{-1}L_{12}<0$

iii) $L_{22}<0, \quad L_{11}-L_{12}L_{22}^{-1}L_{12}^T<0$

Lemma 2 : ([1], [7], [8]) : Consider the problem of finding some matrix K such that

$$\Sigma+\Pi K\Theta^T+\Theta K^T\Pi^T<0. \tag{9}$$

Then (9) is solvable for some K if and only if

$$\Pi_\perp^T\Sigma\Pi_\perp<0, \tag{10}$$

$$\Theta_\perp^T\Sigma\Theta_\perp<0, \tag{11}$$

where $\Pi_\perp$ and $\Theta_\perp$ are orthogonal complements of Π and Θ, respectively.

III. Sufficient conditions of stability and H^∞ norm bound for time-varying delay systems

In this section, we discuss the stability condition of the system (4) and present a sufficient condition which stabilizes the closed-loop system (4) and guarantees the H^∞ norm bound.

Lemma 3 : Consider the time-delay system (4) with $w(t)\equiv 0$. The time-delay system (4) is asymptotically stable for all $d_i(t)\geq 0$, $i=1,2,3$ with the assumption (2), if there exist positive-definite matrices P, R_1, R_2, and R_3 such that

$$\tilde{Q}=\begin{bmatrix} Q_{11} & PA_{10} & PB_{20} & PB_{00}KE_{30} \\ A_{10}^T P & -\tilde{R}_1 & 0 & 0 \\ B_{20}^T P & 0 & -\tilde{R}_2 & 0 \\ E_{30}^T K^T B_{00}^T P & 0 & 0 & -\tilde{R}_3 \end{bmatrix} < 0 \quad (12)$$

where

$$\begin{aligned} Q_{11} &= A_{cl}^T P + PA_{cl} + E_{10}^T R_1 E_{10} \\ &\quad + C_{00}^T K^T E_{20}^T R_2 E_{20} K C_{00} + C_{30}^T R_3 C_{30}, \end{aligned} \quad (13)$$

$$\tilde{R}_i = (1-m_i)R_i, \quad i=1,2,3. \quad (14)$$

Proof : Let's define a Lyapunov functional $V(\xi,t)$ as follows:

$$\begin{aligned} V(\xi,t) &= \xi^{T(t)} P\xi(t) + \int_{t-d_1(t)}^{t} \xi^T(\tau) E_{10}^T R_1 E_{10}\xi(\tau)d\tau \\ &\quad + \int_{t-d_2(t)}^{t} \xi^T(t) C_{00}^T K^T E_{20}^T R_2 E_{20} K C_{00}\xi(\tau)d\tau \\ &\quad + \int_{t-d_3(t)}^{t} \xi^T(\tau) C_{30}^T R_3 C_{30}\xi(\tau)\,d\tau, \end{aligned} \quad (15)$$

then the corresponding Lyapunov derivative is given by

$$\frac{dV(\xi,t)}{dt} = \eta^T(t)Q\eta(t) \quad (16)$$

where

$$\eta(t)=\begin{bmatrix} \xi(t) \\ E_{10}\xi(t-d_1(t)) \\ E_{20}KC_{00}\xi(t-d_2(t)) \\ C_{30}\xi(t-d_3(t)) \end{bmatrix}, \quad (17)$$

$$Q=\begin{bmatrix} Q_{11} & PA_{10} & PB_{20} & PB_{00}KE_{30} \\ A_{10}^T P & -\overline{R_1} & 0 & 0 \\ B_{20}^T P & 0 & -\overline{R_2} & 0 \\ E_{30}^T K^T B_{00}^T P & 0 & 0 & -\overline{R_3} \end{bmatrix}, \quad (18)$$

$$\overline{R_i} = (1-\dot{d}_i(t))R_i, \quad i=1,2,3. \quad (19)$$

From the assumption (2)

$$\frac{dV(\xi,t)}{dt} = \eta^T(t)Q\eta(t) \leq \eta^T(t)\tilde{Q}\eta(t). \quad (20)$$

Therefore the time-delay system (4) is asymptotically stable under the condition (12). ■

Note that there exist many sufficient conditions of the stability for the time-delay system (4), because we can obtain another sufficient condition according to the selection of Lyapunov functional. The sufficient condition in lemma 3 is necessary for lemma 4.

Lemma 4 : Consider the time-delay system (4) and suppose that $\sigma_{\max}(D_{cl}) < \gamma$. The time-delay system (4) is asymptotically stable and $\|z(t)\|_2 < \gamma\|w(t)\|_2$, if there exist positive-definite matrices P, R_1, R_2, and R_3 such that

$$\begin{bmatrix} S & PB_{cl} & C_{cl}^T & PA_{10} & PB_{20} & PB_{00}KE_{30} \\ B_{cl}^T P & -\gamma I & D_{cl}^T & 0 & 0 & 0 \\ C_{cl}^T & D_{cl} & -\gamma I & 0 & 0 & 0 \\ A_{10}^T P & 0 & 0 & -\tilde{R}_1 & 0 & 0 \\ B_{20}^T P & 0 & 0 & 0 & -\tilde{R}_2 & 0 \\ E_{30}^T K^T B_{00}^T P & 0 & 0 & 0 & 0 & -\tilde{R}_3 \end{bmatrix} < 0 \quad (21)$$

where

$$\begin{aligned} S &= A_{cl}^T P + PA_{cl} + E_{10}^T R_1 E_{10} \\ &\quad + C_{00}^T K^T E_{20}^T R_2 E_{20} K C_{00} + C_{30}^T R_3 C_{30}. \end{aligned} \quad (22)$$

Proof : The positive-definite matrices P, R_1, R_2, and R_3 which satisfy (21) also satisfy (12). In order to establish the upper bound $\gamma\|w(t)\|_2$ for $\|z(t)\|_2$, we introduce

$$J=\int_0^\infty \{\gamma^{-1}z^T(t)z(t) - \gamma w^T(t)w(t) + \dot{V}(\xi(t),t)\}dt. \quad (23)$$

From the initial condition of the state in (4)

$$J=\int_0^\infty \{\gamma^{-1}z^T(t)z(t) - \gamma w^T(t)w(t)\}dt + V(\xi(\infty),\infty) \quad (24)$$

because $V(\xi(0),0)=0$. Therefore the proof is completed if $J<0$. From lemma 1, the inequality (21) is equivalent to

$$\tilde{\Phi} := \begin{bmatrix} \Phi_{11} & PA_{10} & PB_{20} & PB_{00}KE_{30} & \Phi_{15} \\ A_{10}^T P & -\tilde{R}_1 & 0 & 0 & 0 \\ B_{20}^T P & 0 & -\tilde{R}_2 & 0 & 0 \\ E_{30}^T K^T B_{00}^T P & 0 & 0 & -\tilde{R}_3 & 0 \\ \Phi_{15}^T & 0 & 0 & 0 & -\Phi_{55} \end{bmatrix} < 0. \quad (25)$$

where

$$\Phi_{11} = S + \gamma^{-1}C_{cl}^T C_{cl},$$
$$\Phi_{15} = PB_{cl} + \gamma^{-1}C_{cl}^T D_{cl},$$
$$\Phi_{55} = \gamma I - \gamma^{-1}D_{cl}^T D_{cl}.$$

The performance measure (23) can be rewritten as follows:

$$J = \int_0^\infty \tilde{\eta}^T(t)\Phi\tilde{\eta}(t)\,dt \quad (26)$$

where

$$\tilde{\eta}(t)=\begin{bmatrix} \xi(t) \\ E_{10}\xi(t-d_1(t)) \\ E_{20}KC_{00}\xi(t-d_2(t)) \\ C_{30}\xi(t-d_3(t)) \\ w(t) \end{bmatrix}, \quad (27)$$

$$\Phi := \begin{bmatrix} \Phi_{11} & PA_{10} & PB_{20} & PB_{00}KE_{30} & \Phi_{15} \\ A_{10}^T P & -\overline{R_1} & 0 & 0 & 0 \\ B_{20}^T P & 0 & -\overline{R_2} & 0 & 0 \\ E_{30}^T K^T B_{00}^{TP} & 0 & 0 & -\overline{R_3} & 0 \\ \Phi_{15}^T & 0 & 0 & 0 & -\Phi_{55} \end{bmatrix}. \quad (28)$$

We can easily obtain the relation $\Phi \le \tilde{\Phi}$, so $J<0$. ■

The matrix inequality (21) in lemma 4 is similar to the matrix inequality of BRL for non-delay systems except terms related time delays. That is, lemma 4 presents a sufficient condition that the time-delay system (4) is asymptotically stable independently of time delays and the H^∞ norm of the time-delay system is less than given $\gamma > 0$.

IV. Sufficient condition for the existence of H^∞ controllers

By applying the result of lemma 4 developed in the previous section, we present a sufficient condition for the existence of H^∞ controllers of the linear time-delay system (1) and explain how to construct H^∞ controllers.

Using lemma 1, the condition (21) can be changed to

$$\begin{bmatrix} A_{cl}^TP+PA_{cl} & PB_{cl} & C_{cl}^T & PA_{10} & PB_{20} & PB_{00}KE_{30} & E_{10}^T & C_{00}^TK^TE_{20}^T & C_{30}^T \\ B_{cl}^TP & -\gamma I & D_{cl}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{cl} & D_{cl} & -\gamma I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{10}^TP & 0 & 0 & -\tilde{R}_1 & 0 & 0 & 0 & 0 & 0 \\ B_{20}^TP & 0 & 0 & 0 & -\tilde{R}_2 & 0 & 0 & 0 & 0 \\ E_{30}^TK^TB_{00}^TP & 0 & 0 & 0 & 0 & -\tilde{R}_3 & 0 & 0 & 0 \\ E_{10} & 0 & 0 & 0 & 0 & 0 & -R_1^{-1} & 0 & 0 \\ E_{20}KC_{00} & 0 & 0 & 0 & 0 & 0 & 0 & -R_2^{-1} & 0 \\ C_{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_3^{-1} \end{bmatrix} < 0. \tag{29}$$

equivalently, this matrix inequality (29) with the notation (8) can be represented as

$$G+UKV^T+VK^TU^T<0 \tag{30}$$

where

$$U=[B_{00}^TP \;\; 0 \;\; D_{10}^T \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; E_{20}^T \;\; 0]^T, \tag{31}$$

$$V=[C_{00} \;\; D_{20} \;\; 0 \;\; 0 \;\; 0 \;\; E_{30} \;\; 0 \;\; 0 \;\; 0]^T, \tag{32}$$

and

$$G=\begin{bmatrix} A_{00}^TP+PA_{00} & PB_{10} & C_{10}^T & PA_{10} & PB_{20} & 0 & E_{10}^T & 0 & C_{30}^T \\ B_{10}^TP & -\gamma I & D_{11}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{10} & D_{11} & -\gamma I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{10}^TP & 0 & 0 & -\tilde{R}_1 & 0 & 0 & 0 & 0 & 0 \\ B_{20}^TP & 0 & 0 & 0 & -\tilde{R}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\tilde{R}_3 & 0 & 0 & 0 \\ E_{10} & 0 & 0 & 0 & 0 & 0 & -R_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_2^{-1} & 0 \\ C_{30} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_3^{-1} \end{bmatrix}. \tag{33}$$

The lemma 2 cannot be directly applied to (30) because K is a special matrix as (6). Through some matrix manipulations, the inequality (30) will be changed to a useful form. We partition U and V as

$$U=\begin{bmatrix} P\begin{bmatrix} B_2 \\ 0 \end{bmatrix} & P\begin{bmatrix} 0 \\ I \end{bmatrix} \\ 0 & 0 \\ D_{12} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \end{bmatrix}=[U_1 \;\; U_2], \tag{34}$$

$$V=\begin{bmatrix} C_2^T & 0 \\ 0 & I \\ D_{21}^T & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}=[V_1 \;\; V_2]. \tag{35}$$

Substituting (3), (34), and (35) into (30), then

$$\hat{G}+U_2[B_K \;\; A_K]V^T+V[B_K \;\; A_K]^TU_2^T<0 \tag{36}$$

where

$$\hat{G} = G+U_1C_KV_2^T+V_2C_K^TU_1^T. \tag{37}$$

From lemma 2, the inequality (36) is solvable for some $[B_K \;\; A_K]$ if and only if

$$(U_2)_\perp^T\hat{G}(U_2)_\perp<0, \tag{38}$$

$$V_\perp^T\hat{G}V_\perp<0 \tag{39}$$

where $(U_2)_\perp$ and $V_\perp$ are orthogonal complements of U_2 and V, respectively. To simplify the conditions (38) and (39), we partition P and P^{-1} as

$$P=\begin{bmatrix} Y & N \\ N^T & ? \end{bmatrix}, \quad P^{-1}=\begin{bmatrix} X & M \\ M^T & ? \end{bmatrix} \tag{40}$$

where $X, Y \in \mathbf{R}^{n\times n}$, $M, N \in \mathbf{R}^{n\times n}$, and ? means irrelevant. And we can choose orthogonal complements of U_2 and V as follows:

$$(U_2)_\perp=\begin{bmatrix} P^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}, \tag{41}$$

$$V_\perp=\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ -C_2 & -D_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}. \tag{42}$$

Substituting (41) and (42) into the inequalities (38) and (39) gives

$$\begin{bmatrix} XA^T+AX+\tilde{C}_K^TB_2^T+B_2\tilde{C}_K & B_1 & XC_1^T+\tilde{C}_K^TD_{12}^T & A_{d_1} & B_{d_2} & 0 & X & \tilde{C}_K^T & XC_{d_3}^T \\ B_1^T & -\gamma I & D_{11}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ C_1X+D_{12}\tilde{C}_K & D_{11} & -\gamma I & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{d_1}^T & 0 & 0 & -\tilde{R}_1 & 0 & 0 & 0 & 0 & 0 \\ B_{d_2}^T & 0 & 0 & 0 & -\tilde{R}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\tilde{R}_3 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & -R_1^{-1} & 0 & 0 \\ \tilde{C}_K & 0 & 0 & 0 & 0 & 0 & 0 & -R_2^{-1} & 0 \\ C_{d_3}X & & 0 & 0 & 0 & 0 & 0 & 0 & -R_3^{-1} \end{bmatrix} < 0. \tag{43}$$

$$\begin{bmatrix} A^TY+YA-C_2^T\widetilde{R}_3C_2 & YB_1-C_2^T\widetilde{R}_3D_{21} & C_1^T & YA_{d_1} & YB_{d_2} & I & 0 & C_{d_3}^T \\ B_1^TY-D_{21}^T\widetilde{R}_3C_2 & -\gamma I-D_{21}^T\widetilde{R}_3D_{21} & D_{11}^T & 0 & 0 & 0 & 0 & 0 \\ C_1 & D_{11} & -\gamma I & 0 & 0 & 0 & 0 & 0 \\ A_{d_1}^TY & 0 & 0 & -\widetilde{R}_1 & 0 & 0 & 0 & 0 \\ B_{d_2}^TY & 0 & 0 & 0 & -\widetilde{R}_2 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & -R_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_2^{-1} & 0 \\ C_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & -R_3^{-1} \end{bmatrix} < 0, \tag{44}$$

where

$$\widetilde{C_K} = C_KM^T. \tag{45}$$

Using lemma 2, the above inequalities are simplified to

$$\begin{bmatrix} X_{11} & B_1 & XC_1^T+\widetilde{C}_K^TD_{12}^T & X & \widetilde{C}_K^T & XC_{d_3}^T \\ B_1^T & -\gamma I & D_{11}^T & 0 & 0 & 0 \\ C_{1X}+D_{12}\widetilde{C_K} & D_{11} & -\gamma I & 0 & 0 & 0 \\ X & 0 & 0 & -R_1^{-1} & 0 & 0 \\ \widetilde{C_K} & 0 & 0 & 0 & -R_2^{-1} & 0 \\ C_{d_3}X & & 0 & 0 & 0 & -R_3^{-1} \end{bmatrix} < 0, \tag{46}$$

$$\begin{bmatrix} Y_{11} & YB_1-C_2^T\widetilde{R}_3D_{21} & C_1^T & YA_{d_1} & YB_{d_2} \\ B_1^TY-D_{21}^T\widetilde{R}_3C_2 & -\gamma I-D_{21}^T\widetilde{R}_3D_{21} & D_{11}^T & 0 & 0 \\ C_1 & D_{11} & -\gamma I & 0 & 0 \\ A_{d_1}^TY & 0 & 0 & -\widetilde{R}_1 & 0 \\ B_{d_2}^TY & 0 & 0 & 0 & -\widetilde{R}_2 \end{bmatrix} < 0 \tag{47}$$

where

$$X_{11} = XA^T+AX+\widetilde{C_K}^TB_2^T + B_2\widetilde{C_K}+A_{d_1}\widetilde{R}_1^{-1}A_{d_1}^T+B_{d_2}\widetilde{R}_2^{-1}B_{d_2}^T,$$

$$Y_{11} = A^TY+YA-C_2^T\widetilde{R}_3C_2+R_1+C_{d_3}^TR_3C_{d_3}.$$

Theorem 1 : If there exist positive-definite matrices R_1, R_2, R_3, X, and Y satisfying (46), (47), and

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \tag{48}$$

then there exist γ-suboptimal H^∞ controllers of order n for the time-delay system (1).

Proof : There exists a positive-definite matrix P satisfying (40) if and only if the inequality $X-Y^{-1}>0$ holds. This inequality is equivalent to (47). The rest of the proof is mentioned before. ■

Note that theorem 1 does not present the computation of the controller itself, but existence conditions of H^∞ controllers. The inequality (45) is an LMI for X, $\widetilde{C_K}$, R_1^{-1}, R_2^{-1}, R_3^{-1}, and γ and (46) is an LMI for Y, R_1, R_2, R_3, and γ. However (46) and (47) are not LMIs in terms of R_1, R_2, and R_3, simultaneously. Unfortunately, it is not yet known an algorithm solving them at the same time. Here we introduce a procedure for designing H^∞ controllers as follows:

[Procedure]

(P1) Let $\gamma = \gamma_0$.

(P2) Find the regions

$$\overline{R} = \{R_1, R_2, R_3 \mid X>0, (46)\},$$

$$\hat{R} = \{R_1, R_2, R_3 \mid Y>0, R_3>0, (47)\}.$$

(P3) Obtain the intersection of $\overline{R}$ and $\hat{R}$,

$$\tilde{R} = \overline{R} \cap \hat{R}.$$

If $\tilde{R}$ is empty, increase γ and return (P2). If not, go to next step.

(P4) Compute $X>0$, $Y>0$, and $\widetilde{C_K}$ such that $\min_{(R_1,R_2,R_3)\in\tilde{R}} \gamma$ subject to (46)−(48).

If $X>0$, $Y>0$, and $\widetilde{C_K}$ exist, go to next step. If not, increase γ and return (P2).

(P5) Compute two nonsingular matrices $M, N \in \mathbf{R}^{n\times n}$ such that

$$MN^T = I-XY \tag{49}$$

and P from

$$\begin{bmatrix} Y & I \\ N^T & 0 \end{bmatrix} = P\begin{bmatrix} I & X \\ 0 & M^T \end{bmatrix}. \tag{50}$$

(P6) Find C_K from (45) and $[B_K \ \ A_K]$ satisfying (36).

Remark 1 : In the procedure (P1), (P2), and (P3), the set of solution existence $\tilde{R}$ widen as γ is increased, and the existence of the set $\tilde{R}$ does not imply that the matrix inequalities, (46)-(48), are solvable, but a necessary condition for the solvability of (46)-(48).

Remark 2 : The minimization of the procedure 4 is not convex problem in terms of R_1, R_2, and R_3because the inequalities (46) and (47) are not LMIs in terms of them. However, it is not difficult to find the minimum γ because the computation can be executed within the searching regions of R_1, R_2, and R_3 obtained in the procedure (P3).

V. An example

Consider the time-delay system (1) with

$$A=\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_{d_1}=\begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.1 \end{bmatrix},$$

$$B_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2=\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{d_2}=\begin{bmatrix} 0.2 \\ 0 \end{bmatrix},$$

$$C_1=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_{11}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_{12}=\begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix},$$

$$C_2=[1 \ \ 3], \quad C_{d_3}=[0.5 \ \ 0.2], \quad D_{21}=0.5,$$

$$d_1(t)=0.7\sin t+3, \quad d_2(t)=0.8\cos t+3,$$

$$d_3(t)=0.6\sin t+2.$$

Let $\gamma=1$ and let $R_1=aI_2$ for simplicity. From the procedure (P2) and (P3), we can obtain the set

$$\tilde{R} = \{R_1,R_2,R_3 \mid R_1=aI_2, \ 0.2112<a<6.2233, \ 0.0425<R_2<4.0144, \ 0.2749<R_3<41.9436\}.$$

The minimization of the procedure (P4) is attained at

$$R_1=1.7142I_2,\ R_2=0.2907,\ R_3=8.1016,$$

then the minimum value of γ is 0.8182 and X, Y, and $\widetilde{C_K}$ are

$$X=\begin{bmatrix}1.0036 & -0.4816\\ -0.4816 & 0.4686\end{bmatrix},$$

$$Y=\begin{bmatrix}4.0858 & 1.4602\\ 1.4602 & 4.3634\end{bmatrix},$$

$$\widetilde{C_K}=[0.0016\ \ -1.6798].$$

One pair of solutions satisfying (49) is

$$M=\begin{bmatrix}-0.8816 & 0.4720\\ 0.4720 & 0.8816\end{bmatrix},\quad N=\begin{bmatrix}2.7192 & 0\\ -0.7219 & -0.0011\end{bmatrix}$$

and the positive-definite solution of (50) is

$$P=\begin{bmatrix}4.0858 & 1.4602 & 2.7192 & 0\\ 1.4602 & 4.3634 & -0.7219 & -0.0011\\ 2.7192 & -0.7219 & 3.4901 & 0.0007\\ 0 & -0.0011 & 0.0007 & 0.0002\end{bmatrix}.$$

From (45)

$$C_K=[-0.7942\ \ -1.4802],$$

and $[B_K\ \ A_K]$ satisfying (36) is

$$[B_K\ A_K]=\left[\begin{array}{c|cc}-2.0940 & -1.4380 & 0.8061\\ 7174.7176 & -2085.4735 & -6440.0746\end{array}\right].$$

VI. Conclusions

In this paper, we have developed an H^∞ output feedback controller design method for linear systems with delayed states, inputs, and measurement outputs. We have proposed a sufficient condition for the existence of H^∞ output feedback controllers of n-th order in terms of three LMIs for some variables. Based on the positive-definite solutions of three LMIs, the proposed H^∞ controller guarantees not only asymptotic stability but also the H^∞ norm bound for linear time-delay systems independently of the delays. An illustrative example has been given to demonstrate our results.

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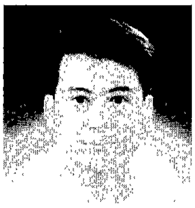

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