Routh Approximants with Arbitrary Order

Younseok Choo and Dongmin Kim

Abstract : It has been pointed out in the literature that the Routh approximation method for order reduction has limitations in treating transfer functions with the denominator-numerator order difference not equal to one. The purpose of this paper is to present a new algorithm based on the Routh approximation method that can be applied to general rational transfer functions, yielding reduced models with arbitrary order.

Keywords : routh approximation method, transfer function, order reduction

I. Introduction

Consider a linear time-invariant system with the transfer function

$$G(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n} \tag{1}$$

where $m < n$, and all coefficients are real numbers. Let $q < p \leq n$ and suppose we want to approximate the system (1) by a model with the transfer function

$$G_{qp}(s) = \frac{d_0 + d_1 s + d_2 s^2 + \cdots + d_q s^q}{c_0 + c_1 s + c_2 s^2 + \cdots + c_p s^p} \tag{2}$$

so that two systems behave as closely as possible in the low-frequency ranges.

During the past three decades such an approximation problem has been an ample area of research, and numerous methods have been reported in the literature. The Routh approximation method introduced by Hutton and Friedland [1] has proven to be one of the most powerful tools for model reduction due to its computational simplicity and efficiency. Furthermore the method has played a key role in solving the instability problem encountered in other approaches including the continued fraction expansion [2]. In other words it yields a stable reduced model whenever a high-order system is stable.

However it has been pointed out in the literature that the Routh approximation method has limitations in treating transfer functions with the denominator-numerator order difference not equal to one [3],[4]. To ameliorate such problems, Langholz and Feinmesser [4] proposed a technique using the Routh approximation method with a modified β-table. But their method still has a shortcoming in that the order of numerator of the reduced model is limited to the order of numerator of the original system, which may inhibit one to get better approximants.

Manuscript received : Feb. 2, 1998., Accepted : Feb. 14, 1999.
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The purpose of this paper is to present a new algorithm for computing reduced models based on the Routh approximation method. Compared to that of [4], the algorithm of this paper has an advantage such that the maximum order of the numerator of the reduced model is not restricted to that of the high-order system.

This paper is organized as follows. In the next two sections, we briefly review the Routh approximation method [1] and its modified form given in [4]. Our main results are presented in Section IV with two numerical examples, and the paper is concluded in Section V.

II. Routh approximation method

As noted in [3] and [4], the Routh approximation in [1] was originally meant to treat the problems with $m = n-1$ and $q = p = -1$ in (1) and (2). For this case, the method can be summarized as follows [1],[5]:

Step 1 : Construct the following α-table

$$\begin{array}{lllll}
a_{10}=a_0 & a_{11}=a_1 & a_{12}=a_2 & a_{13}=a_3 & \cdots \\
a_{11} & 0 & a_{13} & 0 & \cdots \\
a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\
a_{21} & 0 & a_{23} & 0 & \cdots \\
a_{30} & a_{31} & a_{32} & \cdots & \\
a_{31} & 0 & \cdots & & \\
\vdots & \vdots & & & \\
a_{n0} & a_{n1} & & & \\
a_{n1} & & & &
\end{array} \tag{3}$$

[α-Table]

and compute the α parameters by $\alpha_i = a_{i0}/a_{i1}$, $i = 1, 2, \cdots, n$ (the Routh array may be more convenient than the α-table for this purpose. But (3) is used here for the ease of presentation). In the above table, the first row is formed by the denominator coefficients of $G(s)$ starting from the constant term. The entries of all other odd rows are determined by the usual cross multiplication rule, i.e., for $i = 2, 3, \cdots, n$

$$a_{ij} = a_{i-1,j+1} \quad (j \text{ even})$$

$$a_{ij} = a_{i-1,j+1} - \alpha_{i-1} a_{i-1,j+2} \quad (j \text{ odd})$$

Step 2 : Construct the β-table as follows:

$$\begin{array}{ccccc} b_{10}=b_0 & b_{11}=b_1 & b_{12}=b_2 & b_{13}=b_3 & \cdots \\ a_{11} & 0 & a_{13} & 0 & \cdots \\ b_{20} & b_{21} & b_{22} & b_{23} & \cdots \\ a_{21} & 0 & a_{23} & 0 & \cdots \\ \vdots & \vdots & \vdots & & \\ b_{n0} & & & & \\ a_{n1} & & & & \end{array} \tag{4}$$

[β-Table]

The numerator coefficients of $G(s)$ form the first row and all even rows are copied from the α-table. Again all items of odd rows are determined by the cross multiplication rule. Obtain the β parameters by $\beta_i = b_{i0}/a_{i1},\ i=1,2,\cdots,n$.

Step 3 : Compute $A_k(s)$ and $B_k(s)$ recursively by [5]

$$A_k(s) = s^2 A_{k-2}(s) + \alpha_k A_{k-1}(s) \tag{5}$$

$$B_k(s) = \beta_k s^{k-1} + s^2 B_{k-2}(s) + \alpha_k B_{k-1}(s) \tag{6}$$

for $k=1,2,\cdots,p$ using the initial conditions $B_{-1}(s) = B_0(s) = 0$, $A_{-1}(s)=1/s$ and $A_0(s)=1$. Then the pth-order approximant of $G(s)$ is given by

$$G_{p-1,p}(s) = \frac{B_p(s)}{A_p(s)}$$

III. Modified routh approximation method

Now suppose that m, n in (1) and q, p in (2) are given arbitrarily with $m<n$ and $q<p\leq n$. The Routh approximation method cannot be applied in this case, and so Langholz and Feinmesser [4] suggested a modified method using a different β-table. In their method, the denominator $A_p(s)$ of $G_{qp}(s)$ is the same as that of [1]. To obtain the numerator $B_{q+1}(s)$, we first build a modified β-table as follows:

$$\begin{array}{ccccc} b_{10}=b_0 & b_{11}=b_1 & b_{12}=b_2 & b_{13}=b_3 & \cdots \\ a_{n-m,1} & 0 & a_{n-m,3} & 0 & \cdots \\ b_{20} & b_{21} & b_{22} & b_{23} & \cdots \\ a_{n-m+1,1} & 0 & a_{n-m+1,3} & 0 & \cdots \\ \vdots & \vdots & \vdots & & \\ b_{m0} & & & & \\ a_{n1} & & & & \end{array} \tag{7}$$

[Modified β-Table]

From the above table, we can determine the β parameters by

$$\beta_k = b_{k0}/a_{j1}$$

for $k=1,\cdots,m+1$, where $j=n-m+k-1$. Next compute $B_{q+1}(s)$ by recursively applying (6) from $k=1$ to $q+1$ with α_k replaced by α_j ($j=n+m+k-1$). Then the reduced model is given by

$$G_{qp}(s) = K_{qp}\frac{B_{q+1}(s)}{A_p(s)} \tag{8}$$

where the constant K_{qp} is multiplied so that dc gain of the original system is preserved in the low-order model. Note that $K_{k-1,k}=1$ for all $k\geq 1$ since

$$G_{k-1,k}(0) = \beta_1/\alpha_1 = G(0).$$

It is not difficult to see that we can compute at most ($m+1$) β parameters from the modified β-table, which in turn implies the use of above algorithm is limited to the cases where the order of numerator of the reduced model is not greater than the order of numerator of the original system. In approxi- mation point of view, it is a strong restriction since higher order reduced models, in general, may behave more closely to original systems than lower order ones.

IV. New algorithm

We now propose a new algorithm that can overcome the limitation discussed in the previous section. Even if $m\neq n-1$, the α- and β-tables of Steps 1-2 in Section II can be built again. A caution should be taken in this case. When even rows are copied from the α-table in Step 2, no element should be omitted. Using the computed values of α parameters, form the D-table given below by expanding $\alpha_1,\cdots,\alpha_p$ from bottom to top, and from left to right (i.e., through the reverse procedure of (3)) [6]. The expansion always starts from 1 placed at the left and bottom of the table. Then the denominator coefficients are determined from the first row by setting $c_i = e_{1i}$, $i=0,1,\cdots,p$.

$$\begin{array}{cccccc} e_{10} & e_{11} & e_{12} & e_{13} & \cdots & e_{1p}=1 \\ e_{11} & 0 & e_{13} & 0 & \cdots & \\ \vdots & \vdots & \vdots & & & \\ e_{q-1,0} & e_{q+1,1} & e_{q+1,2} & e_{q+1,3} & \cdots & \\ e_{q+1,1} & 0 & e_{q+1,3} & 0 & \cdots & \\ \vdots & \vdots & \vdots & & & \\ \alpha_{p-1}\alpha_p & \alpha_p & 1 & & & \\ \alpha_p & 0 & & & & \\ \alpha_p & 1 & & & & \\ 1 & & & & & \end{array} \tag{9}$$

[D-Table]

Similarly, expanding $\beta_1,\cdots,\beta_{q+1}$ from bottom to top, and from left to right yields the N-table

$$\begin{array}{cccccc} f_{10} & f_{11} & f_{12} & f_{13} & \cdots & f_{1q} \\ e_{11} & 0 & e_{13} & 0 & \cdots & \\ \vdots & \vdots & \vdots & & & \\ \beta_q e_{q1} & \beta_{q+1} e_{q+1,1} & & & & \\ e_{q1} & 0 & & & & \\ \beta_{q+1} e_{q+1,1} & & & & & \\ e_{q-1,1} & & & & & \end{array} \tag{10}$$

[N Table]

All even rows in (10) are partially moved from the D-table given in (9). Note that the number of elements moved decreases from $q+1$ to 1. The numerator coefficients of the reduced model are, then, given by $d_i = f_{1i},\ i = 0, 1, \cdots, q$.

Example 1 : Consider a system with the transfer function

$$G(s) = \frac{900 + 248s}{120 + 180s + 102s^2 + 18s^3 + s^4} \tag{11}$$

It is wanted to find a third-order approximant, i.e. $p = 3$. The modified Routh approximation in [4] can yield $G_{03}(s)$ and $G_{13}(s)$ only. Between the two, we compute $G_{13}(s)$. Three α parameters are easily computed from the α-table by

$$\alpha_1 = 0.6667,\quad \alpha_2 = 2,\quad \alpha_3 = 5.625$$

On the other hand, two β parameters computed from the modified β-table are

$$\beta_1 = 56.25,\quad \beta_2 = 248$$

Then $G_{13}(s)$ is given by

$$G_{13}(s) = \frac{56.25 + 15.5s}{7.5 + 11.25s + 6.2917s^2 + s^3} \tag{12}$$

It is possible to get more accurate model $G_{23}(s)$ with the same denominator using the algorithm of this paper. To determine $G_{23}(s)$, it is necessary to calculate three β parameters, i.e.,

$$\beta_1 = 5,\quad \beta_2 = 124/45,\quad \beta_3 = -45/8$$

Then, from (9) and (10), we have

$$G_{23}(s) = \frac{56.25 + 15.5s - 0.625s^2}{7.5 + 11.25s + 6.2917s^2 + s^3} \tag{13}$$

The impulse responses and Nyquist diagrams are compared in Fig. 1 and Fig. 2, respectively. The integral-squared error (ISE) of impulse responses for (12) and (13) are 0.075 and 0.0229, respectively. It is

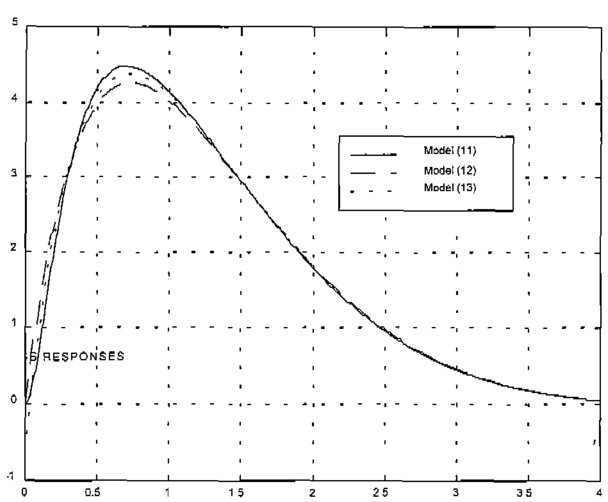


Fig. 1. Comparison of impulse responses.

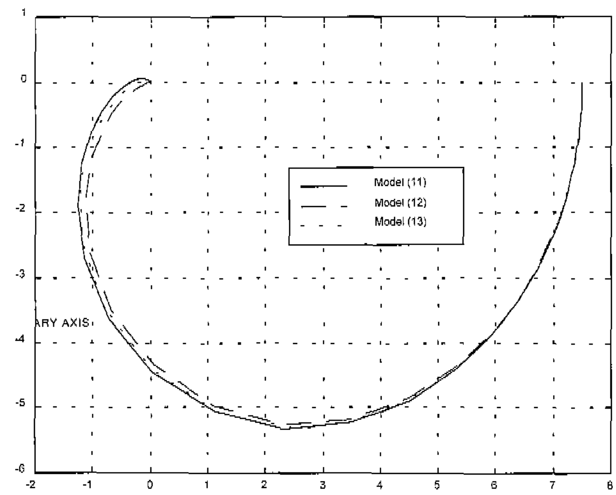


Fig. 2. Comparison of Nyquist diagrams.

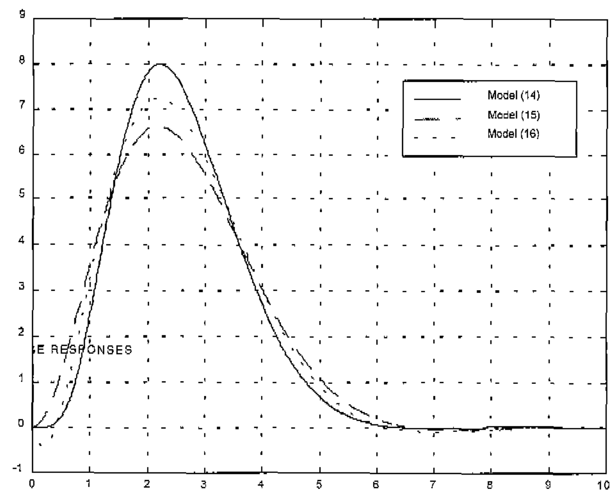


Fig. 3. Comparison of impulse responses.

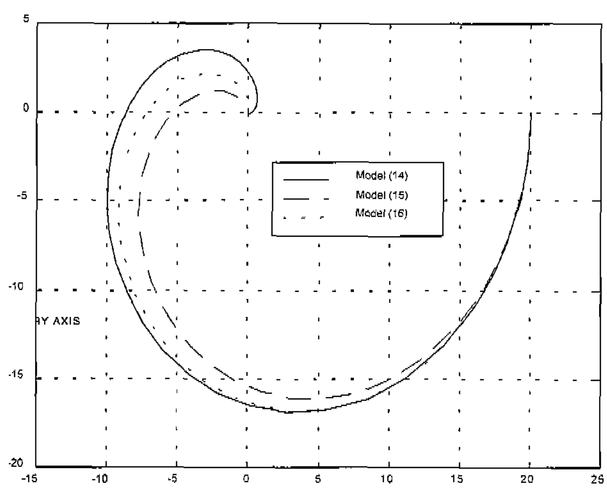


Fig. 4. Comparison of Nyquist diagrams.

obvious that the reduced model (13) approximates (11) more accurately than (12).

Example 2 : An eighth-order system has the transfer function

$$G(s) = \frac{192000 + 80000s}{D(s)} \tag{14}$$

where

$$D(s) = 9600 + 28880s + 37492s^2 + 27470s^3 + 11870s^4 + 3017s^5 + 437s^6 + 33s^7 + s^8$$

Let $p=4$. Then the method of [4] can yield $G_{04}(s)$ and $G_{14}(s)$ only. For example, $G_{14}(s)$ is given by

$$G_{14}(s) = \frac{29.9937 + 12.4967s}{1.4996 + 4.5115s + 5.6208s^2 + 3.5809s^3 + s^4} \tag{15}$$

On the other hand, $G_{24}(s)$ computed by the proposed algorithm is

$$G_{24}(s) = \frac{29.9937 + 12.4967s - 4.7231s^2}{1.4996 + 4.5115s + 5.6208s^2 + 3.5809s^3 + s^4} \tag{16}$$

The impulse responses and Nyquist diagrams are compared in Fig. 3 and Fig. 4, respectively. In this case, ISE of impulse responses for (15) and (16) are 3.2764 and 1.0526, respectively. Again the reduced model (16) approximates (14) much more accurately than (15).

V. Conclusions

This paper presented a new algorithm based on the Routh approximation method with focus placed on the numerator of the reduced model. The maximum order of the numerator of the reduced model is not limited to that of the original system, which extends an existing result. Two examples were given for illustration.

References

[1] M. F. Hutton and B. F. Friedland, "Routh approximations for reducing order of linear, timeinvariant systems," *IEEE Trans. Automat. Control.*, vol. AC-20, pp. 329-337, 1975.

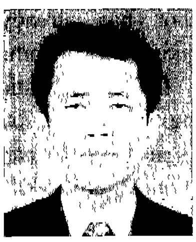
[2] C. F. Chen and L. S. Shieh, "A novel approach to linear model simplification," *Int. J. Control.*, vol. 8, pp. 561 570, 1968.

[3] Y. Shamash, "Model reduction using the Routh stability criterion and the Pade approximation technique," *Int. J. Control.*, vol. 21, pp. 475-484, 1975.

[4] G. Langholz and D. Feinmesser, "Model reduction by Routh approximations," *Int. J. Sys Sci.*, vol. 9, pp. 493-496, 1978.

[5] V. Krishnamurthy and V. Seshadri, "A simple and direct method of reducing order of linear systems using Routh approximants in frequency domain," *IEEE Trans. Automat. Control.*, AC-21, pp. 797-799, 1976.

[6] E. Kahoraho, J. L. Gutierrez and S. Dormido, "Inversion algorithm to construct Routh approximants," *Electron. Lett.*, vol. 21, pp. 424-425, 1985.



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