Control Design for Flexible Joint Manipulators with Mismatched Uncertainty: Adaptive Robust Scheme

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Abstract : Adaptive robust control scheme is introduced for flexible joint manipulator with nonlinearities and uncertainties. The system does not satisfy the matching condition due to insufficient actuators for each node. The control only relies on the assumption that the bound of uncertainty exists. Thus, the bounded value does not need to be known a prior. The control utilizes the update law by estimating the bound of the uncertainties. The control scheme uses the backstepping method and constructs a state transformation. Also, stability analysis is done for both transformed system and original system.

Keywords : mismatched uncertainty, lyapunov approach, state transformation, implanted control, adaptive robust control, flexible joint manipulators

I. Introduction

A control problem for flexible joint manipulators which are nonlinear uncertain systems is considered. The experimental work regarding to the effect of joint flexibility [1] shows a significant influence on system performance compared with rigid manipulators. So far there have been a lot of works related to the study of the control for flexible joint manipulators. Spong [2] cited references of these works. These are exact model based approach, which includes singular perturbation [3], feedback linearization scheme [4], and invariant manifold scheme [5][6], robust control [7], and adaptive control [8][9]. Feedback linearization requires exact knowledge of the robot parameters. However, from practical aspects we need to consider the issue in the presence of uncertainty. As for robust control based on Lyapunov approach we need the bound of uncertainty *a priori*. This may occur practical concern whether we can appropriately estimate the bound of the uncertainty. Insufficient knowledge of the uncertainty may arise unnecessary control cost or saturation in controller. Consequently, an adaptive control scheme is developed. Earlier works on adaptive control schemes for flexible joint manipulators have been conducted by several authors [9]-[11]. These reports introduced control schemes which require feedback of acceleration or "jerk". Measurement noise in the system however prohibits the implementation of these schemes. Basically, the idea of the adaptive control is to reduce the level of uncertainty by estimating unknown parameters. On the other hand, robust control is to design a controller that can tolerate some level of uncertainty and provide satisfactory performance. In many cases, with only adaptive control there may be excessive transientresponses even if parameter adaptation converges. Therefore, it is worth while to investigate a controller which combines adaptive and robust scheme to enhance system performance. To utilize robust control scheme we have to overcome mismatched uncertainty issue which includes the current system. Since flexible joint manipulator system does not have control input in each mode the system is not matched uncertain system any longer. This paper use the state transformation via implanted control to overcome this issue, which is also shown in [12][16]. The control using the *state transformation via implanted control* [12] relies on the possible bound of uncertainty. Sometimes, the control can be conservative with using a high upper-bound of uncertainty. The control scheme in [16] can be applied to general type of manipulators but the constraint imposed on the boundedness of inertia matrix is a drawback in control. Also the control has the conservativeness issue. In this paper the issues on the conservativeness and uniform bound ball adjustment are addressed by introducing a adaptive version for flexible joint manipulator system.

The major development of the proposed adaptive robust control in this paper is divided into two parts. A state transformation via implanted control is used for the development. First, by proposing adaptive version we overcome a practical concern that the possible bound of uncertainty is to be given a priori. By using an adaptive robust scheme we try to estimate the bound of the uncertainty. The proposed adaptive approach satisfies some properties that include uniform stability and uniform boundedness. It also satisfies a property that transformed states approach zero. Furthermore, by this scheme the original states approach zero in case the gravitational force is absent or the system is coordinated such that gravitational force approaches zero as link position converges to zero. We demonstrate the procedure to design control schemes and apply those controls to a 2-link flexible joint manipulator.

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II. Flexible joint manipulators

Consider an n serial link mechanical manipulator. The links are assumed rigid. The joints are however flexible. All joints are revolute or prismatic and are directly actuated by DC-electric motors. The dynamic equation of motion of the flexible joint manipulator can be expressed in terms of the partition of the generalized coordinates [7]:

$$\begin{bmatrix} D(q_l) & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{q}_l \\ \ddot{q}_j \end{bmatrix}+\begin{bmatrix} C(q_l,\dot{q}_l)\dot{q}_l \\ 0 \end{bmatrix}+\begin{bmatrix} G(q_l) \\ 0 \end{bmatrix} + \begin{bmatrix} K(q_l-q_j) \\ -K(q_l-q_j) \end{bmatrix}=\begin{bmatrix} 0 \\ u \end{bmatrix}, \tag{1}$$

where $D(q_l)$ is the link inertia matrix and J is a constant diagonal matrix representing the inertia of actuator. $C(q_l,\dot{q}_l)\dot{q}_l$ represents the Coriolis and centrifugal force, $G(q_l)$ represents the gravitational force, and u denotes the input force from the actuators. Also, joint stiffness by K(hence K^{-1} exists) is presented. $q_l=[q_2\ q_4 \cdots q_{2n-2}\ q_{2n}]^T$ $q_j=[q_1\ q_3 \cdots q_{2n-3}\ q_{2n-1}]^T$, q_2, $q_4 \cdots$ are link angles and q_1, q_3 $\cdots$ are joint angles.

III. Adaptive robust control

We consider an adaptive version of robust control for a flexible joint manipulator system. This approach is based on the state transformation via implanted control and on combining state vectors and parameters of bounds. This control does not need the bound of the uncertainty a priori. In this approach we have properties on system performance, which are uniform stability and uniform boundedness for the states and parameters to be estimated. Furthermore, by this approach we see that both transformed states and original states approach zero.

Let $X_1=q_l$, $X_2=\dot{q}_l$, $X_3=q_j$ and $X_4=\dot{q}_j$ also let $x_1=[X_1^T\ X_2^T]^T$, $x_2=[X_3^T\ X_4^T]^T$ and $x=[x_1^T\ x_2^T]^T$.

We construct the following two subsystems for the flexible joint manipulator system by using the state variables x_1, x_2:

$$N_1:\ \dot{x}_1(t) = f_1(x_1(t),\sigma_1(t)) + B_1(x_1(t),\sigma_1(t))x_2(t), \tag{2}$$

$$N_2:\ \dot{x}_2(t) = f_2(x(t),\sigma_2(t)) + B_2(\sigma_2(t))u(t), \tag{3}$$

where the functions and matrices of (2) and (3) are same with those of [16].

Here, $\sigma_1 \in R^{o_1}$ and $\sigma_2 \in R^{o_2}$ are uncertainty parameter vectors in N_1 and N_2. Suppose we do not need to know the possible bound of uncertainty but the bound should be "compact". Thus, we propose the following Assumption.

Assumption 1 : For each subsystem, the mappings $\sigma_1(\cdot):R\rightarrow\Sigma_1\subset R^{o_1}$, $\sigma_2(\cdot):R\rightarrow\Sigma_2\subset R^{o_2}$, $\dot{\sigma}_1:R\rightarrow\Sigma_{1t}\subset R^{o_1}$ are Lebesgue measurable with $\Sigma_1,\Sigma_2,\Sigma_{1t}$ unknown but compact.

From now on, if no confusion arises we omit argument for the uncertainty in $D(\sigma_1,q_l)$, $C(\sigma_1,q_l,\dot{q}_l)$, etc. Now, we premultiply K^{-1} on both sides of the first part of (1) and construct two subsystems as follows:

$$N_1:\ \hat{D}(q_l)\ddot{q}_l+\hat{C}(q_l,\dot{q}_l)\dot{q}_l+\hat{G}(q_l)+q_l=q_j, \tag{4}$$

$$N_2:\ J\ddot{q}_j+K(q_j-q_l)=u, \tag{5}$$

where

$$\begin{aligned}\hat{D}(q_l)&=K^{-1}D(q_l),\\ \hat{C}(q_l,\dot{q}_l)&=K^{-1}C(q_l,\dot{q}_l),\\ \hat{G}(q_l)&=K^{-1}G(q_l).\end{aligned} \tag{6}$$

The problem is to design control u which renders the systems N_1, N_2 to have good performance. Notice that the uncertainty does not meet the matching condition [15] of the total system. Thus, we divide the total system into two subsystems as shown in (4)-(5) and introduce an implanted control for the subsystem N_1. Therefore both subsystems have "inputs". Let us rewrite (4)-(5) as

$$\begin{aligned}N_1 :&\ \hat{D}(q_l)\ddot{q}_l+\hat{C}(q_l,\dot{q}_l)\dot{q}_l+\hat{G}(q_l)+q_l\\ &=u_1+q_j-u_1,\end{aligned} \tag{7}$$

$$N_2 :J\ddot{q}_j+K(q_j-q_l)=u, \tag{8}$$

where the "control" u_1 is implanted. This does not affect the dynamics in N_1.

We now transform the system (N_1, N_2) to a system ($\hat{N}_1$, $\hat{N}_2$) by using a state transformation. First, let $z_1=[Z_1^T\ Z_2^T]^T$, $z_2=[Z_3^T\ Z_4^T]^T$ and $z=[z_1^T\ z_2^T]^T$, where

$$\begin{aligned}Z_1 &:= q_l, \quad Z_2 := \dot{q}_l,\\ Z_3 &:= q_j-u_1, \quad Z_4 := \dot{q}_j-\dot{u}_1.\end{aligned} \tag{9}$$

This implies that $z_1=x_1$ and $z_2=x_2-[u_1\ \dot{u}_1]^T$. The dynamics of the manipulator can be expressed in terms of z :

$$\hat{N}_1:\ \hat{D}(Z_1)\ddot{Z}_1$$

$$=-\hat{C}(Z_1,\dot{Z}_1)\dot{Z}_1-\hat{G}(Z_1)-Z_1+Z_3+u_1, \tag{10}$$

$$\hat{N}_2 :J\ddot{Z}_3=-J\ddot{u}_1-KZ_3+KZ_1-Ku_1+u. \tag{11}$$

Let

$$\begin{aligned}&\phi_1(q_l, \dot{q}_l, \sigma_1, \dot{\sigma}_1)\\ &:= -\frac{1}{2}\dot{\hat{D}}(q_l, \dot{q}_l, \sigma_1, \dot{\sigma}_1)(\dot{q}_l + S_1 q_l)\\ &- \hat{C}(q_l, \dot{q}_l, \sigma_1)\dot{q}_l - \hat{G}(q_l, \sigma_1)\\ &- q_l + \hat{D}(q_l, \sigma_1)S_1\dot{q}_l,\end{aligned} \tag{12}$$

for given $S_1 = diag[S_{1i}]_{n\times n}$, $S_{1i} > 0$. Then, we see that there exists an uncertain function $\rho_1: R^n \times R^n \to R_+$ such that for all $q_l \in R^n$, $\dot{q}_l \in R^n$, $\sigma_1 \in \Sigma_1$, $\dot{\sigma}_1 \in \Sigma_{1t}$,

$$\|\phi_1(q_l, \dot{q}_l, \sigma_1, \dot{\sigma}_1)\| \le \rho_1(q_l, \dot{q}_l). \tag{13}$$

Let

$$\begin{aligned}&\phi_2(z_1, z_2, \sigma_1, \sigma_2, \dot{\sigma}_1)\\ &:= -J(\sigma_2)\ddot{u}_1(z_1, z_2, \sigma_1, \sigma_2) - K(\sigma_2)Z_3\\ &\quad + K(\sigma_2)Z_1 - K(\sigma_2)u_1 + J(\sigma_2)S_2\dot{Z}_3,\end{aligned} \tag{14}$$

for given $S_2 = diag[S_{2i}]_{n\times n}$, $S_{2i} > 0$. Here, the implanted control u_1 is described later. Then, we see that there exists an uncertain function $\rho_2: R^{2n} \times R^{2n} \to R_+$, such that for all $z_1 \in R^{2n}$, $z_2 \in R^{2n}$, $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$, $\dot{\sigma}_1 \in \Sigma_{1t}$,

$$\|\phi_2(z_1, z_2, \sigma_1, \sigma_2, \dot{\sigma}_1)\| \le \rho_2(z_1, z_2). \tag{15}$$

Assumption 2 [14] : 1) There exist an unknown constant vector $\beta_1 \in (0, \infty)^k$ and a known function $\Pi_1: R^n \times R^n \times (0,\infty)^k \to R_+$ such that for all $q_l \in R^n$, $\dot{q}_l \in R^n$

$$\rho_1(q_l, \dot{q}_l) = \Pi_1(q_l, \dot{q}_l, \beta_1). \tag{16}$$

2) The function $\Pi_1(q_l, \dot{q}_l, \cdot): (0,\infty)^k \to R_+$ is C^2 (i.e., 2-times continuously differentiable) and concave (i.e., $-\Pi_1(q_l, \dot{q}_l, \cdot)$ is convex), and nondecreasing with respect to each coordinate of argument, β_1.

3) The functions $\Pi_1(\cdot)$ and $\frac{\partial \Pi_1}{\partial \beta_1}(\cdot)$ are both continuous.

4) There exist an unknown constant $\beta_2 \in (0, \infty)^j$ and a known function $\Pi_2: R^{2n} \times R^{2n} \times (0,\infty)^j \to R_+$, such that for all $z_1 \in R^{2n}$, $z_2 \in R^{2n}$,

$$\rho_2(z_1, z_2) = \Pi_2(z_1, z_2, \beta_2). \tag{17}$$

5) The function $\Pi_2(z_1, z_2, \cdot): (0,\infty)^j \to R_+$ is C^1, concave and nondecreasing with respect to each coordinate of argument, β_2.

6) The functions $\Pi_2(\cdot)$ and $\frac{\partial \Pi_2}{\partial \beta_2}(\cdot)$ are both continuous.

Let

$$\mu_1 = (\dot{Z}_1 + S_1 Z_1)^T \Pi_1(z_1, \hat{\beta}_1), \tag{18}$$

$$\mu_1 = [\mu_{11}\ \mu_{12} \ldots \mu_{1n}]^T, \tag{19}$$

$$p_1 = [p_{11}\ p_{12} \ldots p_{1n}]^T. \tag{20}$$

We construct controller for the subsystem $\widehat{N}_1$:

$$\begin{aligned}u_1(t) = &-K_{p1}Z_1(t) - K_{v1}\dot{Z}_1(t)\\ &+ p_1(z_1(t), \hat{\beta}_1(t), \varepsilon_1),\end{aligned} \tag{21}$$

where

$$\begin{aligned}\dot{\hat{\beta}}_1(t) = &\ T_1^{-1}\frac{\partial \Pi_1^T}{\partial \beta_1}(z_1(t), \hat{\beta}_1(t))\\ &\times \|\dot{Z}_1(t) + S_1 Z_1(t)\|,\end{aligned} \tag{22}$$

$$\begin{aligned}&K_{p1} := diag[k_{p1i}]_{n\times n},\ k_{p1i} > 0,\\ &K_{v1} := diag[k_{v1i}]_{n\times n},\ k_{v1i} > 0,\end{aligned} \tag{23}$$

$$\dot{\varepsilon}_1(t) = -\frac{n}{4l_1}\varepsilon_1(t), \tag{24}$$

$$\hat{\beta}_1(t_0) \in (0,\infty)^k,\ \varepsilon_1(t_0) \in (0,\infty),\ l_1 > 0.$$

Here T_1 is a nonsingular diagonal matrix with positive elements and n corresponds to the number of links.

For given $\varepsilon_1 > 0$, p_{1i} is chosen to be

$$p_{1i} = \begin{cases} -\dfrac{\mu_{1i}}{\|\mu_{1i}\|}\Pi_1(z_1, \hat{\beta}_1), & \text{if } \|\mu_{1i}\| > \varepsilon_1\\ -\sin\left(\dfrac{\pi\mu_{1i}}{2\varepsilon_1}\right)\Pi_1(z_1, \hat{\beta}_1), & \text{if } \|\mu_{1i}\| \le \varepsilon_1\end{cases} \tag{25}$$

$i = 1, 2, \ldots, n$. Note that

$$p_{1i}\begin{cases} \le -\dfrac{\mu_{1i}}{\varepsilon_1}\Pi_1(z_1, \hat{\beta}_1), & \text{if } 0 \le \mu_{1i} \le \varepsilon_1\\ \ge -\dfrac{\mu_{1i}}{\varepsilon_1}\Pi_1(z_1, \hat{\beta}_1), & \text{if } -\varepsilon_1 \le \mu_{1i} < 0,\end{cases} \tag{26}$$

and $\|p_{1i}\| \le \Pi_1(y_1, \hat{\beta}_1)$.

Next, for given $\varepsilon_2 > 0$ we design control for the subsystem $\widehat{N}_2$ as follows:

$$\begin{aligned}u(t) = &-K_{p2}Z_3(t) - K_{v2}\dot{Z}_3(t)\\ &+ p_2(z_1(t), z_2(t), \hat{\beta}_2(t), \varepsilon_2(t)),\end{aligned} \tag{27}$$

where

$$\begin{aligned}&p_2(z_1, z_2, \hat{\beta}_2, \varepsilon_2)\\ &= \begin{cases} -\dfrac{\mu_2(z_1, z_2, \hat{\beta}_2)}{\|\mu_2(z_1, z_2, \hat{\beta}_2)\|}\Pi_2(z_1, z_2, \hat{\beta}_2),\\ \quad \text{if } \|\mu_2(z_1, z_2, \hat{\beta}_2)\| > \varepsilon_2\\ \dfrac{\mu_2(z_1, z_2, \hat{\beta}_2)}{\varepsilon_2}\Pi_2(z_1, z_2, \hat{\beta}_2),\\ \quad \text{if } \|\mu_2(z_1, z_2, \hat{\beta}_2)\| \le \varepsilon_2\end{cases}\end{aligned} \tag{28}$$

and K_{p2} and K_{v2} are positive diagonal matrices. Here $\hat{\beta}_2(t)$ and $\varepsilon_2(t)$ are determined by

$$\dot{\hat{\beta}}_2(t) = T_2^{-1}\|\dot{Z}_3 + S_2 Z_3\|\frac{\partial \Pi_2^T}{\partial \beta_2}(z_1, z_2, \hat{\beta}_2), \tag{29}$$

$$\dot{\varepsilon}_2(t) = -\frac{1}{4l_2}\varepsilon_2(t), \quad (30)$$

$$\hat{\beta}_2(t_0) \in (0,\infty)^j,\ \varepsilon_2(t_0) \in (0,\infty),\ l_2 > 0,$$

where T_2 is a positive diagonal matrix. In this section, we construct ε_1 and ε_2 dynamics instead of selecting them as constants. These will be shown in proof to help to cancel the remaining terms of the derivative of Lyapunov functions.

The selection of K_{p1}, K_{v1}, K_{p2} and K_{v2} can be conducted as follows.

i) After choosing S_1, select $\underline{\lambda}_1$ such that for $w_1 > 0$,

$$\underline{\lambda}_1 - \frac{1}{2} w_1 \lambda_{\max}(\overline{S}_1) > 0, \quad (31)$$

where

$$\underline{\lambda}_1 = \min[\lambda_{\min}(K_{v1}),\ \lambda_{\min}(S_1 K_{p1})], \quad (32)$$

$$\overline{S}_1 = \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix}. \quad (33)$$

ii) Based on $\underline{\lambda}_1$ we select values for K_{v1}, K_{p1}.

The selection of is shown as the following subsequent steps.

iii) Let

$$\underline{\lambda}_2 = \min[\lambda_{\min}(K_{v2}),\ \lambda_{\min}(S_2 K_{p2})]. \quad (34)$$

iv) After choosing S_2 select $\underline{\lambda}_2$ such that for $w_1 > 0$,

$$\underline{\lambda}_2 - \frac{1}{2} w_1^{-1} > 0. \quad (35)$$

v) Based on $\underline{\lambda}_2$ we select K_{v2} and K_{p2}.

Assumption 3 : There exist unknown positive constants $\underline{\sigma}_k$, $\overline{\sigma}_k$ such that

$$\underline{\sigma}_k I \le \hat{D}(\sigma_1, q_l) \le \overline{\sigma}_k I,\ \forall q_l \in R^n,\ \forall \sigma_1 \in \Sigma_1. \quad (36)$$

Define the parameter estimate vectors

$$\begin{aligned} \hat{\psi}_1(t) &= [\ \hat{\beta}_{11}(t)\ \hat{\beta}_{21}(t) \ldots \hat{\beta}_{k1}(t)\ \varepsilon_1(t)]^T \\ &\in (0,\infty)^{k+1} =: \Psi_1, \\ \hat{\psi}_2(t) &= [\ \hat{\beta}_{12}(t)\ \hat{\beta}_{22}(t) \ldots \hat{\beta}_{j2}(t)\ \varepsilon_2(t)]^T \\ &\in (0,\infty)^{j+1} =: \Psi_2, \\ \hat{\psi} &= [\ \hat{\psi}_1^T\ \hat{\psi}_2^T]^T, \\ \Psi &= \Psi_1 \cup \Psi_2, \end{aligned} \quad (37)$$

and the parameter vectors

$$\psi_1 = [\beta_{11}\ \beta_{21} \ldots \beta_{k1}\ 0]^T > 0,$$

$$\psi_2 = [\beta_{12}\ \beta_{22} \ldots \beta_{j2}\ 0]^T > 0. \quad (38)$$

The controlled system can be described by

$$\begin{aligned} \hat{D}(Z_1)\ddot{Z}_1 &= -\hat{C}(Z_1, \dot{Z}_1)\dot{Z}_1 - \hat{G}(Z_1) \\ &\quad - Z_1 + Z_3 + u_1, \end{aligned} \quad (39)$$

$$J\ddot{Z}_3 = -J\ddot{u}_1 - KZ_3 + KZ_1 - Ku_1 + u, \quad (40)$$

$$\dot{\hat{\psi}}(t) = \begin{bmatrix} T_1^{-1}\|\dot{Z}_1(t) + S_1 Z_1(t)\| \\ \times \frac{\partial \Pi_1^T}{\partial \beta_1}(z_1(t),\ \hat{\beta}_1(t)) \\ -\frac{n}{4l_1}\varepsilon_1(t) \\ T_2^{-1}\|\dot{Z}_3(t) + S_2 Z_3(t)\| \\ \times \frac{\partial \Pi_2^T}{\partial \beta_2}(z_1(t), z_2(t),\ \hat{\beta}_2(t)) \\ -\frac{1}{4l_2}\varepsilon_2(t) \end{bmatrix}. \quad (41)$$

Here, arguments on the uncertainty in $\hat{D}$, $\hat{C}$, $\hat{G}$, and J are omitted for simplicity.

Theorem 1 : Suppose Assumptions 1-3 are met, then the system (39)-(41) under the control (27) has the following properties.

Property 1 : Existence of Solutions: For each $(z_0, \hat{\psi}_0, t_0) \in R^{4n} \times \Psi \times R$ there exists a solution $(z, \hat{\psi}) : [t_0, t_1) \to R^{4n} \times \Psi$ of (39)-(41) with $(z(t_0), \hat{\psi}(t_0)) = (z_0, \psi_0)$.

Property 2 : Uniform Stability: For each $\eta > 0$ there exists $\delta > 0$ such that if $(z(\cdot), \psi(\cdot))$ is any solution of (39)-(41) with $\|z(t_0)\|$, $\|\hat{\psi}(t_0) - \psi\| < \delta$ then $\|z(t)\|$, $\|\hat{\psi}(t) - \psi\| < \eta$ for all $t \in [t_0, t_1)$.

Property 3 : Uniform Boundedness of Solutions: For each r_1, $r_2 > 0$ there exist $d_1(r_1, r_2)$, $d_2(r_1, r_2) \ge 0$ such that if $(z(\cdot), \hat{\psi}(\cdot))$ is any solution of (39)-(41) with $\|z(t_0)\| \le r_1$ and $\|\hat{\psi}(t_0) - \psi\| \le r_2$ then $\|z(t)\| \le d_1(r_1, r_2)$ and $\|\hat{\psi}(t) - \psi\| \le d_2(r_1, r_2)$ for all $t \in [t_0, t_1)$.

Property 4 : Extension of Solutions: Every solution of (39)-(41) can be extended into a solution defined on $[t_0, \infty)$.

Property 5 : Convergence of $z(\cdot)$ to 0: If (z, ψ): $[t_0, \infty) \to R^{4n} \times \Psi$ is a solution of (39)-(41) then $\lim_{t\to\infty} z(t) = 0$.

Proof : Choose functions $V_{1T}(z_1, \hat{\psi}_1)$ and $V_{2T}(z_2, \hat{\psi}_2)$ as follows:

$$V_{1T}(z_1, \hat{\psi}_1, \varepsilon_1) = V_1(z_1) + V_{\beta_1}(\hat{\beta}_1) + l_1\varepsilon_1, \quad (42)$$

$$V_{2T}(z_2, \hat{\psi}_2, \varepsilon_2) = V_2(z_2) + V_{\beta_2}(\hat{\beta}_2) + l_2\varepsilon_2, \quad (43)$$

where

$$\begin{aligned} V_1(z_1) &= \frac{1}{2}(Z_2 + S_1 Z_1)^T \hat{D}(Z_2 + S_1 Z_1) \\ &\quad + \frac{1}{2} Z_1^T (K_{p1} + S_1 K_{v1}) Z_1, \end{aligned} \quad (44)$$

$$V_{\beta_1}(\hat{\beta}_1) = \frac{1}{2}(\hat{\beta}_1 - \beta_1) T_1 (\hat{\beta}_1 - \beta_1), \quad (45)$$

$$V_2(z_2) = \frac{1}{2}(Z_4+S_2Z_3)^T\hat{D}(Z_4+S_2Z_3) + \frac{1}{2}Z_3^T(K_{p2}+S_2K_{v2})Z_3, \tag{46}$$

$$V_{\beta_2}(\tilde{\beta}_2) = \frac{1}{2}(\hat{\beta}_2-\beta_2)T_3(\hat{\beta}_2-\beta_2), \tag{47}$$

To show that V_{1T} and V_{2T} are legitimate Lyapunov function candidates, we prove that both V_{1T} and V_{2T} are positive definite and decrescent. Based on Assumption 3,

$$\begin{aligned} V_1(z_1) &\geq \frac{1}{2}\underline{\sigma}_k\|Z_2+S_1Z_1\|^2 + \frac{1}{2}Z_1^T(K_{p1}+S_1K_{v1})Z_1 \\ &= \frac{1}{2}\underline{\sigma}_k\sum_{i=1}^{n}(Z_{2i}^2+2S_{1i}Z_{2i}Z_{1i}+S_{1i}^2Z_{1i}^2) + \frac{1}{2}\sum_{i=1}^{n}(k_{p1i}+S_{1i}k_{v1i})Z_{1i}^2 \\ &=: \frac{1}{2}\sum_{i=1}^{n}[Z_{1i}\ Z_{2i}]\,\underline{\Omega}_{1i}\begin{bmatrix} Z_{1i} \\ Z_{2i}\end{bmatrix}, \end{aligned} \tag{48}$$

$$V_{\beta_1}(\hat{\beta}_1-\beta_1) \geq \frac{1}{2}\lambda_{\min}(T_1)\,\|\hat{\beta}_1-\beta_1\|^2, \tag{49}$$

where

$$\underline{\Omega}_{1i} = \begin{bmatrix} \underline{\sigma}_kS_{1i}^2+k_{p1i}+S_{1i}k_{v1i} & \underline{\sigma}_kS_{1i} \\ \underline{\sigma}_kS_{1i} & \underline{\sigma}_k \end{bmatrix}. \tag{50}$$

Here, Z_{1i} and Z_{2i} are the i-th components of Z_1 and Z_2, respectively. Since $\underline{\Omega}_{1i}>0$, $\forall i$ and $T_1>0$, V_1 and V_{β_1} are positive definite. Therefore, by combining (53) and (54), we have

$$\begin{aligned} V_1 &\geq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\min}(\underline{\Omega}_{1i})(Z_{1i}^2+Z_{1i}^2) \\ &\geq \gamma_1^{(1)}\|z_1\|^2, \end{aligned} \tag{51}$$

$$\begin{aligned} V_{\beta_1} &\geq \frac{1}{2}\lambda_{\min}(T_1)\|\hat{\beta}_1-\beta_1\|^2 \\ &= \gamma_7^{(1)}\|\hat{\beta}_1-\beta_1\|^2, \end{aligned} \tag{52}$$

where

$$\gamma_1^{(1)} := \frac{1}{2}\min_i\left[\min_{\sigma_1}\lambda_{\min}(\underline{\Omega}_{1i}),\ i=1,2,\cdots,n\right], \tag{53}$$

$$\gamma_7^{(1)} := \frac{1}{2}\lambda_{\min}(T_1), \tag{54}$$

and γ_1^1 is an unknown constant. Let $\tilde{\beta}_1=\hat{\beta}_1-\beta_1$. Next, in conjunction with Assumption 3 it can be seen that

$$\begin{aligned} V_1(z_1) &\leq \frac{1}{2}\overline{\sigma}_k\|Z_2+S_1Z_1\|^2 + \frac{1}{2}Z_1^T(K_{p1}+S_1K_{v1})Z_1 \\ &= \frac{1}{2}\overline{\sigma}_k\sum_{i=1}^{n}(Z_{2i}^2+2S_{1i}Z_{2i}Z_{1i}+S_{1i}^2Z_{1i}^2) \end{aligned}$$

$$\begin{aligned} &+\frac{1}{2}\sum_{i=1}^{n}(k_{p1i}+S_{1i}k_{v1i})Z_{1i}^2 \\ &=: \frac{1}{2}\sum_{i=1}^{n}[Z_{1i}\ Z_{2i}]\,\overline{\Omega}_{1i}\begin{bmatrix} Z_{1i} \\ Z_{2i}\end{bmatrix}, \end{aligned} \tag{55}$$

$$V_{\beta_1}(\tilde{\beta}_1) \leq \lambda_{\max}(T_1)\|\tilde{\beta}_1\|^2, \tag{56}$$

where

$$\overline{\Omega}_{1i} = \begin{bmatrix} \overline{\sigma}_kS_{1i}^2+k_{p1i}+S_{1i}k_{v1i} & \overline{\sigma}_kS_{1i} \\ \overline{\sigma}_kS_{1i} & \overline{\sigma}_k \end{bmatrix}. \tag{57}$$

Furthermore, we have

$$\begin{aligned} V_1 &\leq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\min}(\overline{\Omega}_{1i})(Z_{1i}^2+Z_{1i}^2) \\ &\leq \gamma_2^{(1)}\|z_1\|^2, \end{aligned} \tag{58}$$

$$\begin{aligned} V_{\beta_1} &\leq \frac{1}{2}\lambda_{\max}(T_1)|\hat{\beta}_1-\beta_1|^2 \\ &= \gamma_8^{(1)}\|\hat{\beta}_1-\beta_1\|^2, \end{aligned} \tag{59}$$

where

$$\gamma_2^{(1)} := \frac{1}{2}\max_i\left[\max_{\sigma_1}\lambda_{\max}(\overline{\Omega}_{1i}),\ i=1,2,\cdots,n\right] \tag{60}$$

$$\gamma_8^{(1)} := \frac{1}{2}\lambda_{\max}(T_1), \tag{61}$$

and $\gamma_2^{(1)}, \gamma_8^{(1)}$ are unknown constants.

Similar to V_1 and V_{β_1}, V_2 and V_{β_2} is also positive and decrescent. This follows since

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{n}\lambda_{\min}(\underline{\Omega}_{2i})(Z_{3i}^2+Z_{3i}^2) &\leq V_2(z_2) \\ \leq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\max}(\overline{\Omega}_{2i})(Z_{3i}^2+Z_{3i}^2), \end{aligned} \tag{62}$$

$$\begin{aligned} \frac{1}{2}\lambda_{\min}(T_2)\|\tilde{\beta}_2\|^2 &\leq V_{\beta_2}(\tilde{\beta}_2) \\ \leq \frac{1}{2}\lambda_{\max}(T_2)\|\tilde{\beta}_2\|^2. \end{aligned} \tag{63}$$

where

$$\underline{\Omega}_{2i} := \begin{bmatrix} \underline{\theta}S_{2i}^2+k_{p2i}+S_{2i}k_{v2i} & \underline{\theta}S_{2i} \\ \underline{\theta}S_{2i} & \underline{\theta} \end{bmatrix},$$

$$\overline{\Omega}_{2i} := \begin{bmatrix} \overline{\theta}S_{2i}^2+k_{p2i}+S_{2i}k_{v2i} & \overline{\theta}S_{2i} \\ \overline{\theta}S_{2i} & \overline{\theta} \end{bmatrix},$$

$$\begin{aligned} \overline{\theta} &:= \lambda_{\max}(J), \\ \underline{\theta} &:= \lambda_{\min}(J). \end{aligned} \tag{64}$$

Thus,

$$\gamma_1^{(2)}\|z_2\|^2 \leq V_2(z_2) \leq \gamma_2^{(2)}\|z_2\|^2, \tag{65}$$

$$\gamma_7^{(2)}\|\hat{\beta}_2-\beta_2\|^2\le V_{\beta_2}(\hat{\beta}_2)\le\gamma_8^{(2)}\|\hat{\beta}_2-\beta_2\|^2, \quad (66)$$

where

$$\begin{aligned}
\gamma_1^{(2)} &:= \frac{1}{2}\min_i\left[\min_{\sigma_2}\lambda_{\min}(\underline{\Omega}_{2i}),\ i=1,2,\cdots,n\right],\\
\gamma_2^{(2)} &:= \frac{1}{2}\max_i\left[\max_{\sigma_2}\lambda_{\max}(\overline{\Omega}_{2i}),\ i=1,2,\cdots,n\right],\\
\gamma_7^{(2)} &:= \frac{1}{2}\lambda_{\min}(T_2),\\
\gamma_8^{(2)} &:= \frac{1}{2}\lambda_{\max}(T_2),
\end{aligned} \quad (67)$$

and, $\gamma_1^{(2)},\gamma_2^{(2)}$ are unknown constants.

The derivative of V_{1T} is given by

$$\dot{V}_{1T}=\dot{V}_1+\dot{V}_{\beta_1}+l_1\dot{\varepsilon}_1. \quad (68)$$

Concerning $\dot{V}_1$, it can be seen that

$$\begin{aligned}
\dot{V}_1 &= (\dot{Z}_1+S_1Z_1)^T\hat{D}(\ddot{Z}_1+S_1Z_1)^T\\
&+\frac{1}{2}(\dot{Z}_1+S_1Z_1)^T\dot{\hat{D}}(\dot{Z}_1+S_1Z_1)\\
&+Z_1^T(K_{p1}+S_1K_{v1})\dot{Z}_1.
\end{aligned} \quad (69)$$

From (14), we obtain

$$\begin{aligned}
\dot{V}_1 &= (\dot{Z}_1+S_1Z_1)^T(-\hat{C}\dot{Z}_1-\hat{G}-Z_1+Z_3\\
&+u_1+\hat{D}S_1Z_1+\frac{1}{2}\dot{\hat{D}}\dot{Z}_1+\frac{1}{2}\dot{\hat{D}}S_1Z_1)\\
&+Z_1^T(K_{p1}+S_1K_{v1})\dot{Z}_1\\
&=(\dot{Z}_1+S_1Z_1)^T\frac{1}{2}\dot{\hat{D}}(\dot{Z}_1+S_1Z_1)\\
&-\hat{C}\dot{Z}_1-\hat{G}-Z_1+\hat{D}S_1\dot{Z}_1)\\
&+(\dot{Z}_1+S_1Z_1)^Tu_1+(\dot{Z}_1+S_1Z_1)^TZ_3\\
&+Z_1^T(K_{p1}+S_1K_{v1})\dot{Z}_1.
\end{aligned} \quad (70)$$

By the "control" u_1 in (20), (16), and (31) it can be shown that

$$\begin{aligned}
\dot{V}_1 &\le (\dot{Z}_1+S_1Z_1)^T(-K_{p1}Z_1-K_{v1}\dot{Z}_1+p_1)\\
&+\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)\\
&+(\dot{Z}_1+S_1Z_1)^TZ_3+Z_1^T(K_{p1}+S_1K_{v1})\dot{Z}_1\\
&\le-\underline{\lambda}_1\|z_1\|^2+\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)\\
&+(\dot{Z}_1+S_1Z_1)^Tp_1+\|\dot{Z}_1+S_1Z_1\|\|Z_3\|.
\end{aligned} \quad (71)$$

For $\|\mu_{1i}\|>\varepsilon_1$, the second and third term in (71) becomes

$$\begin{aligned}
&\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^Tp_1\\
&\le\sum_{i=1}^n\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|\Pi_1(z_1,\beta_1)\\
&+\sum_{i=1}^n(\dot{Z}_{1i}+S_{1i}Z_{1i})\left(-\frac{\dot{Z}_{1i}+S_{1i}Z_{1i}}{\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|}\Pi_1(z_1,\hat{\beta}_1)\right)\\
&=\sum_{i=1}^n\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|(\Pi_1(z_1,\beta_1)-\Pi_1(z_1,\hat{\beta}_1)).
\end{aligned} \quad (72)$$

When $\|\mu_{1i}\|\le\varepsilon_1$, then

$$\begin{aligned}
&\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^Tp_1\\
&\le\sum_{i=1}^n\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|\Pi_1(z_1,\beta_1)\\
&+\sum_{i=1}^n(\dot{Z}_{1i}+S_{1i}Z_{1i})^2\left(-\Pi_1^2(z_1,\hat{\beta}_1)\frac{1}{\varepsilon_1}\right).
\end{aligned} \quad (73)$$

Concerning $\dot{V}_{\beta_1}$, it follows

$$\begin{aligned}
\dot{V}_{\beta_1} &= (\hat{\beta}_1-\beta_1)^TT_1\\
&\times\left(T_1^{-1}\|\dot{Z}_1+S_1Z_1\|\frac{\partial\Pi_1^T}{\partial\beta_1}(z_1,\hat{\beta}_1)\right).
\end{aligned} \quad (74)$$

Since $-\Pi_1(z_1,\cdot)$ is convex for all $z_1\in R^{2n}$, the first and third term in (79) becomes

$$\begin{aligned}
&\frac{\partial\Pi_1}{\partial\beta_1}(z_1,\hat{\beta}_1)(\hat{\beta}_1-\beta_1)\\
&\le\Pi_1(z_1,\hat{\beta}_1)-\Pi_1(z_1,\beta_1)
\end{aligned} \quad (75)$$

Therefore, it becomes

$$\dot{V}_{\beta_1}\le(\Pi_1(z_1,\hat{\beta}_1)-\Pi_1(z_1,\beta_1))\,\|\dot{Z}_1+S_1Z_1\|. \quad (76)$$

If $\|\mu_{1i}\|>\varepsilon_1$, by using (71)-(72) and (76) we obtain

$$\begin{aligned}
&\dot{V}_1+\dot{V}_{\beta_1}\\
&\le-\underline{\lambda}_1\|z_1\|^2\\
&+\sum_{i=1}^n\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|(\Pi_1(z_1,\beta_1)-\Pi_1(z_1,\hat{\beta}_1))\\
&+\sum_{i=1}^n\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|(\Pi_1(z_1,\hat{\beta}_1)-\Pi_1(z_1,\beta_1))\\
&+\|\dot{Z}_1+S_1Z_1\|\,\|Z_3\|\\
&=-\underline{\lambda}_1\|z_1\|^2+\|\dot{Z}_1+S_1Z_1\|\,\|Z_3\|.
\end{aligned} \quad (77)$$

If $\|\mu_{1i}\|\le\varepsilon_1$, then by using (71), (73), and (76) we obtain

$$\begin{aligned}
&\dot{V}_1+\dot{V}_{\beta_1}\\
&\le-\underline{\lambda}_1\|z_1\|^2+\sum_{i=1}^n\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|\Pi_1(z_1,\hat{\beta}_1)\\
&-\sum_{i=1}^n(\dot{Z}_{1i}+S_{1i}Z_{1i})^2\Pi_1^2(z_1,\hat{\beta}_1)\frac{1}{\varepsilon_1}\\
&+\|\dot{Z}_1+S_1Z_1\|\,\|Z_3\|
\end{aligned} \quad (78)$$

Therefore, $\dot{V}_1+\dot{V}_{\beta_1}$ is upper bounded by:

$$\begin{aligned}
&\dot{V}_1+\dot{V}_{\beta_1}\\
&\le-\underline{\lambda}_1\|z_1\|^2+\sum_{i=1}^n\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|\Pi_1(z_1,\hat{\beta}_1)\\
&-\sum_{i=1}^n(\dot{Z}_{1i}+S_{1i}Z_{1i})^2\Pi_1^2(z_1,\hat{\beta}_1)\frac{1}{\varepsilon_1}\\
&+\|\dot{Z}_1+S_1Z_1\|\,\|Z_3\|
\end{aligned} \quad (79)$$

Based on the inequalities $ab \le \frac{1}{2}(a^2+b^2)$, $a, b \in R$, $\|Z_3\|^2 \le \|z_2\|^2$, we have the inequality condition for $\|\dot{Z}_1+S_1Z_1\| \, \|Z_3\|$ with any constant $w_1>0$.

$$\begin{aligned} &\|\dot{Z}_1+S_1Z_1\| \, \|Z_3\| \\ &\le \frac{1}{2} w_1\|\dot{Z}_1+S_1Z_1\|^2+\frac{1}{2} w_1^{-1}\|Z_3\|^2 \\ &\le \frac{1}{2} w_1\|\dot{Z}_1+S_1Z_1\|^2+\frac{1}{2} w_1^{-1}\|z_2\|^2 \\ &\le \frac{1}{2} w_1\lambda_{\min}(\overline{S}_1)\|z_1\|^2+\frac{1}{2} w_1^{-1}\|z_2\|^2. \end{aligned} \tag{80}$$

From (84), and inequality (85), we get

$$\begin{aligned} \dot{V}_{1T} \le& -\underline{\lambda}_1\|z_1\|^2 \\ &+\sum_{i=1}^{n}\|\dot{Z}_{1i}+S_{1i}Z_{1i}\|\Pi_1(z_1, \hat{\beta}_1) \\ &-\sum_{i=1}^{n}(\dot{Z}_{1i}+S_{1i}Z_{1i})^2\Pi_1^2(z_1, \hat{\beta}_1)\frac{1}{\varepsilon_1} \\ &+l_1\dot{\varepsilon}_1+\|\dot{Z}_1+S_1Z_1\| \, \|Z_3\| \\ \le& -\underline{\lambda}_1\|z_1\|^2+\frac{n\varepsilon_1}{4}-\frac{n\varepsilon_1}{4} \\ &+\|\dot{Z}_1+S_1Z_1\| \, \|Z_3\|. \\ \le& -\underline{\lambda}_1\|z_1\|^2+\frac{1}{2} w_1\lambda_{\max}(\overline{S}_1)\|z_1\|^2 \\ &+\frac{1}{2} w_1^{-1}\|z_2\|^2. \end{aligned} \tag{81}$$

Next, the derivative of V_{2T} is given by

$$\dot{V}_{2T}=\dot{V}_2+\dot{V}_{\beta_2}+l_2\dot{\varepsilon}_2. \tag{82}$$

Concerning $\dot{V}_2$, it follows from (11) and (14)

$$\begin{aligned} \dot{V}_2 =& (\dot{Z}_3+S_2Z_3)^T(\ddot{Z}_3+S_2\dot{Z}_3) \\ &+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3 \\ =& (\dot{Z}_3+S_2Z_3)^T(-J\ddot{u}_1-KZ_3 \\ &+KZ_1-Ku_1+JS_2\dot{Z}_3+u) \\ &+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3 \\ =& (\dot{Z}_3+S_2Z_3)^T(\phi_2+u) \\ &+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3. \end{aligned} \tag{83}$$

It follows from (15),(17),(27) and (34)

$$\begin{aligned} \dot{V}_2 \le& \|\dot{Z}_3+S_2Z_3\| \, \|\phi_2\|+(\dot{Z}_3+S_2Z_3)^Tu \\ &+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3 \\ \le& \|\dot{Z}_3+S_2Z_3\| \, \Pi_2(z_1,z_2,\beta_2) \\ &+(\dot{Z}_3+S_2Z_3)^T(-K_{p1}Z_3-K_{v1}\dot{Z}_3+p_2) \\ &+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3 \\ \le& -\underline{\lambda}_2\|z_2\|^2+\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\beta_2) \\ &+(\dot{Z}_3+S_2Z_3)^Tp_2. \end{aligned} \tag{84}$$

Concerning $\dot{V}_{\beta_2}$, it follows from (29)

$$\begin{aligned} \dot{V}_{\beta_2} &= (\hat{\beta}_2-\beta_2)^T T_1(\dot{\hat{\beta}}_2-\dot{\beta}_2) \\ &= (\hat{\beta}_2-\beta_2)^T T_1 \left(T_1^{-1}\frac{\partial\Pi_2^T}{\partial\beta_2}(z_1,z_2,\hat{\beta}_2)\right). \end{aligned} \tag{85}$$

Since $-\Pi_2(z_1,z_2,\cdot)$ is convex for all $(z_1,z_2)\in R^{2n}\times R^{2n}$, we get

$$\begin{aligned} &(\hat{\beta}_2-\beta_2)^T\frac{\partial\Pi_2^T}{\partial\beta_2}(z_1,z_2,\hat{\beta}_2) \\ &\le \Pi_2(z_1,z_2,\hat{\beta}_2)-\Pi_2(z_1,z_2,\beta_2). \end{aligned} \tag{86}$$

Therefore, it becomes

$$\dot{V}_{\beta_2} \le (\Pi_2(z_1,z_2,\hat{\beta}_2)-\Pi(z_1,z_2,\beta_2))\,\|\dot{Z}_3+S_2Z_3\|. \tag{87}$$

By adding (84) and (87), we obtain

$$\begin{aligned} \dot{V}_{2T} =& \dot{V}_2+\dot{V}_{\beta_2}+l_2\dot{\varepsilon}_2 \\ =& (\dot{Z}_3+S_2Z_3)^T\Pi_2(z_1,z_2,\hat{\beta}_2) \\ &+(\dot{Z}_3+S_2Z_3)^Tu+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3 \\ &+l_2\dot{\varepsilon}_2. \end{aligned} \tag{88}$$

From the control u in (27) and (34) it can be seen that

$$\begin{aligned} \dot{V}_{2T} =& (\dot{Z}_3+S_2Z_3)^T\Pi_2(z_1,z_2,\hat{\beta}_2) \\ &+(\dot{Z}_3+S_2Z_3)^T(-K_{p2}Z_3-K_{v2}\dot{Z}_3+p_2) \\ &+Z_3^T(K_{p2}+S_2K_{v2})\dot{Z}_3+l_2\dot{\varepsilon}_2 \\ \le& -\underline{\lambda}_2\|z_2\|^2+\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\hat{\beta}_2) \\ &+(\dot{Z}_3+S_2Z_3)^Tp_2+l_2\dot{\varepsilon}_2. \end{aligned} \tag{89}$$

For $\|\mu_2(z_1,z_2,\hat{\beta}_2\|>\varepsilon_2$, the second and third term in (89) gives

$$\begin{aligned} &\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\hat{\beta}_2)+(\dot{Z}_3+S_2Z_3)^Tp_2 \\ &=\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\hat{\beta}_2) \\ &-\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\hat{\beta}_2) \\ &=0. \end{aligned} \tag{90}$$

When $\|\mu_2(z_1,z_2,\hat{\beta}_2\|\le\varepsilon_2$, then

$$\begin{aligned} &\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\hat{\beta}_2)+(\dot{Z}_3+S_2Z_3)^Tp_2 \\ &=\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\hat{\beta}_2) \\ &-\|\dot{Z}_3+S_2Z_3\|^2\Pi_2^2(z_1,z_2,\hat{\beta}_2)\frac{1}{\varepsilon_2} \\ &\le\frac{\varepsilon_2}{4}. \end{aligned} \tag{91}$$

Therefore, we have

$$\dot{V}_{2T} \le -\underline{\lambda}_2\|z_2\|^2+\frac{\varepsilon_2}{4}+l_2\dot{\varepsilon}_2$$

$$=-\lambda_2\|z_2\|^2+\frac{\varepsilon_2}{4}+l_2\left(-\frac{\varepsilon_2}{4l_2}\right) \\ =-\lambda_2\|z_2\|^2. \tag{92}$$

This shows that $\dot{V}_{2T}$ is bounded from above. By above results (81) and (92), we get

$$\begin{aligned}\dot{V}_T &= \dot{V}_{1T}+\dot{V}_{2T} \\ &\le -\left(\lambda_1-\frac{1}{2}w_1\lambda_{\max}(\bar{S}_1)\right)\|z_1\|^2 \\ &\quad -\left(\lambda_2-\frac{1}{2}w_1^{-1}\right)\|z_2\|^2.\end{aligned} \tag{93}$$

If we choose λ_1 and λ_2 such that

$$\lambda_1-\frac{1}{2}w_1\lambda_{\max}(\bar{S}_1)>0, \\ \lambda_2-\frac{1}{2}w_1^{-1}>0, \tag{94}$$

then we have

$$\begin{aligned}\dot{V}_T &\le -\min\left[\lambda_1-\frac{1}{2}w_1\lambda_{\max}(\bar{S}_1),\right. \\ &\quad \left.\lambda_2-\frac{1}{2}w_1^{-1}\right]\|z\|^2 \\ &=: -\gamma_3(\|z\|) \text{ a.e. on } [t_0,t_1),\end{aligned} \tag{95}$$

where

$$\gamma_3(\|z\|)=\min\left[\lambda_1-\frac{1}{2}w_1\lambda_{\max}(\bar{S}_1),\right. \\ \left.\lambda_2-\frac{1}{2}w_1^{-1}\right]\|z\|^2. \tag{96}$$

With the arguments given above, Properties 2-4 follow directly as we apply the results in [14]. For the Property 5, we consider from (71), and (84)

$$\begin{aligned}&\dot{V}_1+\dot{V}_2 \\ &\le\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^Tp_1 \\ &-\lambda_1\|z_1\|^2 \\ &+\frac{1}{2}\lambda_{\min}(\bar{S}_1)w_1\|z_1\|^2 \\ &+\frac{1}{2}w_1^{-1}\|z_2\|^2-\lambda_2\|z_2\|^2 \\ &+\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\beta_2)+(\dot{Z}_3+S_2Z_3)^Tp_2 \\ &\le\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^Tp_1 \\ &+\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\beta_2)+(\dot{Z}_3+S_2Z_3)^Tp_2\end{aligned} \tag{97}$$

For the first and second term of (97) it can be seen that

$$\begin{aligned}&\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^Tp_1 \\ &=\|\dot{Z}_1+S_1Z_1\|(\Pi_1(z_1,\beta_1)-\Pi_1(z_1,\hat{\beta}_1)) \\ &+\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\hat{\beta}_1)+(\dot{Z}_1+S_1Z_1)^Tp_1.\end{aligned} \tag{98}$$

For the first two terms in (98) we get

$$\begin{aligned}&\|\dot{Z}_1+S_1Z_1\|(\Pi_1(z_1,\beta_1)-\Pi_1(z_1,\hat{\beta}_1)) \\ &\le\|\dot{Z}_1+S_1Z_1\|\frac{\partial\Pi_1}{\partial\beta_1}(z_1,\hat{\beta}_1)(\beta_1-\hat{\beta}_1) \\ &\le\sum_{i=1}^{k}\|\dot{Z}_1+S_1Z_1\|\pi(\beta_{1i}-\hat{\beta}_{1i})\frac{\partial\Pi_1}{\partial\beta_1}(z_1,\hat{\beta}_1) \\ &=: a_1^1(t),\end{aligned} \tag{99}$$

where $\pi: R\rightarrow R_-$ is given by

$$\pi(a)=\begin{cases}0, & a<0 \\ a, & a\ge 0\end{cases}. \tag{100}$$

We see that $a_1^1(t)\ge 0$ for all $t\in[t_0,\infty)$. Also, utilizing (22) and (99), we obtain for each $t\in[t_0,\infty)$

$$\int_{t_0}^{t}a_1^1(\tau)d\tau=\int_{t_0}^{t}\sum_{i=1}^{k}\pi(\beta_{1i}-\hat{\beta}_{1i})T_1\ \hat{\beta}_{1i}(\tau)d\tau \\ \le\eta(t_0)-\eta(t), \tag{101}$$

where

$$\eta(t):=\sum_{i=1}^{k}\frac{1}{2}T_1\pi^2(\beta_{1i}-\hat{\beta}_{1i}(t)). \tag{102}$$

Since $\eta(t)\ge 0$, we see that

$$\int_{t_0}^{t}a_1^1(\tau)d\tau\le\eta(t_0),\quad\forall t\in[t_0,\infty). \tag{103}$$

Hence, $\int_{t_0}^{\infty}a_1^1(\tau)d\tau$ is finite. For the last two terms of (98) it can be shown

$$\begin{aligned}&\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\hat{\beta}_1)+(\dot{Z}_1+S_1Z_1)^Tp_1 \\ &\le\frac{n\varepsilon_1(t)}{4} \\ &\le\frac{n\varepsilon_1(t_0)}{4}.\end{aligned} \tag{104}$$

Therefore, by (99) and (104) the first term in (97) is given by

$$\|\dot{Z}_1+S_1Z_1\|\Pi_1(z_1,\beta_1)+(\dot{Z}_1+S_1Z_1)^Tp_1 \\ \le a_1^1(t)+\frac{n\varepsilon_1(t_0)}{4}. \tag{105}$$

Similar to (105), the last two terms in (97) can be seen to satisfy

$$\|\dot{Z}_3+S_2Z_3\|\Pi_2(z_1,z_2,\beta_2)+(\dot{Z}_3+S_2Z_3)^Tp_2 \\ \le a_1^2(t)+\frac{\varepsilon_2(t_0)}{4}, \tag{106}$$

where

$$a_1^2(t):=\sum_{i=1}^{i}\pi(\beta_{2i}-\hat{\beta}_{2i})\frac{\partial\Pi_2}{\partial\beta_{2i}}(z_1,z_2,\hat{\beta}_2) \\ \times\|\dot{Z}_3+S_2Z_3\|. \tag{107}$$

Also, we see that $a_1^2(t) \geq 0$ and $\int_{t_0}^{\infty} a_1^2(\tau)d\tau$ is finite. Therefore, we get

$$\dot{V}_1 + \dot{V}_2 \leq a_1^1(t) + \frac{n\varepsilon_1(t_0)}{4} + a_1^2(t) + \frac{\varepsilon_2(t_0)}{4} =: a_1(t) + a_2, \tag{108}$$

where

$$a_1(t) = a_1^1(t) + a_1^2(t), \quad a_2 = \frac{n\varepsilon_1(t_0)}{4} + \frac{\varepsilon_2(t_0)}{4}. \tag{109}$$

From (55), (58) and (62), we obtain

$$\hat{\gamma}_1\|z\|^2 \leq V_1(z_1) + V_2(z_2) \leq \hat{\gamma}_2\|z\|^2, \tag{110}$$

where $\hat{\gamma}_1 = \min(\gamma_1^{(1)}, \gamma_1^{(2)})$, $\hat{\gamma}_2 = \max[\gamma_2^{(1)}, \gamma_2^{(2)}]$.

In view of (95), we have the following result

$$\int_{t_0}^{t} \gamma_3(\|z(\tau)\|)d\tau \leq V_T(t_0) - V_T(t) \leq V_T(t_0) \tag{111}$$

for all $t \in [t_0, \infty)$. Hence, $\int_{t_0}^{t} \gamma_3(\|z(\tau)\|)d\tau$ is finite. Here, we see that $\int_{t_0}^{\infty} a_1(\tau)d\tau$ is finite, and $a_1(t) \geq 0$, $\forall t \in [t_0, \infty)$. The result of (108), (110), and finiteness of $\int_{t_0}^{\infty} \gamma_3(\|z(\tau)\|)d\tau$, and $\int_{t_0}^{\infty} a_1(\tau)d\tau$ satisfy Lemma 3 in [14]. Hence $\lim_{t\to\infty}\|z(t)\| = 0$. Thus, $\lim_{t\to 0} z(t) = 0$, and we see that this fact satisfies Property 5. Q.E.D.

IV. Performance of system (N_1, N_2)

We now investigate the corresponding performance of thc original system based on the performance of the transformed system. The analysis for the performance of the original system follows the similar approach [16].

Finally, we obtain

$$\|x\| = (\|x_1\|^2 + \|x_2\|^2)^{\frac{1}{2}} \leq (d^2 + (c_{13}d + c_{14}(d) + c_{12})^2)^{\frac{1}{2}} =: \omega(d) < \infty. \tag{112}$$

This enables us to investigate the uniform boundedness of x based on the performance of the transformed system.

Theorem 2. Suppose that Assumptions 1-3 are met, then the system (8)-(9), (26), (28), and (34)-(35) under the control (32) satisfies Properties 1-4.

Proof. The system performance has been shown as above. Q.E.D.

V. Illustrative example

Consider a 2-link flexible joint manipulator (Figure 1). Let link angle vectors $q_l = [q_2 \; q_4]^T$ and joint angle vectors $q_j = [q_1 \; q_3]^T$. Then we have $D(q_l)$, $C(q_l, \dot{q}_l)$, $G(q_l)$, J, K as follows and all parameters are unknown in [16].

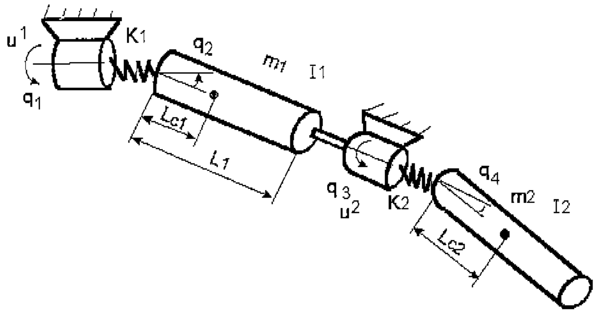

Fig. 1. 2-link flexible joint manipulator.

Select each value as follows:

$$S_1 = S_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \rho_k = 1, \; T_1 = T_2 = I_{2\times 2}, \quad T_3 = T_4 = I_{4\times 4}, \; w_1 = 0. \tag{113}$$

We choose $\varepsilon_1 = \varepsilon_2 = 10$ for the case 1. Based on the above values we can choose $\lambda_1 = 1$, and $\lambda_2 = 2$, to satisfy (31) and (35). So we select $K_{p1} = 1$, $K_{v1} = 2$, $K_{p2} = 2$ and $K_{v2} = 3$. Next, set

$$\Pi_1(z_1, \hat{\beta}_1) = \hat{\beta}_{11}(q_2^2 + q_4^2) + \hat{\beta}_{21}(\dot{q}_2^2 + \dot{q}_4^2), \tag{114}$$

$$\Pi_2(z_1, z_2, \hat{\beta}_2) = \hat{\beta}_{12}(q_2^2 + q_4^2) + \hat{\beta}_{22}(q_1^2 + q_3^2) + \hat{\beta}_{32}(\dot{q}_2^2 + \dot{q}_4^2) + \hat{\beta}_{42}(\dot{q}_1^2 + \dot{q}_3^2). \tag{115}$$

Now we have following controllers:

$$u_1 = -K_{p1}q_l - K_{v1}\dot{q}_l + p_1, \tag{116}$$

$$p_1 = [p_{11} \, p_{12}]^T, \tag{117}$$

$$p_{11} = \begin{cases} -\frac{\mu_{11}}{\|\mu_{11}\|}\Pi_1(q_l, \hat{\beta}_1), & \text{if } \|\mu_{11}\| > \varepsilon_1 \\ -\sin(\frac{\pi\mu_{11}}{2\varepsilon_1})\Pi_1(q_l, \hat{\beta}_1), & \text{if } \|\mu_{11}\| \leq \varepsilon_1, \end{cases} \tag{118}$$

$$p_{12} = \begin{cases} -\frac{\mu_{12}}{\|\mu_{12}\|}\Pi_1(q_l, \hat{\beta}_1), & \text{if } \|\mu_{12}\| > \varepsilon_1 \\ -\sin(\frac{\pi\mu_{12}}{2\varepsilon_1})\Pi_1(q_l, \hat{\beta}_1), & \text{if } \|\mu_{12}\| \leq \varepsilon_1, \end{cases} \tag{119}$$

$$[\mu_{11} \;\; \mu_{12}]^T = (\dot{q}_l + S_1 q_l)^T \Pi_1(q_l, \hat{\beta}_1), \tag{120}$$

$$u = -K_{p2}(q_j - u_1) - K_{v2}(\dot{q}_j - \dot{u}_1) + p_2, \tag{121}$$

where

$$\mu_2 = (\dot{q}_j - \dot{u}_1 + S_2(q_j - u_1))\Pi_2(z_1, z_2, \hat{\beta}_2). \tag{122}$$

$$p_2 = \begin{cases} -\dfrac{\mu_2}{\|\mu_2\|}\Pi_2 & \text{if } \|\mu_2\| > \varepsilon_2 \\ -\dfrac{\mu_2}{\varepsilon_2}\Pi_2 & \text{if } \|\mu_2\| \le \varepsilon_2, \end{cases} \tag{123}$$

The update laws of parameters is shown:

$$\begin{aligned} \hat{\beta}_1 &= [\,\hat{\beta}_{11}\ \ \hat{\beta}_{21}]^T \\ &= \begin{bmatrix} \|\dot{Z}_1 + S_1 Z_1\|(\,q_2^2 + q_4^2) \\ \|\dot{Z}_1 + S_1 Z_1\|(\,\dot{q}_2^2 + \dot{q}_4^2) \end{bmatrix}, \end{aligned} \tag{124}$$

$$\begin{aligned} \hat{\beta}_2 &= [\,\hat{\beta}_{12}\ \ \hat{\beta}_{22}\ \ \hat{\beta}_{32}\ \ \hat{\beta}_{42}]^T \\ &= \begin{bmatrix} \|\dot{Z}_3 + S_2 Z_3\|(\,q_2^2 + q_4^2) \\ \|\dot{Z}_3 + S_2 Z_3\|(\,q_1^2 + q_3^2) \\ \|\dot{Z}_3 + S_2 Z_3\|(\,\dot{q}_2^2 + \dot{q}_4^2) \\ \|\dot{Z}_3 + S_2 Z_3\|(\,\dot{q}_1^2 + \dot{q}_3^2) \end{bmatrix}. \end{aligned} \tag{125}$$

$\varepsilon_1(\cdot)$, $\varepsilon_2(\cdot)$ are chosen as:

$$\dot{\varepsilon}_1 = -\frac{2}{1.5}\varepsilon_1, \quad \dot{\varepsilon}_2 = -\frac{1}{1.5}\varepsilon_2. \tag{126}$$

For simulations, we choose $m_1 = 1$, $m_2 = 0.5 + 0.3\sin(0.2t)$, $L_1 = 1$, $L_{c2} = 0.5$, $K_1 = K_2 = 1 + 0.5\sin(0.2t)$, $J_1 = J_2 = 0.15$, $I_1 = I_2 = 1$, These parameters are unknown but the upper bounds exist without necessarily knowing those values. Simulation results are shown in Fig. 2-8. We decide to apply a nonlinear control which is designed via feedback linearization of the nominal system. Here, we adopt the input-output feedback linearization where the input is the joint torque u and output is the link angle q_l. Fig. 2-4 show the response of the system (151) with the nominal system based input-output feedback linearization control for the time-varying case. We choose nominal values as $\overline{m}_2 = 0.3$, $\overline{K}_1 = \overline{K}_2 = 0.5$, and $\overline{J}_1 = \overline{J}_2 = 0.1$. The others are identical to those as chosen above. Therefore, the control performance is not satisfactory as expected viewing the simulation results Fig. 2-4. This is since the design for the feedback linearization only utilizes the "nominal" part of parameters. Therefore, the further the true uncertain parameter is from the nominal one, the less likely the system performs close to when it is with the nominal parameter. Fig. 5-8 show the performance improvement by using the adaptive robust control. We see that both link and joint angles and angular velocities approach zero and have a satis-

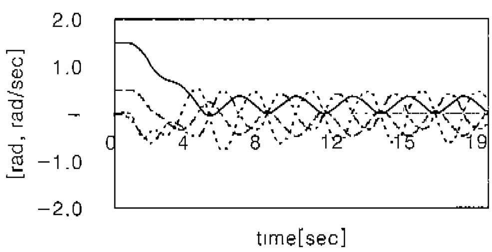

Fig. 2. History of link angles and angular velocities with feedback linearization control (—— q_2, ---- q_4, ——— $\dot{q}_2$, ... $\dot{q}_4$).

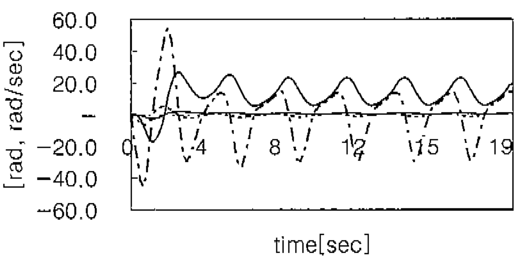

Fig. 3. History of joint angles and angular velocities with feedback linearization control (—— q_1, ---- q_3, ——— $\dot{q}_1$, ... $\dot{q}_3$).

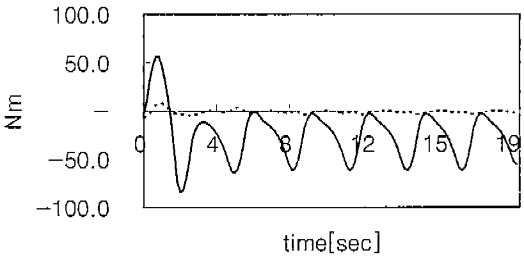

Fig. 4. Input torques at actuators with feedback linearization control (—— u_1, ... u_2).

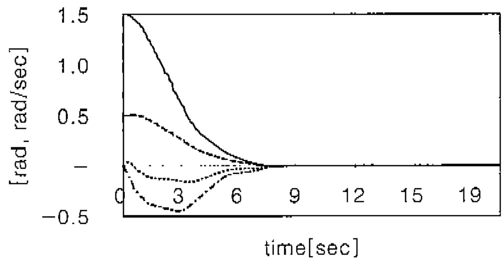

Fig. 5. History of link angles and angular velocities with adaptive robust control (—— q_2, ---- q_4, ——— $\dot{q}_2$, ... $\dot{q}_4$).

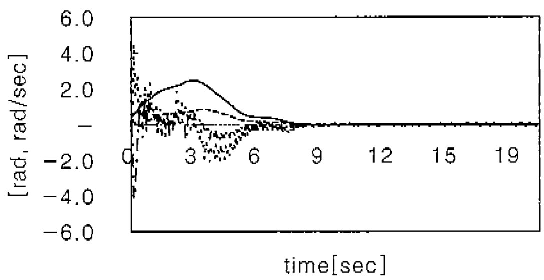

Fig. 6. History of joint angles and angular velocities with adaptive robust control (—— q_1, ——— q_3, ---- $\dot{q}_1$, ... $\dot{q}_3$).

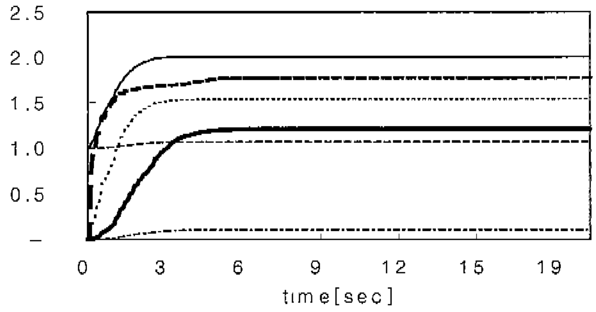

Fig. 7. History of parameters (—— β_{11}, ----- β_{21}, ... β_{12}, ------- β_{22}, —— β_{32}, - — β_{42}).

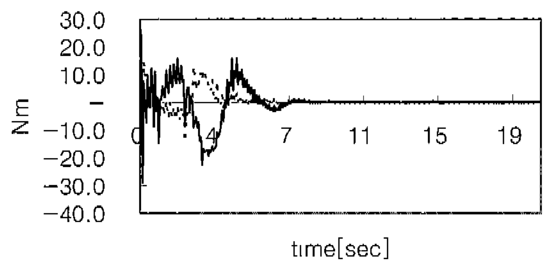

Fig. 8. Input torques at actuators with adaptive robust control (—— u_1, ... u_2).

factory transient performance. Fig. 7 shows the parameter histories and we see that all parameters remain bounded. All parameters converge to certain values, which are true ones or not, as time elapses. Nevertheless, by the Properties 1 through 5 we see that the parameter errors are uniformly bounded and stable. Fig. 8 shows the input torque histories. With the use of the adaptive robust controls, an improved system performance in terms of smaller settling time and steady state error is achieved when comparing to the nominal system based feedback linearization control.

VI. Conclusion

An adaptive robust control has been constructed for flexible joint manipulators which are nonlinear, time-varying and mismatched. State transformation via implanted control is introduced. No statistical property of the uncertainty is assumed and utilized. Only the existence of the bound of uncertainty is assumed, although the bound is not given a priori. The control has been utilized by combining states and parameters to be estimated, and guarantees 5 properties mentioned in section II regarding to uniform boundedness and uniform stability etc. Since ε_1 and ε_2 will decay, a careful selection of l_1 and l_2 to overcome chattering in practical implementations is recommended.

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