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SPACE-LIKE COMPLEX SUBMANIFOLDS OF AN INDEFINITE KÄHLER MANIFOLD

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ABSTRACT. The purpose of this paper is to study the complete submanifolds with restricted space-like and time-like holomorphic bisectional curvatures in an indefinite locally symmetric Kähler manifold.

1. Introduction

The theory of indefinite complex submanifolds of an indefinite complex space form is one of interesting topics in differential geometry and it has been investigated by many geometers from the various different points of view ([1]-[3], [6], [7], [9] and [15]).

Now, let M be an n-dimensional space-like complex hypersurface of an $(n+1)$-dimensional indefinite Kähler manifold M' of index 2. We denote by $H'(P', Q')$ the holomorphic bisectional curvature of M' for any holomorphic planes P' and Q'. In particular, the holomorphic bisectional curvature $H'(P', Q')$ for any two space-like holomorphic planes P' and Q' is said to be *space-like* and that for any space-like holomorphic plane P' and any time-like holomorphic plane Q' is said *time-like.* We call it simply a *space-like* or *time-like holomorphic bisectional curvature.* Then the first author and Nakagawa [8] proved the following theorem.

THEOREM A ([8]). *Let M be an $n(\geqq 2)$-dimensional complete space-like complex hypersurface of an $(n+1)$-dimensional indefinite Kähler manifold M' of index 2. If the ambient space is locally symmetric and if it has non-negative space-like holomorphic bisectional curvatures and*

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non-positive time-like holomorphic bisectional curvatures, then M is totally geodesic.

Let M' be an $(n+p)$-dimensional indefinite Kähler manifold of constant holomorphic sectional curvature c and of index $2p$, and let M be an n-dimensional space-like complex submanifold of M'. We introduce the concept of the normal curvature of M and the normal curvature operator of M (see Section 5). The time-like totally real bisectional curvature is closely related to the normal curvature of M. It seems to us to be interesting to give an information about the squared norm $|\alpha|_2 = h_2$ of the second fundamental form α of M. The Chern-type problem in the space-like Kähler geometry is as follows;

PROBLEM. For an n-dimensional complete space-like complex submanifold M of an $(n+p)$-dimensional indefinite complex space form $M_p^{n+p}(c)$ of constant holomorphic sectional curvature c of index $2p$ (> 0), does there exists a constant h in such a way that M is totally geodesic, provided $h_2 > h$?

In [7] and [16], the authors recently treated with this problem independently from the mutually different point of view, and they obtained partial solutions under the additional conditions, respectively. The purpose of this paper is to prove the following theorem. In order to fulfill this theorem, we generalize Theorem A in the case where M is a space-like complex submanifold and then, by applying this result, research the Chen-type problem from the view point of the holomorphic bisectional curvatures.

THEOREM. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+2)$-dimensional indefinite locally symmetric Kähler manifold M' of index 4. Assume that the normal connection of M is proper. If M' has non-negative space-like holomorphic bisectional curvatures and non-positive time-like holomorphic bisectional curvatures, then M is totally geodesic.*

2. Semi-definite Kähler manifolds

This section is concerned with recalling basic formulas on semi-definite Kähler manifolds. Let M be a complex $m(\geqq 2)$-dimensional semi-definite Kähler manifold equipped with the semi-definite Kähler metric tensor g and almost complex structure J. For the semi-definite Kähler structure $\{g, J\}$, it follows that J is integrable and the index of g is even, say $2q (0 \leqq q \leqq m)$. In the case where q is contained in the range $0 < q < m$, M is called an *indefinite Kähler manifold* and the structure $\{g, J\}$ is called an *indefinite Kähler structure.* And, in particular, in the case where $q = 0$ or m, M is only called a *Kähler manifold* and the structure $\{g, J\}$ is called a *Kähler structure.* We can choose a local field $\{E_\alpha\} = \{E_A, E_{A^*}\} = \{E_1, \ldots, E_m, E_{1^*}, \ldots, E_{m^*}\}$ of orthonormal frames on a neighborhood of M, where $E_{A^*} = JE_A$ and $A^* = m + A$. Here the indices $A, B, \ldots$ run from 1 to m and the indices $\alpha, \beta, \ldots$ run from 1 to $2m = m^*$. We set $U_A = \frac{1}{\sqrt{2}}(E_A - iE_{A^*})$ and $\overline{U}_A = \frac{1}{\sqrt{2}}(E_A + iE_{A^*})$, where i is the imaginary unit. Then $\{U_A\}$ constitutes a local field of unitary frames on the neighborhood of M. We remark here the fact that U_A is space-like or time-like is equivalent to the result that E_A is space-like or time-like. This is a complex linear frame which is orthonormal with respect to the semi-definite Kähler metric, that is, $g(U_A, \overline{U}_B) = \epsilon_A \delta_{AB}$, where

$$\epsilon_A = -1 \text{ or } 1 \text{ according as } \quad 1 \leqq A \leqq q \quad \text{or} \quad q+1 \leqq A \leqq m.$$

Let $\{\theta_\alpha\}$, $\{\theta_{\alpha\beta}\}$ and $\{\Theta_{\alpha\beta}\}$ be the canonical form, the connection form and the curvature form, respectively on M with respect to the local field $\{E_\alpha\} = \{E_A, E_{A^*}\}$ of orthonormal frames. Then we have the following structure equations.

$$\begin{aligned}
&d\theta_\alpha + \sum_\beta \epsilon_\beta \theta_{\alpha\beta} \wedge \theta_\beta = 0, \ \theta_{\alpha\beta} - \theta_{\alpha^*\beta^*} = 0, \\
&\theta_{\alpha^*\beta} + \theta_{\alpha\beta^*} = 0, \ \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0, \ \theta_{\alpha\beta^*} - \theta_{\beta\alpha^*} = 0, \\
(2.1) \qquad &d\theta_{\alpha\beta} + \sum_\gamma \epsilon_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} = \Theta_{\alpha\beta}, \\
&\Theta_{\alpha\beta} = -\frac{1}{2} \sum_{\gamma,\delta} \epsilon_\gamma \epsilon_\delta K_{\alpha\beta\gamma\delta} \theta_\gamma \wedge \theta_\delta,
\end{aligned}$$

where $K_{\alpha\beta\gamma\delta}$ denotes the components of the Riemannian curvature tensor R of M.

Now, let $\{\omega_A\} = \{\omega_1, \ldots, \omega_m\}$ be the dual coframe field with respect to the local field $\{U_A\}$ of unitary frames on the neighborhood of M.

Then $\{\omega_A\}$ consists of complex valued 1-forms of type $(1,0)$ on M such that $\omega_A(U_B) = \epsilon_A\delta_{AB}$ and $\omega_1,\dots,\omega_m,\overline{\omega}_1,\dots,\overline{\omega}_m$ are linearly independent. The semi-definite Kähler metric g of M can be expressed as $g = 2\sum_A \epsilon_A\omega_A \otimes \overline{\omega}_A$. Associated with the frame field $\{U_A\}$, there exist complex valued forms ω_{AB}, which are usually called *connection forms* on M such that they satisfy the structure equations of M:

$$
\begin{aligned}
& d\omega_A + \sum_B \epsilon_B\omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \overline{\omega}_{BA} = 0, \\
(2.2) \qquad & d\omega_{AB} + \sum_C \epsilon_C\omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\
& \Omega_{AB} = \sum_{C,D} \epsilon_C\epsilon_D R_{\bar{A}BC\bar{D}}\omega_C \wedge \overline{\omega}_D,
\end{aligned}
$$

where Ω_{AB} (resp. $R_{\bar{A}BC\bar{D}}$) denotes the curvature form (resp. the components of the semi-definite Riemannian curvature tensor R) of M. So, by (2.1) and (2.2), we obtain

$$(2.3) \quad R_{\bar{A}BC\bar{D}} = -\{(K_{ABCD} + K_{A^*BC^*D}) + i(K_{A^*BCD} - K_{ABC^*D})\}.$$

The equation (2.2) implies the skew-Hermitian symmetry of Ω_{AB}, which is equivalent to the symmetric condition

$$(2.4) \qquad R_{\bar{A}BC\bar{D}} = \overline{R}_{\bar{B}AD\bar{C}}.$$

Moreover, the first Bianchi identity $\sum_B \epsilon_B\Omega_{AB} \wedge \omega_B = 0$ is given by the exterior differential of the first equation and the third equation of (2.2), which implies the further symmetric relations

$$(2.5) \qquad R_{\bar{A}BC\bar{D}} = R_{\bar{A}CB\bar{D}} = R_{\bar{D}CB\bar{A}} = R_{\bar{D}BC\bar{A}}.$$

Next, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows;

$$S = \sum_{A,B} \epsilon_A\epsilon_B(S_{A\bar{B}}\omega_A \otimes \overline{\omega}_B + S_{\bar{A}B}\overline{\omega}_A \otimes \omega_B),$$

where $S_{A\bar{B}} = \sum_C \epsilon_C R_{\bar{C}CA\bar{B}} = S_{\bar{B}A} = \overline{S}_{\bar{A}B}$. The scalar curvature r of M is also given by $r = 2\sum_A \epsilon_A S_{A\bar{A}}$.

The components $R_{\bar{A}BC\bar{D}:E}$ and $R_{\bar{A}BC\bar{D}:\bar{E}}$ of the covariant derivative of the Riemannian curvature tensor R are given by

(2.6)

$$\sum_E \epsilon_E(R_{\bar{A}BC\bar{D}:E}\omega_E + R_{\bar{A}BC\bar{D}:\bar{E}}\bar{\omega}_E) = dR_{\bar{A}BC\bar{D}}$$
$$-\sum_E \epsilon_E(R_{\bar{E}BC\bar{D}}\bar{\omega}_{EA} + R_{\bar{A}EC\bar{D}}\omega_{EB} + R_{\bar{A}BE\bar{D}}\omega_{EC} + R_{\bar{A}BC\bar{E}}\bar{\omega}_{ED}).$$

The second Bianchi formula is given by

$$R_{\bar{A}BC\bar{D}:E} = R_{\bar{A}BE\bar{D}:C}. \tag{2.7}$$

Let M be an m-dimensional semi-definite Kähler manifold of index $2q(0 \leqq q \leqq m)$. A plane section P of the tangent space T_xM of M at any point x is said to be *non-degenerate*, provided that the restriction of $g_x|_P$ to P is non-degenerate. It is easily seen that P is non-degenerate if and only if it has a basis $\{X, Y\}$ such that $g(X,X)g(Y,Y)-g(X,Y)^2 \neq 0$. If the non-degenerate plane P is invariant by the complex structure J, then it said to be *holomorphic*. It is also trivial that the plane P is holomorphic if and only if it contains a vector X such that $g(X,X) \neq 0$. For the non-degenerate plane P spanned by X and Y in P, the sectional curvature $K(P)$ of P is usually defined by

$$K(P) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

The holomorphic plane spanned by space-like or time-like vectors X and JX is said to be *space-like* or *time-like*, respectively. The sectional curvature $K(P)$ of the non-degenerate holomorphic plane P is called the *holomorphic sectional curvature*, which is denoted by $H(P)$. The semi-definite Kähler manifold M is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for any non-degenerate holomorphic plane P and any point on M. Then M is called a *semi-definite complex space form*, which is denoted by $M_q^m(c)$ provided that it is of constant holomorphic sectional curvature c, of complex dimension m and of index $2q(\geqq 0)$. It is seen in Wolf [17] that the standard models of semi-definite complex space forms are the following three kinds: the semi-definite complex projective space $CP_q^m(c)$, the semi-definite complex Euclidean space C_q^m or the semi-definite complex hyperbolic space $CH_q^m(c)$, according as $c > 0$, $c = 0$ or $c < 0$. For any integer $q(0 \leqq q \leqq m)$, it is seen by [17] that they are complete simply connected semi-definite complex space forms of dimension m and of index $2q$. The Riemannian curvature tensor $R_{\bar{A}BC\bar{D}}$ of $M = M_q^m(c)$ is given by

$$R_{\bar{A}BC\bar{D}} = \frac{c}{2}\epsilon_B\epsilon_C(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}).$$

Given two holomorphic planes P and Q in T_xM at any point x in M, the holomorphic bisectional curvature $H(P,Q)$ determined by the two planes P and Q of M is defined by

$$H(P,Q) = \frac{g(R(X,JX)JY,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{2.8}$$

where X (resp. Y) is a non-zero vector in P (resp. Q). It is a simple matter to verify that the right hand side in (2.8) depends only on P and Q, so, it is well defined. It may be also denoted by $H(P,Q) = H(X,Y)$. It is easily seen that $H(P,P) = H(P) = H(X,X) =: H(X)$ is the holomorphic sectional curvature determined by the holomorphic plane P, where X is a non-zero vector in P. We denote by P_A the holomorphic plane $[E_A, JE_A]$ spanned by E_A and $JE_A = E_{A^*}$. We set

$$H(P_A,P_B) = H(E_A,E_B) = H_{AB}, \quad A \neq B,$$
$$H(P_A,P_A) = H(P_A) = H_{AA} = H_A.$$

The holomorphic bisectional curvature $H_{AB}(A \neq B)$ and the holomorphic sectional curvature H_A are given by

$$H_{AB} = \frac{g(R(E_A,JE_A)JE_B,E_B)}{g(E_A,E_A)g(E_B,E_B)} = -\epsilon_A\epsilon_B K_{AA^*BB^*}, \quad A \neq B,$$
$$H_A = \frac{g(R(E_A,JE_A)JE_A,E_A)}{g(E_A,E_A)g(E_A,E_A)} = -K_{AA^*AA^*}.$$

By (2.3), we have

$$H_{AB} = \epsilon_A\epsilon_B R_{\bar{A}AB\bar{B}} \ (A \neq B), \ H_A = R_{\bar{A}AA\bar{A}}. \tag{2.9}$$

Now, we introduce here a fundamental property for the generalized maximum principal due to Omori [12] and Yau [19].

THEOREM 2.1 ([12], [19]). *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below on M. If a C^2-function f is bounded from above on M, then, for any positive constant ϵ, there exists a point p such that*

$$|\nabla f(p)| < \epsilon, \quad \Delta f(p) < \epsilon, \quad \sup\ f - \epsilon < f(p),$$

where ∇f is the gradient of the function f and Δ denotes the Laplacian operator on M, and sup f *denotes the supremum of f.*

In order to apply the generalized maximum principle to the practical problem the Liouville-type theorem due to Choi, the first author and Suh [5] is very useful. Here a slight more extended property than their theorem is introduced. Since it is proved by the careful checking of their proof, the following theorem is quoted without proof.

THEOREM 2.2 ([5]). *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below and let F be any polynomial of one variable f with constant coefficients $c_0, \cdots, c_{k+1}$ such that*

$$F(f) = c_0 f^n + c_1 f^{n-1} + \cdots + c_k f^{n-k} + c_{k+1},$$

where $n > 1$, $n - k > 0$ and $c_0 > c_{k+1}$. If a C^2-function f satisfies $\Delta f \geqq F(f)$, then we have $F(\sup\ f) \leqq 0$.

3. Space-like complex submanifolds

This section is concerned with space-like complex submanifolds of an indefinite Kähler manifold. First of all, the basic formulas for the theory of space-like complex submanifolds are prepared.

Let M' be an $(n+p)$-dimensional connected indefinite Kähler manifold of index $2p(>0)$ with the indefinite Kähler structure (g', J'). Let M be an n-dimensional connected space-like complex submanifold of M' and let g be the induced Kähler metric tensor on M from g'. We can choose a local field $\{U_A\} = \{U_1, \ldots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that restricted to M, $U_1, \ldots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated:

$$A, B, C, \cdots = 1, \ldots, n, n+1, \ldots, n+p\ ;$$
$$i, j, k, \cdots = 1, \ldots, n\ ; \qquad x, y, z, \cdots = n+1, \ldots, n+p.$$

With respect to the frame field $\{U_A\}$, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame field. Then the indefinite Kähler metric tensor g' of M' is given by $g' = 2\sum_A \epsilon_A \omega_A \otimes \overline{\omega}_A$, where $\{\epsilon_A\} = \{\epsilon_i, \epsilon_x\}$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$\begin{aligned} & d\omega_A + \sum_B \epsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \overline{\omega}_{AB} = 0, \\ & d\omega_{AB} + \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \\ & \Omega'_{AB} = \sum_{C,D} \epsilon_C \epsilon_D R'_{\bar{A}BC\bar{D}} \omega_C \wedge \overline{\omega}_D, \end{aligned} \tag{3.1}$$

where Ω'_{AB} (resp. $R'_{\bar{A}BC\bar{D}}$) denotes the curvature form with respect to the frame field $\{U_A\}$ (resp. the components of the indefinite Riemannian curvature tensor R') of M'. Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced Kähler metric tensor g of M is given by

$$g = 2 \sum_j \epsilon_j \omega_j \otimes \overline{\omega}_j.$$

Then $\{U_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{U_j\}$, which consists of complex valued 1-forms of type (1.0) on M. Moreover, $\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n$ are linearly independent, and $\{\omega_j\}$ is the canonical forms on M. It follows from (3.2) and Cartan's lemma that the exterior derivatives of (3.2) give rise to

$$\omega_{xi} = \sum_j \epsilon_j h_{ij}{}^x \omega_j, \quad h_{ij}{}^x = h_{ji}{}^x. \tag{3.3}$$

The quadratic form $\alpha = \sum_{x,i,j} \epsilon_x \epsilon_i \epsilon_j h_{ij}{}^x \omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle on M in M' is called the *second fundamental form* of the submanifold M. From the structure equations for M', the structure equations for M are similarly given by

$$
(3.4) \qquad \begin{aligned} &d\omega_i + \sum_j \epsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \overline{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \ \Omega_{ij} = \sum_{k,l} \epsilon_k \epsilon_l R_{\bar{i}jk\bar{l}} \omega_k \wedge \overline{\omega}_l, \end{aligned}
$$

where $\Omega = (\Omega_{ij})$ (resp. $R_{\bar{i}jk\bar{l}}$) denotes the curvature form with respect to the unitary frame field $\{U_A\}$ (resp. the component of the semi-definite Riemannian curvature tensor R) of M.

Moreover, the following relationships are obtained:

$$
(3.5) \quad d\omega_{xy} + \sum_z \epsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \quad \Omega_{xy} = \sum_{k,l} \epsilon_k \epsilon_l R_{\bar{x}yk\bar{l}} \omega_k \wedge \overline{\omega}_l,
$$

where Ω_{xy} is called the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1)-(3.4) that we have the Gauss equation

$$
(3.6) \qquad R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \sum_x \epsilon_x h_{jk}{}^x \overline{h}_{il}{}^x.
$$

And by means of (3.1)-(3.3) and (3.5), we have

$$
(3.7) \qquad R_{\bar{x}yk\bar{l}} = R'_{\bar{x}yk\bar{l}} + \sum_j \epsilon_j h_{kj}{}^x \overline{h}_{jl}{}^y.
$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r of M are given by

$$
(3.8) \qquad S_{i\bar{j}} = \sum_k \epsilon_k R'_{\bar{k}ki\bar{j}} - h_{i\bar{j}}{}^2,
$$

$$
(3.9) \qquad r = 2\left(\sum_{j,k} \epsilon_j \epsilon_k R'_{\bar{j}jk\bar{k}} - h_2\right),
$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{x,k} \epsilon_x \epsilon_k h_{ik}{}^x \overline{h}_{kj}{}^x$ and $h_2 = \sum_j \epsilon_j h_{j\bar{j}}{}^2$. Now, the components $h_{ijk}{}^x$ and $h_{ij\bar{k}}{}^x$ of the covariant derivative of the second fundamental form of M are given by

$$(3.10)\quad \sum_k \epsilon_k (h_{ijk}{}^x \omega_k + h_{ij\bar{k}}{}^x \overline{\omega}_k) = dh_{ij}{}^x - \sum_k \epsilon_k (h_{kj}{}^x \omega_{ki} + h_{ik}{}^x \omega_{kj}) + \sum_y \epsilon_y h_{ij}{}^y \omega_{xy}.$$

Then, substituting $dh_{ij}{}^x$ in this definition into the exterior derivative of (3.3) and using (3.1)-(3.4) and (3.10), we have

$$(3.11)\qquad h_{ijk}{}^x = h_{ikj}{}^x, \quad h_{ij\bar{k}}{}^x = -R'_{\bar{x}ij\bar{k}}.$$

Similarly, the components $h_{ijkl}{}^x$ and $h_{ijk\bar{l}}{}^x$ (resp. $h_{ij\bar{k}l}{}^x$ and $h_{ij\bar{k}\bar{l}}{}^x$) of the covariant derivative of $h_{ijk}{}^x$ (resp. $h_{ij\bar{k}}{}^x$) can be defined by

$$(3.12)\quad \sum_l \epsilon_l (h_{ijkl}{}^x \omega_l + h_{ijk\bar{l}}{}^x \overline{\omega}_l) = dh_{ijk}{}^x - \sum_l \epsilon_l (h_{ljk}{}^x \omega_{li} + h_{ilk}{}^x \omega_{lj} + h_{ijl}{}^x \omega_{lk}) + \sum_y \epsilon_y h_{ijk}{}^y \omega_{xy},$$

$$(3.13)\quad \sum_l \epsilon_l (h_{ij\bar{k}l}{}^x \omega_l + h_{ij\bar{k}\bar{l}}{}^x \overline{\omega}_l) = dh_{ij\bar{k}}{}^x - \sum_l \epsilon_l (h_{lj\bar{k}}{}^x \omega_{li} + h_{il\bar{k}}{}^x \omega_{lj} + h_{ij\bar{l}}{}^x \overline{\omega}_{lk}) + \sum_y \epsilon_y h_{ij\bar{k}}{}^y \omega_{xy}.$$

Differentiating (3.10) exteriorly and using the properties $d^2 = 0$, (3.4), (3.5), (3.8), (3.10) and (3.11), we have the following Ricci formula for the second fundamental form

(3.14)

$$h_{ijkl}{}^x = h_{ijlk}{}^x, \quad h_{ij\bar{k}\bar{l}}{}^x = h_{ij\bar{l}\bar{k}}{}^x,$$

$$h_{ijk\bar{l}}{}^x - h_{ij\bar{l}k}{}^x = \sum_r \epsilon_r (R_{\bar{l}ki\bar{r}} h_{rj}{}^x + R_{\bar{l}kj\bar{r}} h_{ri}{}^x) - \sum_y \epsilon_y R_{\bar{l}ky\bar{x}} h_{ij}{}^y.$$

In particular, let the ambient space M' be an $(n+p)$-dimensional semi-definite complex space form $M_{s+t}^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2(s+t)$ $(0 \leqq s \leqq n,\ 0 \leqq t \leqq p)$. Then, we get

$$\begin{aligned} R_{\bar{i}jk\bar{l}} &= \frac{c}{2}\epsilon_j\epsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \epsilon_x h_{jk}{}^x \overline{h}_{il}{}^x, \\ S_{i\bar{j}} &= \frac{c}{2}(n+1)\epsilon_i\delta_{ij} - h_{i\bar{j}}{}^2,\ r = cn(n+1) - 2h_2\ ,\ h_{ij\bar{k}}{}^x = 0, \\ h_{ijk\bar{l}}{}^x &= \frac{c}{2}(\epsilon_k h_{ij}{}^x\delta_{kl} + \epsilon_i h_{jk}{}^x\delta_{il} + \epsilon_j h_{ki}{}^x\delta_{jl}) \\ &\quad - \sum_{y,r}\epsilon_y\epsilon_r(h_{ri}{}^x h_{jk}{}^y + h_{rj}{}^x h_{ki}{}^y + h_{rk}{}^x h_{ij}{}^y)\overline{h}_{rl}{}^y. \end{aligned}$$

4. The Laplacian operator

In this section, the Laplacian of the squared norm of the second fundamental form on a space-like complex submanifold of an indefinite Kähler manifold will be calculated. Let M' be an $(n+p)$-dimensional indefinite Kähler manifold of index $2p$ and let M be an n-dimensional space-like complex submanifold of M'. Let f be any smooth C^2-function on M. The components f_i and $f_{\bar{i}}$ of the exterior derivative df of f are given by

$$df = \sum_i \epsilon_i(f_i\omega_i + f_{\bar{i}}\overline{\omega}_i).$$

Moreover, the components f_{ij} and $f_{i\bar{j}}$ (resp. $f_{\bar{i}j}$ and $f_{\bar{i}\bar{j}}$) of the covariant derivative of f_i (resp. $f_{\bar{i}}$) can be defined by

$$\begin{aligned} \sum_j \epsilon_j(f_{ij}\omega_j + f_{i\bar{j}}\overline{\omega}_j) &= df_i - \sum_j \epsilon_j f_j\omega_{ji}, \\ \sum_j \epsilon_j(f_{\bar{i}j}\omega_j + f_{\bar{i}\bar{j}}\overline{\omega}_j) &= df_{\bar{i}} - \sum_j \epsilon_j f_{\bar{j}}\overline{\omega}_{ji}. \end{aligned}$$

The Laplacian of the function f is by definition given as

$$\Delta f = \sum_j \epsilon_j(f_{j\bar{j}} + f_{\bar{j}j}) = 2\sum_j \epsilon_j f_{j\bar{j}}. \tag{4.1}$$

Now, we calculate the Laplacian of the squared norm $h_2 = |\alpha|_2$ of the second fundamental form α on M. By (3.13) and the second equation of (3.11), we have

$$\begin{aligned}
&\sum_l \epsilon_l (h_{ij\bar{k}l}{}^x \omega_l + h_{ij\bar{k}\bar{l}}{}^x \overline{\omega}_l) \\
&= -dR'_{\bar{x}ij\bar{k}} + \sum_l \epsilon_l (R'_{\bar{x}lj\bar{k}}\omega_{li} + R'_{\bar{x}il\bar{k}}\omega_{lj} + R'_{\bar{x}ij\bar{l}}\overline{\omega}_{lk}) - \sum_y \epsilon_y R'_{\bar{y}ij\bar{k}}\omega_{xy} \\
&= -dR'_{\bar{x}ij\bar{k}} + \sum_A \epsilon_A (R'_{\bar{x}Aj\bar{k}}\omega_{Ai} + R'_{\bar{x}iA\bar{k}}\omega_{Aj} + R'_{\bar{x}ij\bar{A}}\overline{\omega}_{Ak}) \\
&\quad - \sum_A \epsilon_A R'_{\bar{A}ij\bar{k}}\omega_{xA} - \sum_y \epsilon_y (R'_{\bar{x}yj\bar{k}}\omega_{yi} + R'_{\bar{x}iy\bar{k}}\omega_{yj} + R'_{\bar{x}ij\bar{y}}\overline{\omega}_{yk}) \\
&\quad + \sum_l \epsilon_l R'_{\bar{l}ij\bar{k}}\omega_{xl},
\end{aligned}$$

from which together with (2.6), (3.1) and (3.3), it follows that we have

$$\begin{aligned}
&\sum_l \epsilon_l (h_{ij\bar{k}l}{}^x \omega_l + h_{ij\bar{k}\bar{l}}{}^x \overline{\omega}_l) \\
&= -\sum_A \epsilon_A (R'_{\bar{x}ij\bar{k}:A}\omega_A + R'_{\bar{x}ij\bar{k}:\bar{A}}\overline{\omega}_A) \\
&\quad - \sum_{y,l} \epsilon_y \epsilon_l (R'_{\bar{x}yj\bar{k}} h_{il}{}^y \omega_l + R'_{\bar{x}iy\bar{k}} h_{jl}{}^y \omega_l + R'_{\bar{x}ij\bar{y}} \overline{h}_{kl}{}^y \overline{\omega}_l) \\
&\quad + \sum_{l,r} \epsilon_l \epsilon_r R'_{\bar{r}ij\bar{k}} h_{rl}{}^x \omega_l.
\end{aligned}$$

Comparing the coefficients of ω_l in the above equation, we have

(4.2)
$$h_{ij\bar{k}l}{}^x = -R'_{\bar{x}ij\bar{k}:l} - \sum_y \epsilon_y (R'_{\bar{x}yj\bar{k}} h_{il}{}^y + R'_{\bar{x}iy\bar{k}} h_{jl}{}^y) + \sum_r \epsilon_r R'_{\bar{r}ij\bar{k}} h_{rl}{}^x.$$

On the other hand, from (3.14) we get

(4.3)
$$\begin{aligned}
h_{ijk\bar{l}}{}^x - h_{ij\bar{l}k}{}^x =& \sum_r \epsilon_r (R'_{\bar{r}ki\bar{l}} h_{rj}{}^x + R'_{\bar{r}kj\bar{l}} h_{ri}{}^x) - \sum_y \epsilon_y R'_{\bar{l}ky\bar{x}} h_{ij}{}^y \\
&- \sum_{y,r} \epsilon_y \epsilon_r (h_{ik}{}^y \overline{h}_{rl}{}^y h_{rj}{}^x + h_{jk}{}^y \overline{h}_{rl}{}^y h_{ri}{}^x) \\
&- \sum_{y,r} \epsilon_y \epsilon_r h_{kr}{}^x \overline{h}_{rl}{}^y h_{ij}{}^y,
\end{aligned}$$

where we have used (3.6) and (3.7). By (4.2) and (4.3), we obtain

(4.4)

$$\begin{aligned} h_{ijk\bar{l}}{}^{x} = &- R'_{\bar{x}ij\bar{l}:k} - \sum_{y} \epsilon_y (R'_{\bar{x}yj\bar{l}} h_{ik}{}^{y} + R'_{\bar{x}yi\bar{l}} h_{jk}{}^{y} + R'_{\bar{x}yk\bar{l}} h_{ij}{}^{y}) \\ &+ \sum_{r} \epsilon_r (R'_{\bar{r}jk\bar{l}} h_{ri}{}^{x} + R'_{\bar{r}ik\bar{l}} h_{rj}{}^{x} + R'_{\bar{r}ij\bar{l}} h_{rk}{}^{x}) \\ &- \sum_{y,r} \epsilon_y \epsilon_r (h_{ik}{}^{y} \overline{h}_{rl}{}^{y} h_{rj}{}^{x} + h_{jk}{}^{y} \overline{h}_{rl}{}^{y} h_{ri}{}^{x}) \\ &- \sum_{y,r} \epsilon_y \epsilon_r h_{kr}{}^{x} \overline{h}_{rl}{}^{y} h_{ij}{}^{y}. \end{aligned}$$

The matrix $A = (A_y{}^x)$ of order p defined by $A_y{}^x = \sum_{i,j} \epsilon_i \epsilon_j h_{ij}{}^x \overline{h}_{ij}{}^y$ is a Hermitian one. Since M is space-like and the normal space is time-like, it is a positive semi-definite Hermitian matrix of order p. Summing up $k = l$ in (4.4), we have

(4.5)

$$\begin{aligned} &\sum_{k} \epsilon_k h_{ijk\bar{k}}{}^{x} \\ &= - \sum_{k} \epsilon_k R'_{\bar{x}ij\bar{k}:k} - \sum_{y,k} \epsilon_y \epsilon_k (R'_{\bar{x}yj\bar{k}} h_{ik}{}^{y} + R'_{\bar{x}yi\bar{k}} h_{jk}{}^{y} + R'_{\bar{x}yk\bar{k}} h_{ij}{}^{y}) \\ &\quad + \sum_{k,l} \epsilon_k \epsilon_l (R'_{\bar{l}jk\bar{k}} h_{li}{}^{x} + R'_{\bar{l}ik\bar{k}} h_{lj}{}^{x} + R'_{\bar{l}ij\bar{k}} h_{lk}{}^{x}) \\ &\quad - \sum_{k} \epsilon_k (h_{i\bar{k}}{}^{2} h_{kj}{}^{x} + h_{j\bar{k}}{}^{2} h_{ki}{}^{x}) - \sum_{y} \epsilon_y A_y{}^{x} h_{ij}{}^{y}. \end{aligned}$$

Moreover by (4.1), we see

(4.6)

$$\Delta h_2 = \sum_{i,j,k} \epsilon_i \epsilon_j \epsilon_k \left\{ \left(\sum_{x} \epsilon_x h_{ij}{}^{x} \overline{h}_{ij}{}^{x} \right)_{k\bar{k}} + \left(\sum_{x} \epsilon_x h_{ij}{}^{x} \overline{h}_{ij}{}^{x} \right)_{\bar{k}k} \right\}.$$

The first term in the right hand side of (4.6) is given by

$$\sum_{x,i,j,k} \epsilon_x \epsilon_i \epsilon_j \epsilon_k (h_{ijk\bar{k}}{}^{x} \overline{h}_{ij}{}^{x} + h_{ijk}{}^{x} \overline{h}_{ijk}{}^{x} + h_{ij\bar{k}}{}^{x} \overline{h}_{ij\bar{k}}{}^{x} + h_{ij}{}^{x} \overline{h}_{ij\bar{k}k}{}^{x}).$$

And the second term is expressed as

$$\sum_{x,i,j,k} \epsilon_x\epsilon_i\epsilon_j\epsilon_k(h_{ij\bar{k}k}{}^x\overline{h}_{ij}{}^x + h_{ij\bar{k}}{}^x\overline{h}_{ij\bar{k}}{}^x + h_{ijk}{}^x\overline{h}_{ijk}{}^x + h_{ij}{}^x\overline{h}_{ijk\bar{k}}{}^x).$$

On the other hand, by (4.2), we have

$$\sum_k \epsilon_k h_{ij\bar{k}k}{}^x = -\sum_k \epsilon_k R'_{\bar{x}ij\bar{k}:k} - \sum_{y,k}\epsilon_y\epsilon_k(R'_{\bar{x}yj\bar{k}}h_{ik}{}^y + R'_{\bar{x}yi\bar{k}}h_{jk}{}^y)$$
$$+\sum_{k,r}\epsilon_k\epsilon_r R'_{\bar{r}ij\bar{k}}h_{rk}{}^x.$$

Since A is the positive semi-definite Hermitian matrix of order p, its eigenvalues λ_x are all non-negative real valued functions on M and it is easily seen that we have

$$(4.7)\qquad \sum_x \lambda_x = TrA = -h_2,\quad h_2{}^2 \geqq TrA^2 = \sum_x \lambda_x{}^2 \geqq \frac{1}{p}h_2{}^2.$$

Substituting these three relations into (4.6), we obtain the formula for the Laplacian of the squared norm h_2 of the second fundamental form on M. That is, we have

(4.8)

$$\begin{aligned}\Delta h_2 =& 2|\nabla\alpha|_2 - 2\sum_{x,i,j,k}\epsilon_x\epsilon_i\epsilon_j\epsilon_k R'_{\bar{x}ij\bar{k}:k}\overline{h}_{ij}{}^x - 2\sum_{x,i,j,k}\epsilon_x\epsilon_i\epsilon_j\epsilon_k R'_{\bar{i}xk\bar{j}:\bar{k}}h_{ij}{}^x\\ &-8\sum_{x,y,i,j,k}\epsilon_x\epsilon_y\epsilon_i\epsilon_j\epsilon_k R'_{\bar{x}yj\bar{k}}h_{ki}{}^y\overline{h}_{ij}{}^x - 2\sum_{x,y,k}\epsilon_x\epsilon_y\epsilon_k A_x{}^y R'_{\bar{x}yk\bar{k}}\\ &+4\sum_{x,i,j,k,l}\epsilon_x\epsilon_i\epsilon_j\epsilon_k\epsilon_l R'_{\bar{k}ij\bar{l}}h_{kl}{}^x\overline{h}_{ij}{}^x\\ &+4\sum_{i,j,k}\epsilon_i\epsilon_j\epsilon_k R'_{\bar{i}jk\bar{k}}h_{i\bar{j}}{}^2 - 4h_4 - 2TrA^2,\end{aligned}$$

where we have used (2.4), (2.5), (4.5), (4.6) and $h_4 = \sum_{i,j}\epsilon_i\epsilon_j h_{i\bar{j}}{}^2 h_{j\bar{i}}{}^2$, where the squared norm $|\nabla\alpha|_2$ of the covariant derivative $\nabla\alpha$ of the second fundamental form α on M is defined by

$$(4.9)\qquad |\nabla\alpha|_2 = \sum_{x,i,j,k}\epsilon_x\epsilon_i\epsilon_j\epsilon_k(h_{ijk}{}^x\overline{h}_{ijk}{}^x + h_{ij\bar{k}}{}^x\overline{h}_{ij\bar{k}}{}^x).$$

5. Normal curvature tensor

In this section, we introduce the concept the normal curvature tensor on the space-like complex submanifold in an indefinite Kähler manifold and research its properties.

Let M' be an $(n+p)$-dimensional indefinite Kähler manifold of index $2p$ equipped with indefinite Kähler structure $\{g', J'\}$ and let M be an n-dimensional space-like complex submanifold of M' endowed with induced Kähler structure $\{g, J\}$ from the indefinite Kähler structure $\{g', J'\}$. Let us denote by $\nabla^{\perp}$ the normal connection on M, namely, it is the mapping of $TM \times NM$ into NM defined by

$$\nabla^{\perp}(X, V) = \nabla^{\perp}{}_X V = \text{the normal part of } \nabla'{}_X V$$

for any tangent vector field X in TM and any normal vector field V in NM, where ∇' is the Kähler connection on M' and TM and NM are the tangent bundle and the normal bundle of M, respectively (cf. [14]). The normal curvature tensor $R^{\perp}$ on M is defined by

$$R^{\perp}(X, Y)V = (\nabla^{\perp}{}_X \nabla^{\perp}{}_Y - \nabla^{\perp}{}_Y \nabla^{\perp}{}_X - \nabla^{\perp}{}_{[X,Y]})V,$$

where $X, Y \in TM$ and $V \in NM$. If it satisfies

$$R^{\perp}(X, Y)V = f\, g(X, JY)J'V,$$

where f is any function on M, then the normal connection $\nabla^{\perp}$ is said to be *proper*. In particular, if f is a non-zero constant or zero on M, then it is said to be *semi-flat* or *flat*, respectively.

REMARK 5.1. For the justification of the concept of flatness and semi-flatness, see Chen [4] and Yano and Kon [18], respectively.

On the other hand, the proper case is treated by Ki and Nakagawa [6].

REMARK 5.2. In the semi-Riemannian geometry, the shape operator A on the indefinite Einstein hypersurface M of index $2s$ in $M^{n+1}_{s+1}(c)$ can be not necessarily diagonalized. By the classification of the self-adjoint endomorphisms of a scalar product, we have the following properties;

(1) A is diagonalizable,

(2) A is not diagonalizable, but either $\epsilon_{n+1}h_2 < 0$ or $h_2 = 0$ and not totally geodesic.

An indefinite Einstein hypersurface is said to be *proper* if the shape operator A is diagonalizable ([6]). The terminology, "proper" of the normal connection is named after the concept.

Now, in order to consider the normal curvature transformation, we see the local version of the normal curvature tensor. By means of (3.7), we can define a linear transformation T_N on the np-dimensional complex vector space Ξ^{np} consisting of tensors (ξ_{xk}) at each point on M by

$$T_N(\xi_{xk}) = (\eta_{xk}), \quad \eta_{xk} = \sum_{y,l} \epsilon_y \epsilon_l R_{\bar{x}yk\bar{l}} \xi_{yl}.$$

We denote by $({R_{yl}}^{xk})$ the matrix of the linear transformation T_N. The linear operator defined by the $np \times np$ Hermitian matrix $({R_{yl}}^{xk})$ is called the *normal curvature operator* on M. Then T_N is the self-adjoint operator with respect to the definite metric canonically defined on Ξ^{np} (cf. [7]). We assume that the matrix $({R_{yl}}^{xk})$ is diagonalizable. In this case, we can choose suitably an indefinite unitary frame field $\{U_A\} = \{U_j, U_x\}$ in such a way that, it satisfies

$$R_{\bar{x}yk\bar{l}} = \epsilon_x \epsilon_k f_{xk} \delta^{xk}_{yl} = \epsilon_x \epsilon_k f_{xk} \delta_{xy} \delta_{kl}, \tag{5.1}$$

where every eigenvalue f_{xk} of T_N is a real valued function on M. By (3.7) and (5.1), we have

$$R'_{\bar{x}yk\bar{l}} = \epsilon_x \epsilon_k f_{xk} \delta_{xy} \delta_{kl} - \sum_j \epsilon_j {h_{kj}}^x {\overline{h}_{jl}}^y. \tag{5.2}$$

REMARK 5.3. In the space-like complex hypersurface M, the normal connection is always proper.

6. Locally symmetric spaces

In this section, let M' be an $(n+p)$-dimensional indefinite Kähler manifold of index $2p$. For two holomorphic planes $P' = [X, J'X]$ and

$Q' = [Y, J'Y]$, where X and Y are orthogonal vectors, the holomorphic bisectional curvature $H'(P', Q') = H'(X, Y)$ on M' is defined by

$$H'(P', Q') = H'(X, Y) = \frac{g'(R'(X, J'X)J'Y, Y)}{g'(X, X)g'(Y, Y)} .$$

Assume that M' is locally symmetric, the normal connection of M is proper and it satisfies the following two conditions:

(∗1) The space-like holomorphic bisectional curvature is bounded from below by a_1.

(∗2) The time-like holomorphic bisectional curvature is bounded from above by a_2.

Then M' is said to satisfy the condition (∗) if it satisfies the above conditions (∗1) and (∗2).

Let M be an n-dimensional space-like complex submanifold of M'. For the local field $\{E_A, E_{A^*}\}$ of orthonormal frames associated with the manifold chosen in Section 2, we have by (2.9)

$$H'(P'_j, P'_k) = H'(E_j, E_k) = H'_{jk} = \epsilon_j\epsilon_k R'_{\bar{j}jk\bar{k}},$$
$$H'(P'_x, P'_k) = H'(E_x, E_k) = H'_{xk} = \epsilon_x\epsilon_k R'_{\bar{x}xk\bar{k}}.$$

for the holomorphic plane $P'_A = [E_A, J'E_A]$. Then it satisfies

$$H'_{jk} \geqq a_1, \quad H'_{xk} \leqq a_2.$$

REMARK 6.1. Let M' be an $(n+p)$-dimensional indefinite complex space form $M_p^{n+p}(c)$ of index $2p$ and of constant holomorphic sectional curvature c. Then M' is locally symmetric and it satisfies the condition (∗) and we may consider $a_1 = a_2 = \frac{c}{2}$ if c is non-negative and $a_1 = c$, $a_2 = \frac{c}{2}$ if c is non-positive.

First of all, we estimate Δh_2 from the above on the space-like complex submanifold M. In order to estimate the fourth term and the fifth one in the right hand side of (4.8), we prepare for the basic formulas and a few of properties of the normal curvature operator T_N defined on the submanifold.

Let (M', g') be an $(n+p)$-dimensional indefinite Kähler manifold of index $2p$ and let M be an n-dimensional space-like complex submanifold of M'. Now, we check the relation between the normal curvature and the totally real bisectional curvature $H'(P', Q')$ for a space-like holomorphic plane P' and a time-like holomorphic plane Q' in M'. Accordingly, we have by (5.2)

$$H'_{xk} = -R'_{\bar{x}xk\bar{k}} = -\epsilon_x\epsilon_k f_{xk} + \sum_j \epsilon_j h_{kj}{}^x \overline{h}_{kj}{}^x. \tag{6.1}$$

Between the holomorphic bisectional curvature and the normal curvature, we get the following relation. By (6.1) and the condition (*2), the normal curvature satisfies

$$f_{xk} = H'_{xk} - \sum_j \epsilon_j h_{kj}{}^x \overline{h}_{kj}{}^x \leqq a_2 - \sum_j \epsilon_j h_{kj}{}^x \overline{h}_{kj}{}^x \tag{6.2}$$

from which we can estimate the fourth term and the fifth one in the right hand side of (4.8) as follows;

$$\begin{aligned}
\text{the fourth term} &= -8 \sum_{x,y,i,j,k} \epsilon_x\epsilon_y\epsilon_i\epsilon_j\epsilon_k R'_{\bar{x}yj\bar{k}} h_{ki}{}^y \overline{h}_{ij}{}^x \\
&= -8 \sum_{x,y,i,j,k} \epsilon_x\epsilon_y\epsilon_i\epsilon_j\epsilon_k \left(\epsilon_x\epsilon_j f_{xj}\delta_{xy}\delta_{jk} - \sum_l \epsilon_l h_{jl}{}^x \overline{h}_{lk}{}^y \right) h_{ki}{}^y \overline{h}_{ij}{}^x \\
&= -8 \sum_{x,i,j} \epsilon_x\epsilon_i\epsilon_j f_{xj} h_{ij}{}^x \overline{h}_{ij}{}^x + 8 \sum_{j,k} \epsilon_j\epsilon_k h_{j\bar{k}}{}^2 h_{k\bar{j}}{}^2 \\
&\leqq 8 \sum_{x,i,j} \epsilon_i\epsilon_j \left(a_2 - \sum_k \epsilon_k h_{jk}{}^x \overline{h}_{jk}{}^x \right) h_{ij}{}^x \overline{h}_{ij}{}^x + 8h_4 \\
&= -8a_2h_2 + 8h_4 - 8 \sum_{x,j} \left(\sum_k h_{jk}{}^x \overline{h}_{jk}{}^x \right)^2,
\end{aligned}$$

where the second equality follows from (5.2) and the fourth inequality is derived by (6.2). For real numbers $x_1, \ldots, x_m$, since it is easily seen that $\sum_{\alpha=1}^m x_\alpha{}^2 \geqq \frac{1}{m}(\sum_{\alpha=1}^m x_\alpha)^2$, the last term of the above expression can be estimated from the above by $-\frac{8}{p}\sum_j(\sum_{x,k} h_{jk}{}^x \overline{h}_{jk}{}^x)^2$, we have

$$\text{the fourth term} \leqq -8\left(a_2h_2 - h_4 + \frac{1}{p}h_2{}^2 \right). \tag{6.3}$$

On the other hand, let A be the positive semi-definite Hermitian matrix defined by $(A_y{}^x) = (\sum_{j,k} h_{jk}{}^x \bar{h}_{jk}{}^y)$ and let λ_x be its eigenvalue. Then the fifth term can be estimated as follows;

$$\begin{aligned}\text{the fifth term} &= -2\sum_{x,y,k} \lambda_x \delta_{xy} R'_{\bar{x}yk\bar{k}} = -2\sum_{x,k} \lambda_x R'_{\bar{x}xk\bar{k}} \\ &\leqq 2a_2 \sum_{x,k} \lambda_x = 2na_2 \sum_x \lambda_x\end{aligned}$$

with the help of $(*2)$, from which it follows that we have by (4.7)

$$\text{(6.4)} \qquad \text{the fifth term} \leqq -2na_2h_2.$$

Next, we estimate the sixth term and the seventh one in the right hand side of (4.8). For the sake of the estimation, we consider the curvature operator T' on M'. From the symmetric relation (2.5), on the n^2-dimensional complex vector space $\Xi_x^{n^2} = T_xM^C \times T_xM^C$ at each point x on M which consists of symmetric tensor (ξ_{ij}), we can define a linear transformation T' by

$$T'(\xi_{ij}) = (\eta_{ij}), \quad \eta_{ij} = \sum_{k.l} \epsilon_k \epsilon_l R'_{\bar{k}ij\bar{l}} \xi_{kl}.$$

We denote by $(R'_{kl}{}^{ij})$ the matrix of the linear transformation T'. The linear operator T' defined by the $n^2 \times n^2$ matrix $(R'_{kl}{}^{ij})$ is the called the *curvature operator* on the submanifold M. The curvature operator on the Kähler manifold plays an important role in Nakagawa and Takagi [10]. Since T' is the self-adjoint operator with respect to the metric canonically induced on $\Xi_x^{n^2}$, every eigenvalue R'_{ik} of T' is a real valued function. So, we have

$$\text{(6.5)} \qquad R'_{\bar{i}jk\bar{l}} = R'_{il}{}^{jk} = \epsilon_j \epsilon_k R'_{jk} \delta_{ji} \delta_{kl}, \quad R'_{ij} = R'_{\bar{i}ij\bar{j}} = H'(E_i, E_j).$$

By (6.5) and the condition $(*1)$, we have

$$R_{ij} = R_{\bar{i}ij\bar{j}} = H'(E_i, E_j) = H'_{ij} \geqq a_1.$$

Now, we estimate the sixth term of the right hand side of (4.8). By (6.5), we see

$$\text{the sixth term} = -4 \sum_{x,i,j,k,l} R'_{\bar{k}ij\bar{l}} h_{kl}{}^{x} \bar{h}_{ij}{}^{x}$$
$$= -4 \sum_{x,i,j,k,l} R'_{ij} \delta_{ik} \delta_{jl} h_{kl}{}^{x} \bar{h}_{ij}{}^{x}.$$

From which together with (6.5), it follows that we have

$$\text{(6.6)} \qquad \text{the sixth term} = -4 \sum_{i,j} H'_{ij} \sum_{x} h_{ij}{}^{x} \bar{h}_{ij}{}^{x} \leqq 4a_1 h_2.$$

The matrix $(h_{i\bar{j}}{}^2)$ is negative semi-definite Hermitian one, whose eigenvalues λ_i's are non-positive real functions, i.e., $h_{i\bar{j}}{}^2 = \lambda_i \delta_{ij}$. Since $\sum_j \lambda_j = h_2$, the seventh term is estimated as follows;

$$\text{the seventh term} = 4 \sum_{i,j,k} R'_{\bar{i}jk\bar{k}} h_{i\bar{j}}{}^2 = 4 \sum_{j,k} \lambda_j R'_{\bar{j}jk\bar{k}}.$$

Thus, we have

$$\text{(6.7)} \qquad \text{the seventh term} \leqq 4na_1 h_2.$$

Under the above preparation, we can prove the following proposition.

PROPOSITION 6.1. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite Kähler manifold M' of index $2p$. Assume that M' is locally symmetric and it satisfies the condition $(*)$. If the normal connection of M is proper, then the following statements hold:*

(1) *In the case where $p = 1$ or 2 and $2(n+1)a_1 - (n+4)a_2 \geqq 0$, M is totally geodesic.*

(2) *In the case where $p \geqq 3$ and $2(n+1)a_1 - (n+4)a_2 > 0$, if the scalar curvature on M is bounded from above, then there exists a negative constant h in such a way that M is totally geodesic, provided $h_2 > h$.*

Proof. Since the ambient space is locally symmetric and the squared norm $|\nabla\alpha|_2$ of the covariant derivative $\nabla\alpha$ of the second fundamental form α is non-positive by (4.9), the equation (4.8) is estimated by (6.3), (6.4), (6.6) and (6.7) from the above as follows;

$$\Delta h_2 \leqq -8\left(a_2h_2 - h_4 + \frac{1}{p}h_2{}^2\right) - 2na_2h_2 + 4a_1h_2 + 4na_1h_2 - 4h_4 - 2TrA^2.$$

Accordingly, by (4.7), we obtain

$$\Delta h_2 \leqq A_0 h_2{}^2 + A_1 h_2, \tag{6.8}$$

where the coefficients A_0 and A_1 are constants given by

$$A_0 = \frac{2}{p}(2p-5), \quad A_1 = 2\{2(n+1)a_1 - (n+4)a_2\}.$$

Now, since the space-like holomorphic bisectional curvature of M is bounded from below by a constant, the Ricci curvature of M is bounded from below. In fact, we have

$$S_{j\bar{j}} = \sum_k R_{\bar{j}jk\bar{k}} \geqq \sum_k R'_{\bar{j}jk\bar{k}} = \sum_k H'_{jk} \geqq na_1$$

with the help of (3.6). Let f be the non-negative function defined by $-h_2$. Then by (6.8), we have

$$\Delta f \geqq c_0 f^2 + c_1 f + c_2 =: F(f), \; c_0 = -A_0, \; c_1 = A_1, \; c_2 = 0,$$

where F is the polynomial of the variable f with the constant coefficients.

In the first assertion, the coefficients satisfy $c_0 > 0 = c_2$, which implies that we can apply Theorem 2.2 to the function f and hence we get $F(\sup f) \leqq 0$. Accordingly, we have $\sup f \leqq 0$. Since the function f is non-negative, it vanishes identically on M, which means that M is totally geodesic.

In the second assertion, we remark that the first coefficient c_0 is negative. Since the scalar curvature on M is bounded from above by the assumption, the function f is bounded from above. In fact, we see

$$r = 2\sum_{j,k} R_{\bar{j}jk\bar{k}} = 2\sum_{j,k} H'_{jk} + 2f \geqq 2n^2a_1 + 2f$$

with the help of (3.9). Applying Theorem 2.1 to the function f, we obtain $F(\sup\ f) \leqq 0$, from which we get

$$\sup\ f = 0 \quad \text{or} \quad \sup\ f \geqq -\frac{c_1}{c_0} > 0.$$

For a negative constant h such that $h > \frac{c_1}{c_0}$, suppose that $h_2 > h$. Then we get $inf\ h_2 \geqq h$ and hence $\sup\ f \leqq -h < -\frac{c_1}{c_0}$, which means that $\sup\ f = 0$. Hence f vanishes identically on M, which means that M is totally geodesic. It completes the proof. □

In the case where M is a hypersurface, it is natural that the normal connection is proper. So, the first assertion of Proposition 6.1 proves the following

COROLLARY 6.2. *Let M' be an $(n+1)$-dimensional indefinite Kähler manifold of index 2 and let M be an n-dimensional complete space-like complex hypersurface of M'. Assume that M' is locally symmetric and it satisfies $(*)$ with $2(n+1)a_1 - (n+4)a_2 \geqq 0$. Then M is totally geodesic.*

REMARK 6.2. Corollary 6.2 is given by Kown and Nakagawa [8].

Proof of the main theorem. Since the fact that M' has non-positive time-like holomorphic bisectional curvature is equivalent to the fact that it satisfies the condition $(*2)$ with $a_2 = 0$. Furthermore, it satisfies the condition $(*1)$ with $a_1 = 0$. Accordingly, by Proposition 6.1(1), M is becomes totally geodesic. □

COROLLARY 6.3. *Let M' be an $(n+2)$-dimensional indefinite Kähler manifold of index 4 and let M be an n-dimensional complete space-like complex submanifold of M'. Assume that M' is locally symmetric and it satisfies $(*)$ with $a_1 = a_2 = 0$. If the normal connection of M is proper, then M is totally geodesic.*

REMARK 6.3. For the complex coordinate system (z_A, z_{2n+1}) in C_s^{2n+1}, let $M = M(b_j)$ be the complex hypersurface in given by the equation

$$z_{2n+1} = \sum_j (z_j + b_j z_{j^*})^2, \qquad j^* = j + n$$

for any complex number b_j such that $|b_j| = 1$. Then it is seen in [3] and [15] that M is a family of complete indefinite complex hypersurfaces of index $2s$, which are Ricci flat and not flat. Thus we see $c_1 = 0$, but it is not totally geodesic. This means that in Proposition 6.1 the condition that M is space-like is essential.

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