J. Korean Math. Soc. **36** (1999), No. 6, pp. 1115–1132

# CENTRAL LIMIT THEOREMS FOR MULTITYPE AGE-DEPENDENT BRANCHING PROCESSES

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ABSTRACT. We consider a supercritical multitype age dependent branching process. We define a stochastic process $Z_f(t)$ which is a functional of the empirical age distribution. When the limit of the expectation of this functional vanishes we find some sufficient conditions for the asymptotic normality of the mean of $f$ with respect to the empirical age distribution at time $t$.

## 1. Introduction

Let $\mathbf{Z}(t) = (Z_1(t), \cdots, Z_p(t))$ be a multitype age-dependent branching process defined on a probability space $(\Omega, \mathcal{F}, P)$. A type $i$ particle dies at random time $\lambda_i$ which has distribution $G_i$ and on death it creates $\xi_{ij}$ offsprings of type $j$. $Z_i(t)$ denotes the number of $i$ type particles at time $t$. Let $m_{ij} = E(\xi_{ij})$ and let $\mathbf{M} = ((m_{ij}))_{i,j=1}^{p}$ be the particle production mean matrix. We assume that $\mathbf{M}$ is positively regular and nonsingular throughout this paper. Write $\rho(\mathbf{M})$ for its Perron-Frobenius root which is the maximal eigenvalue of $\mathbf{M}$. We assume that the process is supercritical, that is, $\rho(\mathbf{M}) > 1$.

An important and useful aspect of age-dependent branching processes is the limiting behavior of the age distribution. Let $Z_k(t,a)$ be the number of type $k$ particles living at time $t$ with age $\leq a$ and let $|\mathbf{Z}(t)|$ be the total population size at time $t$. Then $\dfrac{Z_k(t,a)}{|\mathbf{Z}(t)|}$ converges to the limiting distribution, on the set of nonextinction with probability 1 under appropriate assumptions on offspring and lifetime distributions.

Received March 5, 1999.

1991 Mathematics Subject Classification: 60J80, 60F05.

Key words and phrases: multitype age-dependent branching process, renewal equation, central limit theorem.

We consider a stochastic process $\{Z_f(t); t \geq 0\}$ defined by

$$Z_f(t) = \sum_{k=1}^{p} \sum_{j=1}^{Z_k(t)} f(a_{kj}(t)),$$

where $\{a_{kj}(t), j = 1, 2, \cdots\}$ is the age-chart of type $k$ particles at time $t$. If we think $Z_k(t, \cdot)$ as a random point measure describing the ages of $k$ type particles at time $t$ then we can write

$$\frac{Z_f(t)}{|\mathbf{Z}(t)|} = \sum_{k=1}^{p} \int_0^\infty f(a) \frac{Z_k(t, da)}{|\mathbf{Z}(t)|}.$$

Hence for any bounded continuous function $f$, $\dfrac{Z_f(t)}{|\mathbf{Z}(t)|}$ converges to an integral with respect to the limiting distribution on the set of nonextinction.

In this paper we develop some limit theorems for the stochastic process $\{Z_f(t); t \geq 0\}$ when the limit is zero. The main result of this paper is given in section 2 with some preliminaries. In section 3 we analyze the first and the second moment of $Z_f(t)$ and we prove Theorem 1 in section 4.

## 2. Preliminaries and Statement of Result

We make the following assumptions on $G_i$ which are valid at all times.

(A 1) $m_{ij} < \infty$, $i, j = 1, \cdots, p$, $G_i(0+) = 0$, $G_i$ is non-lattice.
(A 2) $\int_0^\infty u G_i(du) < \infty$, $i = 1, \cdots, p$.

The assumption (A 1) is standard and guarantees that $|\mathbf{Z}(t)|$ is finite for any finite $t$. Here we introduce an analog to the concept of a Malthusian parameter for Bellman-Harris process. Let

$$\hat{\mathbf{M}}(\alpha) = ((\hat{M}_{ij}(\alpha)))_{i,j=1}^{p},$$

where $\hat{M}_{ij}(\alpha) = m_{ij} \int_0^\infty e^{-\alpha t} G_i(dt)$. The Malthusian parameter $\alpha$ for $\mathbf{M}$ and $(G_1, \cdots, G_p)$ is defined to be the number $\alpha$ which satisfies the equation $\rho(\hat{\mathbf{M}}(\alpha)) = 1$, provided it exists. In the supercritical case, the Malthusian parameter $\alpha$ exists and is positive. Growth rate is related to this Malthusian parameter in the supercritical case. Let $\mathbf{u} = (u_1, \cdots, u_p)$ and $\mathbf{v} = (v_1, \cdots, v_p)$ be the positive left- and right-eigenvector of $\hat{\mathbf{M}}(\alpha)$ corresponding to the eigenvalue 1 such that $\mathbf{1} \cdot \mathbf{v} = 1$, $\mathbf{u} \cdot \mathbf{v} = 1$. Under the usual '$j \log j$' conditions ($E(\xi_{ij} \log \xi_{ij}) < \infty$, $i, j = 1, \cdots, p$) it is known (see Mode (1971)) that there exists a random variable $W$ such that

$$\lim_{t \to \infty} e^{-\alpha t} \mathbf{Z}(t) = \eta W, \quad \text{a.s.} \tag{1}$$

where $\eta = (\eta_1, \cdots, \eta_p)$, and $\eta_i = u_i(1 - \int_0^\infty e^{-\alpha t} G_i(dt))$.

Now let $Z_k(t, a)$ be the number of type $k$ particles living at time $t$ with age $\leq a$ and let $|\mathbf{Z}(t)| = Z_1(t) + \cdots + Z_p(t)$ be the total population size at time $t$. Assume the '$j \log j$' condition on the offspring distributions;

$$E(\xi_{ij} \log \xi_{ij}) < \infty, i, j = 1, \cdots, p.$$

Then it is known that (see Rama Murthy (1976)) for $k = 1, \cdots, p$ on the set of nonextinction

$$\sup_x \left| \frac{Z_k(t, x)}{|\mathbf{Z}(t)|} - c_0 u_k \int_0^x e^{-\alpha u}(1 - G_k(u))du \right| \to 0 \quad \text{a.s.},$$

where $c_0 = [\sum_{j=1}^p u_j \int_0^\infty e^{-\alpha u}(1 - G_j(u))du]^{-1}$. With $x = \infty$, we have for each $k = 1, \cdots, p$,

$$\frac{Z_k(t)}{|\mathbf{Z}(t)|} \xrightarrow{\text{a.s.}} c_0 u_k \int_0^\infty e^{-\alpha u}(1 - G_k(u))du$$

on the set of nonextinction. Since $u_k > 0 (k = 1, \cdots, p)$, we know that $Z_k(t) \xrightarrow{\text{a.s.}} \infty$ on the set of nonextinction, so it is trivial to see that

$$\sup_x \left| \frac{Z_k(t, x)}{Z_k(t)} - A_k(x) \right| \xrightarrow{\text{a.s.}} 0 \quad \text{as} \quad t \to \infty, \tag{2}$$

where $A_k(x) = \int_0^x e^{-\alpha u}(1 - G_k(u))du / \int_0^\infty e^{-\alpha u}(1 - G_k(u))du$.

We add superscript $a$ to random variables and their moments to indicate the case when $P$ is supported by those $\omega$'s which start with one particle of age $a \geq 0$. Put

$$
\begin{aligned}
G^a(t) &= \frac{G(t+a)-G(a)}{1-G(a)}, \\
{}_im_f(t) &= E(Z_f(t)|\mathbf{Z}(0)=\mathbf{e}_i), \\
{}_iD_f(t) &= E((Z_f(t))^2|\mathbf{Z}(0)=\mathbf{e}_i), \\
{}_iD_f^\alpha &= \lim_{t\to\infty} e^{-\alpha t}{}_iD_f(t), \\
\mu_i &= \int_0^\infty e^{-\alpha t}(1-G_i(t))dt, \\
\theta_i &= \sqrt{\frac{\eta_i}{\sum_{j=1}^p \eta_j v_j}} \\
d_{ij} &= E(\xi_{ij}^2)
\end{aligned}
$$

REMARK 1. ${}_iD_f^\alpha$ may not exist in general, but we can show that it does exist when (B 4) and (B 5) below hold (see Proposition 3).

We impose the following assumptions on a measurable function $f; R^+ \to R$ which are not all valid at all times.

(B 1) $f$ is continuous a.e. on the support of $G_i$ for each $i = 1, \cdots, p$.
(B 2) $e^{-\alpha t}(1-G_i(t))f(t)$ is d.R.i. for each $i = 1, \cdots, p$.
(B 3) $\int_0^\infty e^{-\alpha t}(1-G_i(t))f(t)dt = 0$ for each $i = 1, \cdots, p$.
(B 4) $e^{-\alpha t}f^2(t) \to 0$ as $t \to \infty$.
(B 5) $(e^{-\alpha t}({}_km_f \times {}_lm_f) * G_j)(t)$ is d.R.i. for all $j, k, l = 1, \cdots, p$.
(B 6) There exists $s_0 > 0$ such that for $s \geq s_0$,

$$
\begin{aligned}
\sup_{a\geq 0} |f(a+s)(1-G_k^a(s))| &< \infty, \\
\sup_{a\geq 0} |f^2(a+s)(1-G_k^a(s))G_k^a(s)| &< \infty.
\end{aligned}
$$

REMARK 2. 1. (B 4) with (A 2) implies

(B 4)′ $e^{-\alpha t}f^2(t)(1-G_i(t))$ is d.R.i. $i = 1, \cdots, p$.

2. We refer the readers to Karlin and Taylor (1976) for the definition and criteria about directly Riemann integrability.

Now we are ready to state the main theorem of this paper.

THEOREM 1. *Let $\rho(\mathbf{M}) > 1$. Assume $d_{ij} < \infty$, $i, j = 1, \cdots, p$. Let $f$ satisfy (B 1) - (B 6). Then for $0 < x_1 < x_2 < \infty$, $y \in R$,*

$$\lim_{t\to\infty} P_i\left(x_1 \le W \le x_2, \frac{Z_f(t)}{\sqrt{\mathbf{v}\cdot\mathbf{Z}(t)}} \le y\right) = P_i(x_1 \le W \le x_2)\Phi\left(\frac{y}{\sigma_f}\right),$$

*where $\Phi$ is the standard normal distribution and*

$$\sigma_f^2 \equiv \sum_{k=1}^{p} \frac{\theta_k}{\mu_k} \sum_{j=1}^{p} m_{kj}\cdot {}_jD_f^{\alpha} \int_0^{\infty} ue^{-\alpha u} G_j(du).$$

## 3. The Analysis of Moments

In this section we give some estimates for moments of $Z_f(t)$ which is the most important step in proving Theorem 1

PROPOSITION 1. *Assume that (B 2) holds. Then for $i = 1, \cdots, p$,*

$$\lim_{t\to\infty} e^{-\alpha t}{}_im_f(t) = c_0 v_i \sum_{k=1}^{p} u_k \int_0^{\infty} e^{-\alpha u} f(u)(1 - G_k(u))du,$$

*where $c_0 = [\sum_{k=1}^{p}\sum_{j=1}^{p} m_{kj}u_k v_j \int_0^{\infty} te^{-\alpha t} G_i(dt)]^{-1}$.*

*Proof.* Given $\mathbf{Z}(0) = \mathbf{e}_i$, it is easy to see from the additive property of branching processes (see Athreya and Kaplan (1978)) for details) that

$$Z_f(t) = I(\lambda_i > t)f(t) + \sum_{k=1}^{p}\sum_{j=1}^{\xi_{ik}} Z_{f,k,j}(t - \lambda_i), \tag{3}$$

where $\{Z_{f,k,j}(u); u \ge 0\}_{j=1}^{\infty}$ are i.i.d. copies of $\{Z_f(u); u \ge 0\}$ which is initiated by an ancestor of type $k$. Taking expectation we get the following system of renewal equations,

(4)

$${}_im_f(t) = (1 - G_i(t))f(t) + \sum_{k=1}^{p} m_{ik} \int_0^t {}_km_f(t-u)G_i(du), \quad i = 1, \cdots, p.$$

Multiplying $\dfrac{e^{-\alpha t}}{v_i}$ both sides of (4) we get

$$\frac{e^{-\alpha t}{}_im_f(t)}{v_i} = \frac{e^{-\alpha t}f(t)(1 - G_i(t))}{v_i} + \sum_{k=1}^{p}\int_0^t \frac{e^{-(t-u)}{}_km_f(t-u)}{v_k} F_{ik}(du),$$

where $F_{ik}(t) = m_{ik}\frac{v_k}{v_i}\int_0^t e^{-\alpha u}G_i(du)$. For each $i = 1, \cdots, p$,

$$\begin{aligned}\lim_{t\to\infty}\sum_{k=1}^{p} F_{ik}(t) &= \frac{1}{v_i}\sum_{k=1}^{p}\hat{M}_{ik}(\alpha)v_k \\ &= \frac{1}{v_i}(\hat{\mathbf{M}}(\alpha)\cdot\mathbf{v}^t)_i \\ &= \frac{1}{v_i}\times v_i = 1,\end{aligned}$$

since $\mathbf{v} = (v_1, \cdots, v_p)$ is the right eigenvector of $\hat{\mathbf{M}}(\alpha)$ corresponding to 1. That is, if $F_{ij}(\infty) = \lim_{t\to\infty} F_{ij}(t)$

$$\mathbf{F} = ((F_{ij}(\infty)))_{i,j=1}^{p}$$

is a semi-Markov kernel. See Asmussen (1987) for definition and related facts of semi-Markov kernel. Hence there exists a unique stationary measure $\pi = (\pi_1, \cdots, \pi_p)$ such that

$$(5)\qquad \sum_{i=1}^{p}\pi_i\frac{v_k}{v_i}\hat{M}_{ik}(\alpha) = \pi_k, \quad \text{or equivalently,} \quad \sum_{i=1}^{p}\frac{\pi_i}{v_i}\hat{M}_{ik}(\alpha) = \frac{\pi_k}{v_k}.$$

We deduce from (5) that $\left(\frac{\pi_1}{v_1}, \cdots, \frac{\pi_p}{v_p}\right)$ is the left-eigenvector of $\hat{\mathbf{M}}(\alpha)$ corresponding to 1, so

$$\pi_i = v_i u_i, \quad i = 1, \cdots, p.$$

Since $\{e^{-\alpha t}f(t)(1 - G_i(t))\}_{i=1}^{p}$ are d.R.i., we get from the generalized key renewal theorem (see Asmussen (1987), p. 230) that

$$\frac{e^{-\alpha t}{}_i m_f(t)}{v_i} \to c_0\sum_{k=1}^{p} u_k\int_0^\infty e^{-\alpha u}f(u)(1 - G_k(u))du,$$

i.e.,

$$\lim_{t\to\infty} e^{-\alpha t}{}_i m_f(t) = c_0 v_i\sum_{k=1}^{p} u_k\int_0^\infty e^{-\alpha u}f(u)(1 - G_k(u))du.$$

□

PROPOSITION 2. *Assume (B 4) and (B 5) hold. If* $d_{ij} < \infty, i, j = 1, \cdots, p$, *then for* $i = 1, \cdots, p$

$$ {}_iD_f^\alpha \equiv \lim_{t\to\infty} e^{-\alpha t}{}_iD_f(t) = c_0 v_i \sum_{j=1}^{p} u_j \int_0^\infty g_{f,j}^\alpha(u)du, $$

*where*

$$ \begin{aligned} g_{f,j}^\alpha(t) &= e^{-\alpha t} f^2(t)(1-G_j(t)) + e^{-\alpha t}\sum_{k=1}^{p}(d_{jk}-m_{jk})({}_km_f^2 * G_j)(t) \\ &\quad + e^{-\alpha t}\sum_{k,l,k\neq l}^{p} m_{jk}m_{jl}[({}_km_f\cdot {}_lm_f) * G_j](t). \end{aligned} $$

*Proof.* Given $\mathbf{Z}(0) = \mathbf{e}_i$, starting from (3) we have the following equation

$$ \begin{aligned} [Z_f(t)]^2 &= I(\lambda_i > t)f^2(t) + \sum_{k=1}^{p}\sum_{j,l,j\neq l}^{\xi_{ik}} Z_{f,k,j}(t-\lambda_i)Z_{f,k,l}(t-\lambda_i) \\ &\quad + \sum_{k,h,k\neq h}^{p}\sum_{j=1}^{\xi_{ik}}\sum_{l=1}^{\xi_{ih}} Z_{f,k,j}(t-\lambda_i)Z_{f,h,l}(t-\lambda_i) \\ &\quad + \sum_{k=1}^{p}\sum_{j=1}^{\xi_{ik}} (Z_{f,k,j}(t-\lambda_i))^2. \end{aligned} $$

Since $Z_{f,k,j}(t-\lambda_i)$'s are conditionally independent given $\lambda_i$, we have by taking expectation

$$ \begin{aligned} {}_iD_f(t) &= (1-G_i(t))f^2(t) + \sum_{k=1}^{p}(d_{ik}-m_{ik})({}_km_f^2 * G_i)(t) \\ &\quad + \sum_{k\neq h}^{p} m_{ik}m_{ih}[({}_km_f\cdot {}_hm_f) * G_i](t) + \sum_{k=1}^{p} m_{ik}({}_kD_f * G_i)(t). \end{aligned} $$

Multiplying $\frac{1}{v_i}e^{-\alpha t}$ both sides we get the following system of renewal equations

$$ \frac{e^{-\alpha t}{}_iD_f(t)}{v_i} = \frac{g_{f,i}^\alpha(t)}{v_i} + \sum_{k=1}^{p}\int_0^t \frac{e^{-\alpha(t-u)}{}_kD_f(t-u)}{v_k}F_{ik}(du) \tag{6} $$

Combining (B 4)′ and (B 5) we see that $\{g^{\alpha}_{f,j}; j = 1, \cdots, p\}$ are d.R.i. and so

$$e^{-\alpha t}{}_iD_f(t) \to c_0 v_i \sum_{j=1}^{p} u_j \int_0^{\infty} g^{\alpha}_{f,j}(u)du \qquad \text{as} \qquad t \to \infty$$

by the generalized key renewal theorem. □

Now we consider the following representation by appealing to the additive property of branching processes

$$Z_f(t+s) = \sum_{k=1}^{p} \sum_{j=1}^{Z_k(t)} Z_f^{a_{kj}(t)}(s), \tag{7}$$

where $\{a_{kj}(t); j = 1, 2, \cdots\}$ is the age-chart of type $k$ particles at time $t$,

$$Z_f^{a_{kj}(t)}(s) = \sum_{l=1}^{p} \sum_{i=1}^{Z_l(s)} f(a^{li}_{kj}(t+s)),$$

and $\{a^{li}_{kj}(t+s); i = 1, 2, \cdots\}$ is the age-chart at time $t+s$ of type $l$ particles in a line of descent initiated by the $j$th particle of type $k$ of age $a_{kj}(t)$ living at time $t$. Furthermore, $\{Z_f^{a_{kj}(t)}(s), k = 1, \cdots, p, j = 1, 2, \cdots\}$ are independently distributed conditioned on the age chart at time $t$. If $a_{kj}(t) = a$, then the conditional distribution of $Z_f^{a_{kj}(t)}(s)$ is the same as that of $Z_f^a(s)$ which starts with one type $k$ particle whose initial age is $a$. Given age-chart $\{a_{ij}(t); j = 1, 2, \cdots\}$ of type $i$ particles at time $t$ we define $M(s)f$ by

$$(M(s)f)(a_{ij}(t)) = E(Z_f^{a_{ij}(t)}(s)|\mathbf{Z}(0) = \mathbf{e}_i) = {}_im_f^{a_{ij}(t)}(s).$$

PROPOSITION 3. (a) ${}_im_{M(s)f}(t) = {}_im_f(t+s)$
(b) *Assume (B 4) and (B 5) hold. If* $d_{ij} < \infty$, $i, j = 1, \cdots, p$, *then*

$$\lim_{s\to\infty} e^{-\alpha s} \lim_{t\to\infty} e^{-\alpha t}{}_iD_{M(s)f}(t) = 0.$$

*Proof.* (a) We begin with the equation (7). Set $\mathcal{F}_t = \sigma(\mathbf{Z}(s); s \le t)$.

$$\begin{aligned} {}_im_f(t+s) &= E_i\Big(\sum_{k=1}^{p} \sum_{j=1}^{Z_k(t)} Z_f^{a_{kj}(t)}(s)\Big) \\ &= E_i\Big(E_i(\sum_{k=1}^{p} \sum_{j=1}^{Z_k(t)} Z_f^{a_{kj}(t)}(s)|\mathcal{F}_t)\Big) \end{aligned}$$

$$= E_i\left(\sum_{k=1}^{p}\sum_{j=1}^{Z_k(t)} {}_k m_f^{a_{kj}(t)}(s)\right)$$

$$= E_i\left(\sum_{k=1}^{p}\sum_{j=1}^{Z_k(t)} (M(s)f)(a_{kj}(t))\right)$$

$$= {}_i m_{M(s)f}(t).$$

(b) Replacing $f$ by $M(s)f$ in (6), we have

$$\frac{e^{-\alpha t}{}_iD_{M(s)f}(t)}{v_i} = \frac{g^{\alpha}_{M(s)f,i}(t)}{v_i} + \sum_{k=1}^{p}\int_0^t \frac{e^{-\alpha(t-u)}{}_kD_{M(s)f}(t-u)}{v_k}F_{ik}(du).$$

First, we'll show that $g^{\alpha}_{M(s)f,i}$ is d.R.i. for fixed $s$ and for $i = 1, \cdots, p$, so that

$$\frac{e^{-\alpha t}{}_iD_{M(s)f}(t)}{v_i} \longrightarrow c_0 \sum_{j=1}^{p} u_j \int_0^{\infty} g^{\alpha}_{M(s)f,j}(u)du, \quad \text{as} \quad t \to \infty,$$

then we'll prove that for each $j = 1, \cdots, p$

$$e^{-\alpha s}\int_0^{\infty} g^{\alpha}_{M(s)f,j}(u)du \to 0 \quad \text{as} \quad s \to \infty.$$

When the ancestor is of type $i$ with age $t$ at time 0, we have the following identity

$$Z_f^t(s) = I(\lambda_i^t > s)f(t+s) + \sum_{k=1}^{p}\sum_{j=1}^{\xi_{ik}} Z_{f,k,j}(s-\lambda_i^t),$$

where $\lambda_i^t$ is the lifetime of type $i$ particle whose initial age is $t$. Due to the independence of $Z_{f,k,j}(s-\lambda_i^t)$'s given $\lambda_i^t$ we get

$${}_i m_f^t(s) = f(t+s)(1-G_i^t(s)) + \sum_{k=1}^{p} m_{ik}\int_0^s {}_k m_f(s-u)G_i^t(du).$$

Hence,

$$(8) \qquad ({}_i m_f^t(s))^2 \le C[f^2(t+s)(1-G_i^t(s))^2 + \sum_{k=1}^{p}({}_k m_f * G_i^t)^2(s)],$$

where $C$ is a generic constant. Since $G_i^t(s) \leq 1$, by Cauchy-Schwarz inequality we have

$$\begin{aligned}({}_k m_f * G_i^t)^2(s) &\leq (({}_k m_f)^2 * G_i^t)(s) \\ &= \int_0^s [{}_k m_f(s-u)]^2 G_i^t(du) \\ &= \frac{1}{1-G_i(t)} \int_t^{t+s} [{}_k m_f(t+s-u)]^2 G_i(du) \\ &\leq \frac{({}_k m_f^2 * G_i)(t+s)}{1-G_i(t)}.\end{aligned}$$

Combining this with (8), we have

$$\begin{aligned}&e^{-\alpha t}(M(s)f)^2(t)(1-G_i(t)) \\ &\leq Ce^{\alpha s}\{e^{-\alpha(t+s)}(1-G_i(t+s))f^2(t+s) \\ &\quad + \sum_{k=1}^p e^{-\alpha(t+s)}({}_k m_f^2 * G_i)(t+s)\}.\end{aligned} \tag{9}$$

Since the right hand side of (9) is d.R.i. for fixed $s$ from assumption (B 4) and (B 5), we conclude that $e^{-\alpha t}(M(s)f)^2(t)(1-G_i(t))$ is d.R.i. On the other hand,

$$\begin{aligned}&e^{-\alpha t}(|{}_k m_{M(s)f} \cdot {}_l m_{M(s)f}| * G_i)(t) \\ &= e^{-\alpha t} \int_0^t |{}_k m_f(t+s-u) \cdot {}_l m_f(s+t-u)| G_i(du) \\ &\leq e^{\alpha s} e^{-\alpha(t+s)}(|{}_k m_f \cdot {}_l m_f| * G_i)(t+s),\end{aligned} \tag{10}$$

where the last one is d.R.i. for fixed $s$ by assumptions. Furthermore, it follows from (9) and (10) that

$$\begin{aligned}&\int_0^\infty g^\alpha_{M(s)f,i}(t)dt \\ &\leq Ce^{\alpha s}\left\{\int_0^\infty e^{-\alpha(t+s)} f^2(t+s)(1-G_i(t+s))\right. \\ &\quad \left. + \sum_{k=1}^p \sum_{l=1}^p e^{-\alpha(t+s)}(|{}_k m_f \cdot {}_l m_f| * G_i)(t+s)dt\right\}\end{aligned}$$

$$= Ce^{\alpha s}\int_s^\infty \Big\{e^{-\alpha t}f^2(t)(1-G_i(t)) + \sum_{k=1}^p\sum_{l=1}^p e^{-\alpha t}(|{}_km_f\cdot{}_lm_f| * G_i(t))\Big\}dt.$$

So

$$\begin{aligned}
&\lim_{s\to\infty} e^{-\alpha s}\lim_{t\to\infty} e^{-\alpha t}{}_iD_{M(s)f}(t)\\
&= c_0v_i\sum_{j=1}^p u_j\lim_{s\to\infty} e^{-\alpha s}\int_0^\infty g^\alpha_{M(s)f,i}(u)du\\
&\le Cv_i\sum_{j=1}^p u_j\lim_{s\to\infty}\int_s^\infty\Big\{e^{-\alpha t}f^2(t)(1-G_i(t))\\
&\quad+\sum_{k=1}^p\sum_{l=1}^p e^{-\alpha t}(|{}_km_f\cdot{}_lm_f| * G_i(t))\Big\}dt\\
&= 0,
\end{aligned}$$

where the last equality comes from the fact that the integrand is integrable. □

## 4. Proofs

We begin with the representation (7)

$$Z_f(t+s) = \sum_{k=1}^p\sum_{j=1}^{Z_k(t)} Z_f^{a_{kj}}(s).$$

We rewrite (7) as follows;

$$Z_f(t+s) = \sum_{k=1}^p\sum_{j=1}^{Z_k(t)}(Z_f^{a_{kj}}(s) - {}_km_f^{a_{kj}}(s)) + Z_{M(s)f}(t) \tag{11}$$

Dividing (11) by $\sqrt{\mathbf{v}\cdot\mathbf{Z}(t+s)}$ and introducing

$$X_{kj}(s) \equiv (Z_f^{a_{kj}}(s) - {}_km_f^{a_{kj}}(s))e^{-\frac{\alpha}{2}s}$$

we get

$$\begin{aligned}\frac{Z_f(t+s)}{\sqrt{\mathbf{v}\cdot\mathbf{Z}(t+s)}} &= \sum_{k=1}^{p}\left(\sqrt{\frac{Z_k(t)e^{-\alpha t}}{\mathbf{v}\cdot\mathbf{Z}(t+s)e^{-\alpha(t+s)}}}-\theta_k\right)\frac{1}{\sqrt{Z_k(t)}}\sum_{j=1}^{Z_k(t)}X_{kj}(s) \\ &\quad +\sum_{k=1}^{p}\frac{\theta_k}{\sqrt{Z_k(t)}}\sum_{j=1}^{Z_k(t)}X_{kj}(s)+\frac{Z_{M(s)f}(t)}{\sqrt{\mathbf{v}\cdot\mathbf{Z}(t+s)}} \qquad (12)\\ &\equiv A_1(t,s)+A_2(t,s)+A_3(t,s), \quad \text{say.}\end{aligned}$$

We first show that $A_3(t,s)$ can be made small in probability uniformly in $t$, by choosing $s$ large, and then with this large but fixed $s$, we show that $A_1(t,s) \xrightarrow{\text{pr}} 0$ as $t \to \infty$. Finally, for this fixed $s$, we use the Lindeberg-Feller theorem to prove that as $t \to \infty$, $A_2(t,s)$ converges to the desired distribution. Let $I = [x_1, x_2]$ be fixed with $0 < x_1 < x_2 < \infty$. In the following sequence of Lemmas we impose the same assumptions as Theorem 1.

LEMMA 1. *Given $\varepsilon > 0, \delta > 0$, there exists $s_0 = s_0(\varepsilon,\delta)$ such that $s \geq s_0$ implies*

$$\lim_{t\to\infty} P_i(W \in I, |A_3(t,s)| > \varepsilon) < \delta, \quad i = 1,\cdots,p.$$

*Proof.* Fix $i$ and recall from (1) that

$$e^{-\alpha t}(\mathbf{v}\cdot\mathbf{Z}(t)) \xrightarrow{\text{a.s.}} (\mathbf{v}\cdot\eta)\mathbf{W}, \quad \text{as} \quad t\to\infty.$$

For $\varepsilon_1 = \frac{x_1}{2}(\mathbf{v}\cdot\eta)$ choose $s_1 = s_1(\delta)$ such that for $s \geq s_1$,

$$\sup_{t\geq 0} P_i(|e^{-\alpha(t+s)}(\mathbf{v}\cdot\mathbf{Z}(t+s)) - (\mathbf{v}\cdot\eta)W| \geq \varepsilon_1) < \frac{\delta}{2}.$$

Then for $s \geq s_1$ and for all $t \geq 0$,

$$\begin{aligned}&P_i(W\in I, |A_3(t,s)|>\varepsilon)\\ &\quad\leq P_i(W\in I, |A_3(t,s)|>\varepsilon, |e^{-\alpha(t+s)}(\mathbf{v}\cdot\mathbf{Z}(t+s))-(\mathbf{v}\cdot\eta)W|<\varepsilon_1)\\ &\qquad + P_i(W\in I, |e^{-\alpha(t+s)}(\mathbf{v}\cdot\mathbf{Z}(t+s))-(\mathbf{v}\cdot\eta)W|\geq\varepsilon_1)\\ &\quad\leq P_i\left(\frac{|Z_{M(s)f}(t)|}{\sqrt{(\mathbf{v}\cdot\eta)x_1-\varepsilon_1}}e^{-\frac{\alpha}{2}(t+s)}>\varepsilon\right)+\frac{\delta}{2}\\ &\quad\leq \frac{2e^{-\alpha(t+s)}}{\varepsilon^2 x_1(\mathbf{v}\cdot\eta)}E_i[(Z_{M(s)f}(t))^2]+\frac{\delta}{2}.\end{aligned}$$

The last inequality comes from Chebyshev's inequality. Now we can find $s_2$ by Proposition 3 such that for $s \geq s_2$

$$e^{-\alpha s} \lim_{t\to\infty} e^{-\alpha t} {}_iD_{M(s)f}(t) \leq \frac{1}{4}\varepsilon^2 x_1(\mathbf{v}\cdot\eta)\delta.$$

Let $s_0 = \max\{s_1, s_2\}$, then for $s \geq s_0$

$$\lim_{t\to\infty} P_i(W \in I, |A_3(t,s)| > \varepsilon) < \delta.$$

□

LEMMA 2. *Fix $s > 0$, then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} P_i(W \in I, |A_1(t,s)| > \varepsilon) = 0, \quad i = 1, \cdots, p.$$

*Proof.* Put

$$U_k(t,s) = \sqrt{\frac{Z_k(t)e^{-\alpha s}}{\mathbf{v}\cdot\mathbf{Z}(t+s)e^{-\alpha(t+s)}}} - \theta_k, \quad V_k(t,s) = \frac{1}{\sqrt{Z_k(t)}}\sum_{j=1}^{Z_k(t)} X_{kj}(s)$$

so that $A_1(t+s) = \sum_{j=1}^{p} U_k(t,s)V_k(t,s)$. Hence it is enough to show that for each $k = 1, \cdots, p$,

$$\lim_{t\to\infty} P(W \in I, |U_k(t,s)V_k(t,s)| > \varepsilon) = 0 \tag{13}$$

By the conditional independence of $X_{kj}(s)$'s, using the fact $E(X_{kj}(s)|\mathcal{F}_t) = 0$, we have the following

$$\begin{aligned} E(V_k^2(t,s)) &= E\left(\frac{e^{-\alpha s}}{Z_k(t)}\sum_{j=1}^{Z_k(t)} \operatorname{Var}(Z_f^{a_{kj}}(s))\right) \\ &= E\left(\frac{e^{-\alpha s}}{Z_k(s)}\sum_{j=1}^{Z_k(t)}\sum_{i=1}^{5} \gamma_{ki}(a_{kj}, s)\right), \end{aligned}$$

where $\gamma_{ki}$'s are given in Lemma 3 below. For fixed $s > 0$

$$\sup_{0\leq u\leq s} {}_jD_f(u) < \infty, \quad \sup_{0\leq u\leq s} {}_jm_f(u) < \infty.$$

Therefore we can see that for $i = 3, 4, 5$

$$\sup_{a\geq 0} \gamma_{k,i} < \infty.$$

Further (B 6) guarantees that for $i = 1, 2$

$$\sup_{a\geq 0} \gamma_{k,i} < \infty.$$

Hence

$$K \equiv \sup_{t,k} E(V_k^2(t,s)) < \infty.$$

Given $\delta > 0$, choose $M$ such that $\dfrac{K}{M^2} < \delta$ then

$$\begin{aligned} & P_i(W \in I, |U_k(t,s)V_k(t,s)| > \varepsilon) \\ & \quad \le \; P_i(W \in I, |U_k(t,s)V_k(t,s)| > \varepsilon, |V_k(t,s)| \le M) + P(|V_k(t,s)| > M) \\ & \quad \le \; P_i\left(W \in I, |U_k(t,s)| > \frac{\varepsilon}{M}\right) + \delta \end{aligned}$$

where the last inequality results from Chebyshev inequality. Since $U_k(t,s) \xrightarrow{\text{a.s.}} 0$ on $\{W \in I\}$ as $t \to \infty$ (see (1)),

$$\lim_{t\to\infty} P_i\left(W \in I, |U_k(t,s)| > \frac{\varepsilon}{M}\right) = 0.$$

Being $\delta > 0$ arbitrary, the proof is complete. □

The following three lemmas are the multitype versions of lemmas in Kang (1999). The proofs can be carried out in the exactly same way as in the proofs of them. The complication comes not from the idea but from the notation, so we omit the proofs.

LEMMA 3. *For fixed $s_0 > 0$,*

$$\lim_{t\to\infty} \mathrm{Var}(A_2(t,s_0)|\mathcal{F}_t) = \sigma_f^2(s_0), \quad \textit{in probability,}$$

*where*

$$\begin{aligned} \sigma_f^2(s) &= \sum_{k=1}^{p} \theta_k^2 e^{-\alpha s} \int_0^\infty \sum_{i=1}^{5} \gamma_{k,i}(a,s) A_k(da), \\ \gamma_{k,1}(a,s) &= f^2(a+s) G_k^a(s)(1 - G_k^a(s)), \\ \gamma_{k,2}(a,s) &= -2 \sum_{j=1}^{p} m_{kj} f(a+s)(1 - G_k^a(s))({}_j m_f * G_k^a)(s), \\ \gamma_{k,3}(a,s) &= \sum_{j=1}^{p} m_{kj} ({}_j D_f * G_k^a)(s), \\ \gamma_{k,4}(a,s) &= \sum_{j=1}^{p} \sum_{l=1}^{p} m_{kj} m_{kl} ({}_j m_f * G_k^a)(s)({}_l m_f * G_k^a)(s), \end{aligned}$$

$$\gamma_{k,5}(a,s) = \sum_{j=1}^{p}\sum_{l=1}^{p} e_{k,j,l}({}_jm_f \cdot {}_lm_f) * G_k^a(s).$$

$$e_{k,j,l} = \begin{cases} d_{kj} - m_{kl} & \text{if } j = l \\ m_{kj}m_{kl} & \text{if } j \neq l \end{cases}$$

LEMMA 4. *For fixed $s_0 > 0$ and $\varepsilon > 0$,*

$$\sup_{0\le a\le t} E_k([X_k^a(s_0)]^2; |X_k^a(s_0)| > \varepsilon e^{\frac{\alpha}{2}t}) \xrightarrow{\text{pr}} 0 \quad \text{as} \quad t \to \infty,$$

*where $X_k^a(s_0) = (Z_f^a(s_0) - {}_km_f^a(s_0))e^{-\frac{\alpha}{2}s_0}$.*

The following lemma concerns the conditional Lindeberg-Feller condition.

LEMMA 5. *Fix $s_0 > 0$, $\varepsilon > 0$, then for each $k = 1, \cdots, p$*

$$\sum_{j=1}^{Z_k(t)} E\left(\frac{(X_{kj}(s_0))^2}{Z_k(t)}; \left|\frac{X_{kj}(s_0)}{\sqrt{Z_k(t)}}\right| > \varepsilon|\mathcal{F}_t\right) \xrightarrow{\text{pr}} 0 \quad \text{as} \quad t \to \infty.$$

Now we examine the limiting behavior of $P(W \in I, A_2(t+s) \le y)$ for fixed $s$. Since $e^{-\alpha t}(\mathbf{v}\cdot\mathbf{Z}(t)) \xrightarrow{\text{a.s.}} (\mathbf{v}\cdot\eta)W$ as $t \to \infty$, for any $\delta_1, \delta_2 > 0$, there exists $t_0 = t_0(\delta_1, \delta_2)$ such that $t \ge t_0$ implies

$$P\left(\left|d\left(e^{-\alpha t}\frac{\mathbf{v}\cdot\mathbf{Z}(t)}{\mathbf{v}\cdot\eta}, I\right) - d(W, I)\right| > \delta_2\right) \le \delta_1,$$

where $d(x, I) = \inf_{y\in I} d(x, y)$. Let

$$E_t = \left\{\left|d\left(e^{-\alpha t}\frac{\mathbf{v}\cdot\mathbf{Z}(t)}{\mathbf{v}\cdot\eta}, I\right) - d(W, I)\right| \le \delta_2\right\},$$

then

$$\begin{aligned}\{W \in I\} \cap E_t &= \left\{d\left(e^{-\alpha t}\frac{\mathbf{v}\cdot\mathbf{Z}(t)}{\mathbf{v}\cdot\eta}, I\right) \le \delta_2\right\} \\ &= \left\{x_1 - \delta_2 \le e^{-\alpha t}\frac{\mathbf{v}\cdot\mathbf{Z}(t)}{\mathbf{v}\cdot\eta} \le x_2 + \delta_2\right\} \equiv E_t'.\end{aligned}$$

So noting that

$$\begin{aligned}0 &\le P(W \in I, A_2(t+s) \le y) - P(E_t', A_2(t+s) \le y) \\ &\le P(E_t^c) \le \delta_1,\end{aligned}$$

we have

$$
\begin{aligned}
P_i(E_t', A_2(t+s) \le y) &\le P_i(W \in I, A_2(t+s) \le y) \\
&\le P_i(E_t', A_2(t+s) \le y) + \delta_1. \qquad (14)
\end{aligned}
$$

Recall that

$$A_2(t+s) = \sum_{k=1}^{p} \frac{\theta_k}{\sqrt{Z_k(t)}} \sum_{j=1}^{Z_k(t)} X_{kj}(s),$$

where $X_{kj}(s) = (Z_f^{a_{kj}}(s) - {}_k m_f^{a_{kj}}(s))e^{-\frac{\alpha}{2}s}$, and $X_{kj}(s)$ are mutually independent conditioned on $\mathcal{F}_t$. Further, for each $k$, $X_{kj}(s), k = 1, \cdots, Z_k(t)$ satisfy the conditions of Lindeberg-Feller theorem (Lemma 5). Hence using (1) and Lindeberg-Feller theorem (see Durrett (1991), p. 98)

$$
\begin{aligned}
\lim_{t\to\infty} P_i(E_t', A_2(t+s) \le y) &= \lim_{t\to\infty} P_i(E_t')P_i(A_2(t+s) \le y) \\
&= P_i(x_1 - \delta_2 < W < x_2 + \delta_2)\Phi\left(\frac{y}{\sigma_f(s)}\right).
\end{aligned}
$$

Since $\delta_1, \delta_2 > 0$ are arbitrary, (14) proves the following

LEMMA 6. *For fixed $s_0 > 0$,*

$$\lim_{t\to\infty} P_i(W \in I, A_2(t, s_0) \le y) = P_i(W \in I)\Phi\left(\frac{y}{\sigma_f(s_0)}\right), \quad i = 1, \cdots, p.$$

*Proof of Theorem* 1. Now let us put all the pieces together. Let $\varepsilon > 0$ be arbitrary and $y$ be fixed. Choose $\eta_\varepsilon > 0$ such that

$$\left|\Phi\left(\frac{y+\eta_\varepsilon}{\sigma_f}\right) - \Phi\left(\frac{y-\eta_\varepsilon}{\sigma_f}\right)\right| < \frac{\varepsilon}{4}. \qquad (15)$$

Since $\lim_{s\to\infty} \sigma_f^2(s) = \sigma_f^2$, there exists $s_1(\varepsilon)$ such that $s \ge s_1(\varepsilon)$ implies

$$\left|\Phi\left(\frac{y+r\eta_\varepsilon}{\sigma_f}\right) - \Phi\left(\frac{y+r\eta_\varepsilon}{\sigma_f(s)}\right)\right| < \frac{\varepsilon}{4} \qquad (16)$$

where $r = 1, -1$. Let $\delta = \frac{\varepsilon}{2}$ and let $s^* = \max\{s_0(\frac{\eta_\varepsilon}{2}, \delta), s_1(\varepsilon)\}$ where $s_0(\frac{\eta_\varepsilon}{2}, \delta)$ is defined as in Lemma 1. Then

$$\begin{aligned}
&\limsup_{t\to\infty} P_i(W \in I, A_1(t,s^*) + A_2(t,s^*) + A_3(t,s^*) \le y) \\
&\le \limsup_{t\to\infty} P_i(W \in I, A_2(t,s^*) \le y + \eta_\varepsilon) \\
&\quad + \limsup_{t\to\infty} P_i\left(W \in I, |A_1(t,s^*)| \ge \frac{\eta_\varepsilon}{2}\right) \\
&\quad + \limsup_{t\to\infty} P_i\left(W \in I, |A_3(t,s^*)| \ge \frac{\eta_\varepsilon}{2}\right) \\
&\le P_i(W \in I)\Phi\left(\frac{y+\eta_\varepsilon}{\sigma_f(s^*)}\right) + \frac{\varepsilon}{2} \quad \text{by Lemma 6, Lemma 2, and Lemma 1} \\
&\le P_i(W \in I)\Phi\left(\frac{y}{\sigma_f}\right) + \varepsilon \quad \text{by (15) and (16)}
\end{aligned}$$

On the other hand, the same arguments as above lead to

$$\begin{aligned}
&\liminf_{t\to\infty} P_i(W \in I, A_1(t,s^*) + A_2(t,s^*) + A_3(t,s^*) \le y) \\
&\ge \liminf_{t\to\infty} P_i(W \in I, A_2(t,s^*) \le y - \eta_\varepsilon) \\
&\quad - \liminf_{t\to\infty} P_i\left(W \in I, |A_1(t,s^*)| \ge \frac{\eta_\varepsilon}{2}\right) \\
&\quad - \liminf_{t\to\infty} P_i\left(W \in I, |A_3(t,s^*)| \ge \frac{\eta_\varepsilon}{2}\right) \\
&\ge P_i(W \in I)\Phi\left(\frac{y-\eta_\varepsilon}{\sigma_f(s^*)}\right) - \frac{\varepsilon}{2} \\
&\ge P_i(W \in I)\Phi\left(\frac{y}{\sigma_f}\right) - \varepsilon.
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ we finish the proof. □

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