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ON POLYGROUP HYPERRINGS AND REPRESENTATIONS OF POLYGROUPS

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Abstract. In this paper we introduce matrix representations of polygroups over hyperrings and show such representations induce representations of the fundamental group over the corresponding fundamental ring. We also introduce the notion of a polygroup hyperring generalizing the notion of a group ring. We establish homomorphisms among various polygroup hyperrings.

1. Introduction

The concept of hypergroup, which is a generalization of the concept of ordinary group, was first introduced by Marty [6]. A hypergroup is a set H equipped with an associative hyperoperation $\cdot : H \times H \longrightarrow \mathcal{P}^*(H)$ which satisfies the property $x \cdot H = H \cdot x = H$, for all $x \in H$. If the hyperoperation $\cdot$ is associative then H is called a semihypergroup.

In the above definition if $A, B \subseteq H$ and $x \in H$ then we define

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, \quad x \cdot B = \{x\} \cdot B \text{ and } A \cdot x = A \cdot \{x\}.$$

A polygroup is a special case of a hypergroup. According to [1] and [2] a polygroup is a system $\mathcal{P} = \langle P, \cdot, e, ^{-1} \rangle$ where $e \in P$, $^{-1}$ is a unary operation on P, $\cdot$ maps $P \times P$ into nonempty subsets of P, and the following axioms hold for all $x, y, z \in P$:

1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
2) $x \cdot e = e \cdot x = x$,
3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

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A hyperring is a hyperstructure with two hyperoperations $+$ and $\cdot$ that satisfies the ring-like axioms: $(R, +, \cdot)$ is a hyperring if $(R, +)$ is a commutative polygroup, $\cdot$ is an associative hyperoperation and the distributive laws $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$, $(x + y) \cdot z \subseteq x \cdot z + y \cdot z$ are satisfied for every $x, y, z \in R$.

If there exists $u \in R$ such that $x \cdot u = u \cdot x = \{x\}$ for all $x \in R$, then u is called the scalar unit of R and is denoted by 1. The element 0 is called zero element of R if $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$.

$(R, +, \cdot)$ is called a semihyperring if $+$, $\cdot$ are associative hyperoperations where $\cdot$ is distributive with respect to $+$.

Let R_1, R_2 be two hyperrings. A map $f : R_1 \longrightarrow R_2$ is called a strong homomorphism if, for all $x, y \in R_1$, the following relations hold:

$$f(x + y) = (f(x) + f(y)) \text{ and } f(x \cdot y) = f(x) \cdot f(y),$$

and f is called an inclusion homomorphism if, for all $x, y \in R$, the following relations hold:

$$f(x + y) \subseteq f(x) + f(y) \text{ and } f(x \cdot y) \subseteq f(x) \cdot f(y).$$

In section 2 of this paper, we introduce the notion of representation of polygroups by hyper matrices, and we define a polystructure on matrices. We will construct representations of extensions of polygroups and also representations induced on the fundamental group of a polygroup.

In section 3, we introduce polygroup hyperring over a hyperring R and study relations between polygroups and polygroup hyperrings.

2. Representations of Polygroups

A hypermatrix is a matrix with entries from a semihyperring. The hyperproduct of two hypermatrices (a_{ij}), (b_{ij}) which are of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner

$$(a_{ij})(b_{ij}) = \left\{ (c_{ij}) \mid c_{ij} \in \sum_{k=1}^{n} a_{ik} b_{kj} \right\}.$$

One of the important problems concerning representation of polygroups is as follows.

For a given polygroup $\mathcal{P} = \langle P, \cdot, e, ^{-1} \rangle$, find a semihyperring R with a scalar unit and a zero element such that one gets a representation of $\mathcal{P}$ by hypermatrices with entries from R. Recall that if $M_R := \{(a_{ij}) | a_{ij} \in R\}$, then a map $T : P \longrightarrow M_R$ is called a representation of P if $T(x_1 \cdot x_2) = \{T(x) | x \in x_1 \cdot x_2\} = T(x_1)T(x_2), \ \ \forall x_1, x_2 \in P$ and $T(e) = I$, where I is the identity matrix.

In the following we will give an example of a polygroup and find a semihyperring R and a representation of the polygroup over R. After that we will obtain a representation of the direct hyperproduct of two polygroups $P_1 \times P_2$ from representations of P_1 and P_2. Some topics related to the above problem is also discussed.

EXAMPLE 2.1. Suppose that the multiplication table for polygroup $\mathcal{P} = \langle P, \cdot, e, ^{-1} \rangle$ where $P = \{e, a, b\}$ is

·	e	a	b
e	e	a	b
a	a	{e,b}	{a,b}
b	b	{a,b}	{e,a}

In Z_3, we define a hyperoperation $\oplus$ as follows:
$1 \oplus 1 = \{0, 2\}, \ 2 \oplus 2 = \{0, 1\}, \ 1 \oplus 2 = \{1, 2\}$ and $\oplus$ be the usual sum for the other cases, and let $\odot$ be the usual product in Z_3. One can see that $(Z_3, \oplus, \odot)$ is a semihyperring. Then the map $T : P \longrightarrow M_R$ with

$$T(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T(a) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T(b) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a representation of the polygroup P.

Generally, if we choose $i_o, \ j_0, \ i_0 \neq j_0, \ 0 \leq i_0, \ j_0 \leq n$ and then put $T(e) = I_n$, $T(a) = A_n$ and $T(b) = B_n$ where

$$A_n = (a_{ij}) \text{ with } \begin{cases} a_{ii} = 1 & i = 1, \cdots, n \\ a_{i_0 j_0} = 1 & \\ a_{ij} = 0 & \text{otherwise.} \end{cases}$$

$$B_n = (b_{ij}) \text{ with } \begin{cases} b_{ij} = a_{ij} & \text{if } i \neq i_0, \ j \neq j_0, \\ b_{i_0 j_0} = 2 & \end{cases}$$

then T is a representation of P.

EXAMPLE 2.2. Suppose that $\langle P_1, \cdot, e_1, ^{-1} \rangle$ and $\langle P_2, *, e_2, ^{-1} \rangle$ are two polygroups. We know $\langle P_1 \times P_2, o, E, ^{-I} \rangle$ is a polygroup where (x_1, y_1) $o(x_2, y_2) = \{(x, y) \mid x \in x_1 \cdot x_2, \ y \in y_1 * y_2 \}, \ \ E = (e_1, e_2), \ \ (x, y)^{-I} =$

(x^{-1}, y^{-1}). Now, if $T_1 : P_1 \longrightarrow M_R$ and $T_2 : P_2 \longrightarrow M_R$ are two representations of P_1 and P_2 respectively, then we have the following representation for $P_1 \times P_2$:

$$T_1 \times T_2 : P_1 \times P_2 \longrightarrow M_R, \qquad T_1 \times T_2(x, y) = \begin{bmatrix} T_1(x) & 0 \\ 0 & T_2(y) \end{bmatrix}.$$

In [2] extensions of polygroups by polygroups have been introduced in the following way. Suppose $\mathcal{A} = \langle A, \cdot, e, ^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, ^{-1} \rangle$ are two polygroups whose elements have been renamed so that $A \cap B = \{e\}$. A new system $\mathcal{A}[\mathcal{B}] = \langle M, *, e, ^{I} \rangle$ called the extension of $\mathcal{A}$ by $\mathcal{B}$ is formed in the following way: Set $M = A \cup B$ and let $e^I = e$, $x^I = x^{-1}$, $e * x = x * e = x$ for all $x \in M$, and for all $x, y \in M - \{e\}$

$$x * y = \begin{cases} x \cdot y & \text{if } x, y \in A \\ x & \text{if } x \in B,\ y \in A \\ y & \text{if } x \in A,\ y \in B \\ x \cdot y & \text{if } x, y \in B,\ y \neq x^{-1} \\ x \cdot y \cup A & \text{if } x, y \in B,\ y = x^{-1}. \end{cases}$$

In this case $\mathcal{A}[\mathcal{B}]$ is a polygroup which is called the extension of $\mathcal{A}$ by $\mathcal{B}$. In the following proposition we will see how a representation of $\mathcal{B}$ gives a representation of $\mathcal{A}[\mathcal{B}]$.

PROPOSITION 2.3. *Let $\mathcal{A} = \langle A, \cdot, e, ^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, ^{-1} \rangle$ be two polygroups. Let T be a representation of $\mathcal{B}$, then $\varphi : \mathcal{A}[\mathcal{B}] \longrightarrow M_R$ where*

$$\varphi(x) = \begin{cases} T(x) & \text{if } x \in B \\ I_n & \text{if } x \in A \end{cases}$$

is a representation of $\mathcal{A}[\mathcal{B}]$.

Proof. If $x, y \in B$ and $y = x^{-1}$ then $\varphi(x)\varphi(y) = T(x)T(y)$ and $\varphi(x * y) = \varphi(x \cdot y \cup A) = \varphi(x \cdot y) \cup \varphi(A) = \varphi(x \cdot y) \cup \{I_n\}$. Since $e \in x \cdot y$, then $\varphi(e) \in \varphi(x \cdot y)$ and so $I_n \in \varphi(x \cdot y)$. Therefore $\varphi(x * y) = \varphi(x)\varphi(y)$. The rest is obvious. □

Let $\mathcal{P} = \langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. We define a relation β^* as the smallest equivalence relation such that the quotient P/β^* is a group. Then β^* is called the fundamental equivalence relation. Let us denote by U the set of all finite products of elements of P and define a relation β on P as follows: $x\beta y$ iff $\{x, y\} \subseteq u$ for some $u \in U$. Freni proved in [4] that for hypergroups we have $\beta^* = \beta$.

The product $\odot$ in P/β^* is defined as follows:

$$\beta^*(a) \odot \beta^*(b) = \beta^*(c) \text{ for all } c \in \beta^*(a) \cdot \beta^*(b).$$

The β^* equivalence relation was introduced on hypergroups by Koskas [5] and studied mainly by Corsini [3]. See also [7].

Now let $(R, +, \cdot)$ be a hyperring. We define a relation γ^* as the smallest equivalence relation such that the quotient R/γ^* is a ring. γ^* is also called the fundamental equivalence relation and R/γ^* is called the fundamental ring (see [7],[8]). If $\mathcal{U}$ denotes the set of all finite polynomials of elements of R over natural numbers, then a relation γ can be defined on R whose transitive closure is the fundamental relation γ^*. Such a relation γ is defined as follows: $x\gamma y$ iff $\{x, y\} \subseteq u$ for some $u \in \mathcal{U}$.

In R/γ^* both the sum $\oplus$ and the product $\odot$ are defined as follow:

$$\gamma^*(a) \oplus \gamma^*(b) = \gamma^*(c) \text{ for all } c \in \gamma^*(a) + \gamma^*(b),$$

$$\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d) \text{ for all } d \in \gamma^*(a) \cdot \gamma^*(b).$$

In the following proposition we obtain a representation of the fundamental group from a representation of a polygroup.

PROPOSITION 2.4. *Every representation $T(a) = (a_{ij})$ of a polygroup $\langle P, \cdot, e, ^{-1} \rangle$ by $n \times n$ hypermatrices over a hyperring $(R, +, \cdot)$ induces an $n \times n$ representation T^* of the fundamental group P/β^* over the fundamental ring R/γ^* by*

$$T^*(\beta^*(a)) = (\gamma^*(a_{ij})) \text{ for all } \beta^*(a) \in P/\beta^*.$$

Proof. The proof is similar to the proofs of Theorem 8.2.1 and 8.2.3 in [7]. □

DEFINITION 2.5. Let $\mathcal{P} = \langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. Two elements $x, y \in P$ are said to be conjugate if there exists an element $z \in P$ such that $y \in z^{-1}xz$.

It is easy to see that conjugation is an equivalence relation on the set P.

Some property of T^* is given in the following proposition.

PROPOSITION 2.6. *Let T be a representation of P over R of degree n and let I_n be the unit matrix over R/γ^*, then*

i) $T^*(\beta^*(e)) = I_n$;

ii) $T^*(\beta^*(x^{-1})) = (T^*(\beta^*(x)))^{-1}$ *for all* $x \in P$;
iii) *If* $x, y \in P$ *are conjugate, then* $T^*(\beta^*(x))$, $T^*(\beta^*(y))$ *are conjugate.*

Proof. (i) For every $x \in P$, we have $\beta^*(x) = \beta^*(x) \odot \beta^*(e) = \beta^*(e) \odot \beta^*(x)$ and from Proposition 2.4 we get $T^*(\beta^*(x)) = T^*(\beta^*(x))T^*(\beta^*(e)) = T^*(\beta^*(e))T^*(\beta^*(x))$. Therefore $T^*(\beta^*(e)) = I_n$. (ii) We have $e \in x \cdot x^{-1}$ and so $\beta^*(e) = \beta^*\ (x \cdot x^{-1}) = \beta^*(x) \odot \beta^*(x^{-1})$ which implies $T^*(\beta^*(e)) = T^*(\beta^*(x) \odot \beta^*(x^{-1})) = T^*(\beta^*(x))T^*(\beta^*(x^{-1}))$ therefore $I_n = T^*(\beta^*(x))\ T^*(\beta^*(x^{-1}))$. So $T^*(\beta^*(x^{-1})) = (T^*(\beta^*(x))^{-1}$. (iii) The proof is similar to (i) and (ii). □

3. Polygroup Hyperrings

Let $\langle P, \cdot, e, ^{-1} \rangle$ be a finite polygroup, and $\langle R, +, 0, - \rangle$ be a commutative polygroup and $(R, +, *)$ be a hyperring with scalar unit and zero element. Suppose that $R[P]$ is the set of all the functions on P with values in R, i.e., $R[P] = \{ f \mid f : P \longrightarrow R \text{ is a function } \}$. On $R[P]$ we consider the hyperoperations defined as follow:

$$f \oplus g = \{ h \mid h(x) \in f(x) + g(x) \}, \quad f \odot g = \left\{ h \mid h(z) \in \sum_{z \in x \cdot y} f(x) * g(y) \right\}.$$

We define the mapping $^{-I} : R[P] \longrightarrow R[P]$, where $f^{-I} : P \longrightarrow R$ is defined by $f^{-I}(p) = -f(p)$ for every $p \in P$ and let f_0 is the zero map.

Our aim in the following lemma and theorem is to show that $R[P]$ is a hyperring with hyperoperations $\oplus$ and $\odot$.

Lemma 3.1. *$\langle R[P], \oplus, f_0, ^{-I} \rangle$ is a polygroup.*

Proof. For every $f, g, h \in R[P]$, obviously we have $(f \oplus g) \oplus h = f \oplus (g \oplus h)$ and $f_0 \oplus f = f \oplus f_0 = f$. Now let $f \in g \oplus h$ then for every $x \in P$ we have $f(x) \in g(x) + h(x)$ and so $g(x) \in f(x) - h(x)$ and $h(x) \in -g(x) + f(x)$. Therefore $g \in f \oplus (-h)$ and $h \in (-g) \oplus f$. □

Theorem 3.2. *$(R[P], \oplus, \odot)$ is a hyperring.*

Proof. By Lemma 3.1 $\langle R[P], \oplus, f_0, ^{-I} \rangle$ is a polygroup. Let $f_1, f_2, f_3 \in R[P]$. Then

$$
\begin{aligned}
f_1 \odot (f_2 \odot f_3) = \;& f_1 \odot \left\{ f \mid f(z) \in \sum_{z \in x \cdot y} f_2(x) * f_3(y) \right\} \\
=\;& \bigcup_f f_1 \odot f \quad \text{where the union is over } f \text{ with } f(z) \\
& \in \sum_{z \in x \cdot y} f_2(x) * f_3(y) \\
=\;& \bigcup_f \left\{ g \mid g(a) \in \sum_{a \in b \cdot c} f_1(b) * f(c) \right\} \\
=\;& \left\{ g \mid g(a) \in \sum_{a \in b \cdot c} f_1(b) * (\sum_{a \in x \cdot y} f_2(x) * f_3(y)) \right\} \\
=\;& \left\{ g \mid g(a) \in \sum_{a \in b \cdot c} \sum_{c \in x \cdot y} f_1(b) * (f_2(x) * f_3(y)) \right\} \\
=\;& \left\{ g \mid g(a) \in \sum_{a \in b \cdot c} \sum_{c \in x \cdot y} (f_1(b) * f_2(x)) * f_3(y) \right\} \\
=\;& \left\{ g \mid g(a) \in \sum_{a \in b \cdot (x \cdot y)} (f_1(b) * f_2(x)) * f_3(y) \right\} \\
=\;& \left\{ g \mid g(a) \in \sum_{a \in (b \cdot x) \cdot y} (f_1(b) * f_2(x)) * f_3(y) \right\} \\
=\;& \left\{ g \mid g(a) \in \sum_{a \in d \cdot y} \sum_{d \in b \cdot x} (f_1(b) * f_2(x)) * f_3(y) \right\} \\
=\;& \bigcup_f f \odot f_3 \quad \text{where the union is over } f \text{ with } f(z) \\
& \in \sum_{z \in x \cdot y} f_1(x) * f_2(y) \\
=\;& (f_1 \odot f_2) \odot f_3.
\end{aligned}
$$

Similarly we have:

$$
\begin{aligned}
f \odot (f_1 \oplus f_2) = \;& \bigcup_{g \in f_1 \oplus f_2} f \odot g = \bigcup_{g \in f_1 \oplus f_2} \left\{ h \mid h(z) \in \sum_{z \in x \cdot y} f(x) * g(y) \right\} \\
=\;& \left\{ h \mid h(z) \in \sum_{z \in x \cdot y} f(x) * (f_1(y) + f_2(y)) \right\} \\
=\;& \left\{ h \mid h(z) \in \sum_{z \in x \cdot y} (f(x) * f_1(y)) + (f(x) * f_2(y)) \right\}
\end{aligned}
$$

$$\subseteq \left\{ h \mid h(z) \in (\sum_{z\in x\cdot y} f(x) * f_1(y)) + (\sum_{z\in x\cdot y} f(x) * f_2(y)) \right\}$$

$$= \bigcup_{h_1,\ h_2} \left\{ h \mid h(z) \in h_1(z) + h_2(z) \right\}$$

where the union is over $h_1 \in f \odot f_1$ and $h_2 \in f \odot f_2$

$$= \bigcup_{h_1,h_2} h_1 \oplus h_2 = \bigcup_{h_2}\bigcup_{h_1}(h_1 \oplus h_2) = \bigcup_{h_2}(\bigcup_{h_1} h_1 \oplus h_2))$$

$$= \bigcup_{h_2}(f \odot f_1) \oplus h_2 = (f \odot f_1) \oplus (f \odot f_2).$$

Hence $f \odot (f_1 \oplus f_2) \subseteq (f \odot f_1) \oplus (f \odot f_2)$. Similarly it can be proved that $(f_1 \oplus f_2) \odot f \subseteq (f_1 \odot f) \oplus (f_2 \odot f)$. Consequently $(R[P], \oplus, \odot)$ is a hyperring. □

Now that we constructed the hyperring $R[P]$ from R and P we will study relation between the polygroup P and the hyperring $R[P]$.

We define $E : R \longrightarrow R[P]$ by $r \mapsto E_r$ where $E_r : P \longrightarrow R$ is defined by

$$E_r(g) = \begin{cases} r, & g = e \\ 0, & g \neq e. \end{cases}$$

It is clear that E is a one to one function and we have
$E(r_1 + r_2) = E(r_1) \oplus E(r_2);\ E(r_1 * r_2) = E(r_1) \odot E(r_2);\ E(0) = E_0 :=$ zero function.
Therefore R is imbedded in $R[P]$.

If H is a subpolygroup of P, then we write

$$R\langle H\rangle = \{\ f \in R[P] \mid \{\ x \mid f(x) \neq 0\ \} \subseteq H\ \}.$$

Then there is a one to one polygroup homomorphism from $R\langle H\rangle$ to $R[P]$.

PROPOSITION 3.3. *Let P_1 and P_2 be two polygroups and $\psi : P_1 \longrightarrow P_2$ be a mapping. Then there exists an inclusion homomorphism of polygroups $\varphi : R[P_2] \longrightarrow R[P_1]$.*

Proof. We define $\varphi(f) = f \circ \psi$. Obviously, φ is well-defined. If $h \in f_1 \oplus f_2$ then for every $x \in P_1$, we have $\varphi(h)(x) = h(\psi(x)) \in f_1(\psi(x)) + f_2(\psi(x))$ or $\varphi(h)(x) \in \varphi(f_1)(x) + \varphi(f_2(x))$ which implies $\varphi(h) \in \varphi(f_1) \oplus \varphi(f_2)$ and so $\varphi(f_1 \oplus f_2) \subseteq \varphi(f_1) \oplus \varphi(f_2)$. □

PROPOSITION 3.4. *Let $\psi : R \longrightarrow S$ be a surjective inclusion homomorphism of hyperrings and $T = Ker\psi$. Then the mapping $\bar{\psi}$:*

$R[P] \longrightarrow S[P]$ defined by $\bar{\psi}(f) = \psi o f$ is a surjective inclusion homomorphism whose kernel is $T[P]$.

Proof. We have

$$\begin{aligned}\bar{\psi}(f_1 \oplus f_2) = &\ \psi \circ (f_1 \oplus f_2) = \psi \circ \{ f \mid f \in f_1 \oplus f_2 \} \\ = &\ \{ \psi \circ f \mid f \in f_1 \oplus f_2 \} \\ = &\ \{ h \mid h(p) = \psi(f(p)),\ f(p) \in f_1(p) + f_2(p),\ \forall p \in P \} \\ = &\ \{ h \mid h(p) \in \psi(f_1(p) + f_2(p)),\ \forall p \in P \} \\ \subseteq &\ \{ h \mid h(p) \in \psi(f_1(p)) + \psi(f_2(p)) \} = \psi \circ f_1 \oplus \psi \circ f_2\end{aligned}$$

and also

$$\begin{aligned}\bar{\psi}(f_1 \odot f_2) = &\ \left\{ \bar{\psi}(f) \mid f(x) \in \sum_{x \in y \cdot z} f_1(y) * f_2(z) \right\} \\ = &\ \left\{ \psi \circ f \mid f(x) \in \sum_{x \in y \cdot z} f_1(y) * f_2(z) \right\} \\ \subseteq &\ \left\{ \psi \circ f \mid \psi \circ f(x) \in \psi(\sum_{x \in y \cdot z} f_1(y) * f_2(z)) \right\} \\ \subseteq &\ \left\{ \psi \circ f \mid \psi \circ f(x) \in \sum_{x \in y \cdot z} \psi(f_1(y)) * \psi(f_2(z)) \right\} \\ \subseteq &\ \bar{\psi}(f_1) \oplus \bar{\psi}(f_2).\end{aligned}$$

Therefore $\bar{\psi}$ is an inclusion homomorphism. Obviously $\bar{\psi}$ is onto, and

$$\begin{aligned}Ker\bar{\psi} = &\ \{ f \in R[P] \mid \psi \circ f = f_0 \} \text{ where } f_0 \text{ is the zero function} \\ = &\ \{ f \in R[P] \mid \psi(f(x)) = 0,\ \forall x \in P \} \\ = &\ \{ f \in R[P] \mid f(x) \in Ker\psi,\ \forall x \in P \} \\ = &\ \{ f \in R[P] \mid f(x) \in T,\ \forall x \in P \} = T[P].\end{aligned}$$

□

Let Γ^* be the fundamental equivalence relation on $R[P]$ and $\mathcal{U}_{RP}$ denote the set of all finite polynomials of elements of $R[P]$ over natural numbers. In the following theorem we will construct a homomorphism between R/γ^* and $R[P]/\Gamma^*$.

THEOREM 3.5. *There exists a homomorphism $g : R/\gamma^* \longrightarrow R[P]/\Gamma^*$.*

Proof. We define $g(\gamma^*(r)) = \Gamma^*(E_r)$. First we prove that g is well defined. We know $a\gamma^*b$ if and only if there exist $x_1, \cdots, x_{m+1};\ u_1, \cdots, u_m \in$

$\mathcal{U}$ with $x_1 = a,\ x_{m+1} = b$ such that $\{x_i,\ x_{i+1}\} \subseteq u_i,\ i = 1, \cdots, m$. Then this implies $E(\{x_i,\ x_{i+1}\}) \subseteq E(u_i)$ or $\{\ E(x_i),\ E(x_{i+1})\} \subseteq E(u_i) \in \mathcal{U}_{RP}$ and so $E(x_i)\Gamma^* E(x_{i+1}),\ i = 1, \cdots, m$. Therefore $E(a)\Gamma^* E(b)$ that is to say $\Gamma^*(E_a) = \Gamma^*(E_b)$.

Now we will show that g is a homomorphism. This is because $g(\gamma^*(a) \oplus \gamma^*(b)) = g(\gamma^*(a+b)) = \Gamma^*(E(a+b)) = \Gamma^*(E(a)+E(b)) = \Gamma^*(E(a)) \oplus \Gamma^*(E(b)) = g(\gamma^*(a)) \oplus g(\gamma^*(b))$. Similarly, we get $g(\gamma^*(a) \odot \gamma^*(b)) = g(\gamma^*(a)) \odot g(\gamma^*(b))$. □

COROLLARY 3.6. *The following diagram is commutative, i.e.,* $\varphi_2 E = g\varphi_1$ *where* φ_1 *and* φ_2 *are canonical maps.*

$$\begin{array}{ccc} R & \xrightarrow{E} & R[P] \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ R/\gamma^* & \xrightarrow{g} & R[P]/\Gamma^*. \end{array}$$

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