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CONDITIONAL EXPECTATIONS GENERATING THE COMMUTANTS OF SUBALGEBRAS OF L^∞

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Abstract. Given a probability space and a subsigma algebra $\mathcal{A}$, each measure equivalent to the probability measure generates a different conditional expectation operator. We characterize those which act boundedly on the original L^2 space, and show there are sufficiently many such conditional expectations to generate the commutant of $L^\infty(\mathcal{A})$.

Let (X, $\mathcal{F}$, μ) be a probability space, and let $\mathcal{A}$ be a sigma subalgebra of $\mathcal{F}$. All measure spaces encountered in this note are assumed to be complete, and all set, function, etc statements are to be interpreted as holding up to sets of μ measure 0. We are concerned here with linear operators acting on $L^2(\mathcal{F}) = L^2(\mathrm{X}, \mathcal{F}, \mu)$. For $f \in L^\infty(\mathcal{A})$, let M_f be the operator of multiplication by f, acting on $L^2(\mathcal{F})$. Similarly, for an $\mathcal{F}$-measurable function f, L_f is the (in general, unbounded) linear transformation of multiplication by f, with domain $\{g \in L^2(\mathcal{F}) : fg \in L^2(\mathcal{F})\}$. Define

$$\mathcal{L}^\infty(\mathcal{A}) = \{M_f : f \in L^\infty(\mathcal{A})\},$$

so that $\mathcal{L}^\infty(\mathcal{A})$ is a von Neumann subalgebra of the maximal abelian von Neumann algebra $\mathcal{L}^\infty(\mathcal{F})$. $(\mathcal{L}^\infty(\mathcal{A}))'$ is then defined as the commutant of $\mathcal{L}^\infty(\mathcal{A})$; that is, the collection of all bounded operators commuting with all $L^\infty(\mathcal{A})$ multiplications.

The symbol $E_\mu^{\mathcal{A}}$ represents the conditional expectation operator with respect to $\mathcal{A}$. As an operator on $L^2(\mathcal{F})$, it is the orthogonal projection onto $L^2(\mathcal{A})$ (viewed as a Hilbert subspace of $L^2(\mathcal{F})$), but $E_\mu^{\mathcal{A}} f$ is also defined for f in any $L^p(\mathcal{F})$, $1 \leq p \leq \infty$, as well as for any $f \geq 0$ a.e. Since we will only be concerned with measures absolutely continuous with respect to μ, we will often write $E^{\mathcal{A}}$ for $E_\mu^{\mathcal{A}}$.

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In [1], R. G. Douglas characterized contractive projections on $L^1(X, \mathcal{F}, \mu)$ in terms of conditional expectations and *weight functions.* In that work, a nonnegative function w is called a weight function for $\mathcal{A}$ if $E_\mu^{\mathcal{A}} w$ is the characteristic function of a set A (necessarily in $\mathcal{A}$). The corresponding *weighted conditional expectation* operator is then given by the rule $f \to w \cdot E_\mu^{\mathcal{A}} f = L_w E_\mu^{\mathcal{A}} f$. We shall see shortly that such operators are the adjoints of certain conditional expectations $E_\nu^{\mathcal{A}}$. Although we will not examine properties of the collection of weight functions in detail in this note, it may be worth noting that for any nonnegative function f with $E_\mu^{\mathcal{A}} f < \infty$ a.e., the function $\frac{f}{E_\mu^{\mathcal{A}} f} \cdot \chi_{\text{support } E_\mu^{\mathcal{A}} f}$ is a weight function for $\mathcal{A}$, and all weight functions have this form.

In [3] we examined the set

$$\begin{aligned}\mathbb{L} &= \mathbb{L}(\mathcal{A}) = \mathbb{L}(\mathcal{A}, \mathcal{F}, \mu) \\ &= \left\{ f \in L^2(X, \mathcal{F}, \mu) : f \cdot L^2(X, \mathcal{A}, \mu) \subset L^2(X, \mathcal{F}, \mu) \right\}.\end{aligned}$$

$\mathbb{L}(\mathcal{A})$ is seen to be a Hilbert $L^\infty(\mathcal{A})$-module, with inner product

$$\langle f, g \rangle_{\mathbb{L}} = E^{\mathcal{A}}(\bar{f} \cdot g).$$

The following two results are established in that paper:

PROPOSITION 1. *$L^\infty(\mathcal{F}) \subset \mathbb{L}(\mathcal{A}) \subset L^2(\mathcal{F})$, and $f \in \mathbb{L}(\mathcal{A})$ if and only if $E^{\mathcal{A}} |f|^2 \in L^\infty(\mathcal{A})$. For $f \in \mathbb{L}$, the corresponding multiplication operator from $L^2(\mathcal{A})$ to $L^2(\mathcal{F})$ is continuous, with operator norm $\left(\left\| E^{\mathcal{A}} |f|^2 \right\|_\infty \right)^{1/2}$. In particular, $\mathbb{L}$ is a Banach space with respect to the norm $\|f\|_{\mathbb{L}} = \left(\left\| E^{\mathcal{A}} |f|^2 \right\|_\infty \right)^{1/2}$.*

PROPOSITION 2. *Let T be a continuous linear transformation on $L^2(\mathcal{F})$. Then the following are equivalent:*

a) $T \in (\mathcal{L}^\infty(\mathcal{A}))'$

b) *There is a constant C such that, for every $f \in L^2(\mathcal{F})$,*

$$E^{\mathcal{A}}\left(|Tf|^2\right) \le C \cdot E^{\mathcal{A}}\left(|f|^2\right) \text{ a.e.}$$

c) *For each $f \in L^2(\mathcal{F})$ there is a constant C_f such that*

$$E^{\mathcal{A}}\left(|Tf|^2\right) \le C_f \cdot E^{\mathcal{A}}\left(|f|^2\right) \text{ a.e.}$$

d) *For each $f \in L^2(\mathcal{F})$, support $Tf \subset$ support $E^{\mathcal{A}} |f|$.*

e) *For each* $f \in L^2(\mathcal{F})$, *define the measure* μ_f *on* $\mathcal{A}$ *by*

$$d\mu_f = |f|^2 \, d\mu|_{\mathcal{A}}.$$

Then for all f, $\mu_{Tf} \ll \mu_f$.

Moreover, with $\mathbb{L}$ and $\| \, \|_{\mathbb{L}}$ as defined previously, for $T \in (\mathcal{L}^\infty(\mathcal{A}))'$ we have $T\mathbb{L} \subset \mathbb{L}$, and T is continuous with respect to $\| \, \|_{\mathbb{L}}$.

Various Hilbert C*-module results are then used to give related characterizations of $(\mathcal{L}^\infty(\mathcal{A}))'$. Our main goal in this work is to present a set of conditional expectation operators related to $\mathbb{L}(\mathcal{A})$ which generates $(\mathcal{L}^\infty(\mathcal{A}))'$ in the following sense:

The statement "$\mathcal{S}$ generates $(\mathcal{L}^\infty(\mathcal{A}))'$" means $\mathcal{S}$ is a set of bounded operators on $L^2(\mathcal{F})$, and the smallest von Neumann algebra containing $\mathcal{S}$ is $(\mathcal{L}^\infty(\mathcal{A}))'$

Let $\mathcal{E}(\mu)$ be the set of all finite measures on $\mathcal{F}$ which are absolutely continuous with respect to μ. It is shown in [L1] and [L, W], respectively, that for $\nu \in \mathcal{E}(\mu)$,

(i) support $\left(E_\mu^{\mathcal{A}} |f|\right)$ is the smallest set in $\mathcal{A}$ containing support f.

(ii) For $\nu \in \mathcal{E}(\mu)$, $E_\nu^{\mathcal{A}}(f) = \frac{E_\mu^{\mathcal{A}}\left(\frac{d\nu}{d\mu} \cdot f\right)}{E_\mu^{\mathcal{A}}\left(\frac{d\nu}{d\mu}\right)}$.

Since $\frac{1}{E_\mu^{\mathcal{A}}\left(\frac{d\nu}{d\mu}\right)}$ is $\mathcal{A}$-measurable, (ii) may be rewritten as

$$E_\nu^{\mathcal{A}}(f) = E_\mu^{\mathcal{A}}\left(\frac{d\nu/d\mu}{E_\mu^{\mathcal{A}}(d\nu/d\mu)} \cdot f\right).$$

It follows from (i) that for any $f \geq 0$ but not 0 a.e., we may define $\Lambda(f) = \Lambda_\mu^{\mathcal{A}}(f) = \frac{f}{E^{\mathcal{A}}(f)}$; and we see that $E^{\mathcal{A}}\left(\Lambda_\mu^{\mathcal{A}}(f)\right) = \chi_{\text{support } E^{\mathcal{A}}(f)}$.

In general, for $\nu \in \mathcal{E}(\mu)$, $E_\nu^{\mathcal{A}}$ need not be a bounded operator on $L^2(X, \mathcal{F}, \mu)$. In fact, such boundedness is directly tied to $\mathbb{L}(A, \mu)$:

THEOREM 3. *Let* $\nu \in \mathcal{E}(\mu)$. *Then* $E_\nu^{\mathcal{A}}$ *is bounded on* $L^2(X, \mathcal{F}, \mu)$ *if and only if* $\Lambda\left(\frac{d\nu}{d\mu}\right) \in \mathbb{L}$. *If* $E_\nu^{\mathcal{A}}$ *is bounded on* $L^2(X, \mathcal{F}, \mu)$ *then its adjoint is given by the formula* $f \longrightarrow \Lambda\left(\frac{d\nu}{d\mu}\right) \cdot E_\mu^{\mathcal{A}} f$.

Proof. Suppose first that $\Lambda\left(\frac{d\nu}{d\mu}\right) \in \mathbb{L}$; i.e., $E^{\mathcal{A}}\left(\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right)^2\right) \in L^\infty$. For f in $L^2(\mathcal{F},\mu)$,

$$\begin{aligned}\left|E_\mu^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot f\right)\right|^2 &\leq \left(E_\mu^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right)^2\right)\cdot E_\mu^{\mathcal{A}}\left(|f|^2\right) \ a.e. \\ &\leq \left\|\Lambda\left(\frac{d\nu}{d\mu}\right)\right\|_{\mathbb{L}}^2 \cdot E_\mu^{\mathcal{A}}\left(|f|^2\right) \ a.e.\end{aligned}$$

Hence

$$\begin{aligned}\left\|E_\mu^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot f\right)\right\|_{L^2}^2 &\leq \left\|\Lambda\left(\frac{d\nu}{d\mu}\right)\right\|_{\mathbb{L}}^2 \cdot \left\|E_\mu^{\mathcal{A}}|f|^2\right\|_{L^1} \\ &= \left\|\Lambda\left(\frac{d\nu}{d\mu}\right)\right\|_{\mathbb{L}}^2 \cdot \|\ |f|\|_{L^2}^2 .\end{aligned}$$

Note that

$$\begin{aligned}\int_X \left|\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot E^{\mathcal{A}} f\right|^2 d\mu &\leq \int_X \left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right)^2 \cdot E^{\mathcal{A}}\left(|f|^2\right) d\mu \\ &= \int_X \left(E^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right)^2\right)\cdot\left(|f|^2\right) d\mu \\ &\leq \left\|\Lambda\left(\frac{d\nu}{d\mu}\right)\right\|_{\mathbb{L}}^2 \cdot \|f\|_{L^2}^2 ;\end{aligned}$$

so that the linear transformation G given by the formula $Gf = \Lambda\left(\frac{d\nu}{d\mu}\right)\cdot E_\mu^{\mathcal{A}} f$ is bounded on $L^2(\mathcal{F},\mu)$. In fact, G is the adjoint of $E_\nu^{\mathcal{A}}$. To see this, let ϕ be a nonnegative, essentially bounded, $\mathcal{F}$-measurable function for which $\phi\cdot\Lambda\left(\frac{d\nu}{d\mu}\right) \in L^2(\mathcal{F},\mu)$ (The set of such ϕ has dense linear span since $\Lambda\left(\frac{d\nu}{d\mu}\right)$ is finite a.e.). Then for $f \in L^2(\mathcal{F},\mu)$,

$$\begin{aligned}\langle Gf,\phi\rangle_{L^2} &= \int_X \Lambda\left(\frac{d\nu}{d\mu}\right)\cdot E_\mu^{\mathcal{A}} f \cdot \phi d\mu \\ &= \left\langle f, E_\mu^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot\phi\right)\right\rangle_{L^2} .\end{aligned}$$

This shows that $G^*\phi = E_\mu^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot\phi\right)$, and this formula extends to all $\phi \in L^2$.

Now suppose only that $E_\nu^{\mathcal{A}}$ is bounded on $L^2(\mathcal{F},\mu)$. For f and g in $L^2(\mathcal{F},\mu)$, with $\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot f$ in $L^2(\mathcal{F},\mu)$, we have the $L^2(\mathcal{F},\mu)$ inner product chain of equalities

$$\begin{aligned}\langle E_\nu^{\mathcal{A}} f, g\rangle &= \left\langle E_\mu^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot f\right), g\right\rangle \\ &= \left\langle \Lambda\left(\frac{d\nu}{d\mu}\right)\cdot f, E_\mu^{\mathcal{A}} g\right\rangle.\end{aligned}$$

Since there is a dense set of such functions f, it follows that $E_\nu^{\mathcal{A}*} g = \Lambda\left(\frac{d\nu}{d\mu}\right)\cdot E_\mu^{\mathcal{A}} g$. But this implies that $\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot L^2(\mathcal{A},\mu)\subset L^2(\mathcal{F},\mu)$, and that is precisely the condition for $\Lambda\left(\frac{d\nu}{d\mu}\right)\in\mathbb{L}$. □

REMARKS. (1) When different measures are in play, care must be taken in working with conditional expectations. For example, when working on $L^2(X,\mathcal{F},\mu)$, bounded $E_\nu^{\mathcal{A}}$ need not be contractive or self adjoint. However $E_\nu^{\mathcal{A}}$ does retain its projection status:

$$\begin{aligned}E_\nu^{\mathcal{A}}\left(E_\nu^{\mathcal{A}} f\right) &= E_\mu^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\cdot\left(E_\mu^{\mathcal{A}}\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right)\cdot f\right)\right) \\ &= \left(E_\mu^{\mathcal{A}}\Lambda\left(\frac{d\nu}{d\mu}\right)\right)\cdot E_\mu^{\mathcal{A}}\left(\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right)\cdot f\right) \\ &= \chi_{\text{support } E_\mu^{\mathcal{A}}\frac{d\nu}{d\mu}}\cdot E_\mu^{\mathcal{A}}\left(\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right)\cdot f\right).\end{aligned}$$

A judicious use of the conditional Cauchy-Schwarz inequality and the comments about supports made previously shows that $E_\mu^{\mathcal{A}}\left(\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right)\cdot f\right) = 0$ *a.e.* off support $E_\mu^{\mathcal{A}}\frac{d\nu}{d\mu}$. Thus $E_\nu^{\mathcal{A}}\left(E_\nu^{\mathcal{A}} f\right) = E_\nu^{\mathcal{A}} f$. This property of idempotency appears in Douglas' use of weighted conditional expectation operators in [1].

(2) Each $E_\nu^{\mathcal{A}}$ which is bounded on $L^2(X,\mathcal{F},\mu)$ is in $(\mathcal{L}^\infty(\mathcal{A}))'$. The remaining task in this work is to show that there are sufficiently many such $E_\nu^{\mathcal{A}}$ to generate $(\mathcal{L}^\infty(\mathcal{A}))'$.

LEMMA 4. *The linear span of* $\left\{\Lambda\left(\frac{d\nu}{d\mu}\right) : \nu\in\mathcal{E}(\mu) \text{ and } \Lambda\left(\frac{d\nu}{d\mu}\right)\in\mathbb{L}\right\}$ *is dense in* $L^2(X,\mathcal{F},\mu)$.

Proof. Suppose that $\left\langle g, \Lambda\left(\frac{d\nu}{d\mu}\right)\right\rangle = 0$ for every $\Lambda\left(\frac{d\nu}{d\mu}\right) \in \mathbb{L}$. Let u be a bounded positive function which is invertible in $L^\infty(\mathcal{F})$, and define the measure $\nu \in \mathcal{E}(\mu)$ by $d\nu = ud\mu$. The functions u and $E_\mu^A u$ are invertible elements of $L^\infty(\mathcal{F})$ and $L^\infty(\mathcal{A})$, respectively, so that

$$\Lambda(u) = \frac{u}{E_\mu^A u} \in L^\infty(\mathcal{F}) \subset \mathbb{L};$$

hence $\langle g, \Lambda(u)\rangle = 0$. Let $A \in \mathcal{A}$ with $\mu(\mathcal{A}) > 0$. Then

$$\chi_A \cdot \Lambda(u) = \frac{u}{E_\mu^A u}\chi_A = \frac{u \cdot \chi_A}{(E_\mu^A u) \cdot \chi_A} = \frac{u \cdot \chi_A}{(E_\mu^A u \cdot \chi_A)} \in \mathbb{L},$$

hence $\langle g, \chi_A \cdot \Lambda(u)\rangle = 0$; i.e.,

$$\begin{aligned} 0 &= \int_X \chi_A \cdot \Lambda(u) \cdot g d\mu = \int_A \Lambda(u) \cdot g d\mu \\ &= \int_A E_\mu^A(\Lambda(u) \cdot g)\, d\mu. \end{aligned}$$

But A was chosen arbitrarily in $\mathcal{A}$, and $E_\mu^A(\Lambda(u) \cdot g)$ is $\mathcal{A}$-measurable. Therefore $E_\mu^A(\Lambda(u) \cdot g) = 0$ a.e. But then

$$0 = E_\mu^A(\Lambda(u) \cdot g) = E_\mu^A\left(\frac{u}{E_\mu^A u} \cdot g\right) = \frac{E_\mu^A(u \cdot g)}{E_\mu^A u}.$$

Since $E_\mu^A u \neq 0$ a.e., we see that $E_\mu^A(u \cdot g) = 0$ a.e., and (by integrating 0) we have $\langle g, u\rangle = 0$. Since the linear span of the positive, invertible elements of L^∞ is dense in L^2, we must conclude that $g = 0$. □

THEOREM 5. *Let* $\mathcal{S} = \left\{E_\nu^A : \nu \in \varepsilon(\mu) \text{ and } \Lambda\left(\frac{d\nu}{d\mu}\right) \in \mathbb{L}\right\}$. *Then* $\mathcal{S}$ *generates* $(\mathcal{L}^\infty(\mathcal{A}))'$.

Proof. Via the von Neumann double commutant theorem, it suffices to show that if T is a bounded operator for which T and T^* commute with $\mathcal{S}$, then $T \in \mathcal{L}^\infty(\mathcal{A})'$. Let T be such an operator. For $\nu \in \varepsilon(\mu)$ and $\Lambda\left(\frac{d\nu}{d\mu}\right) \in \mathbb{L}$, and $f \in L^2(\mathcal{F}, \mu)$,

$$\begin{aligned} T\left(\Lambda\left(\frac{d\nu}{d\mu}\right) \cdot E_\mu^A f\right) &= TE_\nu^{A*} f \\ &= E_\nu^{A*} Tf = \Lambda\left(\frac{d\nu}{d\mu}\right) \cdot E_\mu^A Tf. \end{aligned}$$

Let $t = E_\mu^{\mathcal{A}}(T1) \in L^2(\mathcal{A}, \mu)$. Then (with $f = 1$ in the displayed equation above) $T\left(\Lambda\left(\frac{d\nu}{d\mu}\right)\right) = t \cdot \Lambda\left(\frac{d\nu}{d\mu}\right)$. Now the linear transformation L_t is closed, and, by Lemma 4, agrees with the bounded operator T on a dense set. Thus L_t is in fact equal to T, and necessarily $t \in L^\infty(\mathcal{A})$. □

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