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ON THE STABILITY OF IMMERSED MANIFOLDS IN E^4

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ABSTRACT. This work is concerned mainly with the variational problem on an immersion $x : M \to E^4$. A new approach is introduced depends on the normal variation in any arbitrary normal direction in the normal bundle. The results of this work are considered as a continuation and an extension to that obtained in [1], [2] and [3], [4] respectively. The methods adapted here are based on Cartan's methods of moving frames and the calculus of variations.

1. Geometric Preliminaries

Let M be a surface, immersed in E^4 (4-dimensional Euclidean space) given by the immersion $x = x(u^1, u^2) : M \to E^4$. Here, we choose a local frame $\{e_A\}$ in E^4, such that, restricted to M, the vectors $e_3, e_4 \varepsilon N_x(M) = T_x^{\perp}(M)$ (the normal bundle) are the unit normal vectors to M at x and $e_1, e_2 \varepsilon T_x(M)$ (the tangent bundle) are the tangent vectors to M at x. We shall use the following convention on the ranges of indices $1 \leq A, B, C, \cdots \leq 4, 1 \leq i, j, k, \cdots \leq 2, 3 \leq \alpha, \beta, \gamma, \cdots \leq 4$, and all sums extend always over the respective ranges of repeated indices. The fundamental equations of the frame $\{x, e_A\}$ are

$$dx = \omega^i e_i,\ de_A = \omega_A^B e_B,\ \omega_A^B + \omega_B^A = 0 \tag{1}$$

where ω^i and ω^A are the dual forms to the tangent vectors e_i and e_A on M and E^4 respectively. The structure equations of E^4 are [5]

$$d\omega^A = \omega^B \wedge \omega_B^A, d\omega_B^A = \omega_B^C \wedge \omega_C^A, \omega_B^A + \omega_A^B = 0. \tag{2}$$

From (1) we have

$$\omega^\gamma = 0,\ d\omega^\gamma = \omega^i \wedge \omega_i^\gamma = 0,\ \gamma = 3, 4.$$

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Using Cartan's lemma, we write

(3) $$\omega_i^\gamma = h_{ij}^\gamma \omega^j, h_{ij}^\gamma = h_{ji}^\gamma$$

where $h_{ij}^\gamma = \omega_i^\gamma(e_j)$ are the $2\underline{nd}$ fundamental quantities associated with the connection forms ω_i^γ on M. From the stationary conditions (1) we have $\omega_3^4 + \omega_4^3 = 0$ and thus

(4) $$\omega_\eta^\gamma = \ell_{\eta i}^\gamma \omega^i, \; \ell_{\eta i}^\gamma + \ell_{\gamma i}^\eta = 0$$

where $\ell_{\eta i}^\gamma = \omega_\eta^\gamma(e_i)$ are the $3\underline{rd}$ fundamental quantities associated with the normal connection ω_η^γ in the normal bundle $T_x^\perp(M)$ [6].
From (1), (2), (3) and (4) we have

(5) $$\left.\begin{aligned} de_\gamma &= \omega_\gamma^i e_i + \omega_\gamma^\eta e_\eta && \text{(Weingarten formula)} \\ de_i &= \omega_i^j e_j + \omega_i^\gamma e_\gamma && \text{(Gauss formula)} \\ d\omega^i &= \omega^i \wedge \omega_j^i \;,\; d\omega_j^i = \omega_j^k \wedge \omega_k^i + \Omega_j^i && \end{aligned}\right\}$$

where

(6) $$\Omega_j^i(e_k, e_\ell) = R_{jk\ell}^i = \sum_\gamma (h_{ik}^\gamma h_{j\ell}^\gamma - h_{i\ell}^\gamma h_{jk}^\gamma)$$

is the Gauss curvature.
The exterior differentiation of the $1\underline{st}$ equation of (1) gives

(7) $$d^2x = \Omega^i e_i + h_{ij}^\gamma \omega^i \omega^j e_\gamma$$

where $\langle d^2x(e_i, e_j), e_\gamma\rangle = h_{ij}^\gamma$. The 2-form Ω^i is given as $\Omega^i = d\omega^i + \omega^j \wedge \omega_j^i$ which is called the tangential component of d^2x on M.

A general unit normal vector field $n = n(\theta)\varepsilon T_x^\perp(M)$ is defined as [7]

(8) $$n(\theta) = \frac{[x, x_{,1}, x_{,2}]}{W} = \sum_\gamma \cos\theta_\gamma e_\gamma, \; \theta_4 = \pi/2 - \theta_3$$

where W is the length of the vector $[x, x_{,1}, x_{,2}]$ (the vector product of the vectors $x, x_{,1}, x_{,2}$) and

(9) $$\left.\begin{aligned} &x_{,i} = \lambda_i^j e_j \varepsilon T_x(M), \; x_{,i} = \frac{\partial x}{\partial u^i}, \; \lambda_i^j = \lambda_i^j(u^1, u^2) \\ &W = \sqrt{g}\Psi(u^1, u^2), \; g = Det(g_{ij}), \; g_{ij} = \langle x_{,i}, x_{,j}\rangle \end{aligned}\right\}$$

where $\Psi = \Psi(u^1,u^2) = \delta_{\eta\beta}x^\beta(u^1,u^2)x^\eta(u^1,u^2)$ is a positive definite function on M and $\langle\ ,\ \rangle$ is the induced metric by the immersion x.

The infinitesimal displacements of the normal $n(\theta)$ are given by

$$dn(\theta) = -\sum_i h^\gamma_{ij}\cos\,\theta_\gamma\omega^j e_i + (k_i + \ell^4_{3i})\omega^i n^\perp(\theta) \tag{10}$$

$$\begin{aligned} d^2n(\theta) \equiv -\Bigg\{&\sum_k h^\gamma_{ki}h^\gamma_{kj}\cos\,\theta_\gamma + \sum_{k,\gamma} h^3_{ki}h^4_{kj}\cos\,\psi_\gamma \\ &+\sum_\gamma (k_i+\ell^4_{3i})(k_j+\ell^4_{3j})\cos\,\theta_\gamma\Bigg\}\omega^i\omega^j e_\gamma \\ &+(d^2\theta + d\omega^4_3)(-1)^\gamma\cos\,\psi_\gamma e_\gamma, (\mathrm{mod}\ e_i)\end{aligned} \tag{11}$$

where $\psi_\gamma = \theta_3 + \frac{\gamma\pi}{2}$ and $n^\perp(\theta) = \cos\,\theta_4 e_3 - \cos\,\theta_3 e_4$.

Using (1), (7), (8) and the 1st equation of (3) one can see that the 1st and 2nd fundamental forms in the normal section n of the immersed manifold M respectively are

$$I = \langle dx, dx\rangle = \delta_{ij}\omega^i\omega^j \tag{12}$$

$$II(\theta) = \langle n(\theta), d^2x\rangle = \sum_\gamma h^\gamma_{ij}(\theta)\omega^i\omega^j \tag{13}$$

where δ_{ij} and δ^{ij} are the well-known kronker deltas and $h^\gamma_{ij}(\theta) = h^\gamma_{ij}\cos\,\theta_\gamma$, $\gamma = 3,4$.

Here, we introduce a quadratic form III related to the normal $n(\theta)$ as the following

$$\left.\begin{aligned} &III(\theta) = \langle dn(\theta), dn(\theta)\rangle = u_{jm}(\theta)\omega^j\omega^m,\ \text{where} \\ &u_{jm}(\theta) = \bar{u}_{jm}(\theta) + \sum_{\gamma\neq\beta}\sum_\ell h^\gamma_{\ell j}h^\beta_{\ell m}\cos\,\theta_\gamma\cos\theta_\beta \\ &\qquad\qquad + ((k_j+\ell^4_{3j})(k_m+\ell^4_{3m})) \\ &\bar{u}_{jm}(\theta) = u^\gamma_{jm}\cos^2\theta_\gamma,\ u^\gamma_{jm} = \sum_\ell h^\gamma_{j\ell}h^\gamma_{\ell m}\end{aligned}\right\} \tag{14}$$

We define the mean curvature vector $H(\theta)$ and the Lipschitz-Killing curvature $G(\theta)$ in the normal section $n(\theta)$ respectively as [5]

$$H(\theta) = h^{\gamma}(\theta)e_{\gamma} \tag{15}$$

$$\begin{aligned} G(\theta) &= \mathrm{Det}(h^{\gamma}_{ij}(\theta)) \\ &= G^{\gamma}\cos^2\theta_{\gamma} + \left(\sum_{i,j} h^3_{ii}h^4_{jj} - \sum_{i\neq j} h^3_{ij}h^4_{ij}\right)\cos\theta_3\cos\theta_4 \end{aligned} \tag{16}$$

where, $H^{\gamma}(\theta) = H^{\gamma}\cos\theta_{\gamma}$, $H^{\gamma} = \frac{1}{2}h^{\gamma}_{ij}\delta^{ij}$, $\gamma = 3, 4$.
Here, H^{γ}, are the components of the mean curvature vector and $G^{\gamma} = \mathrm{Det}(h^{\gamma}_{ij})$ are the Lipschitz-Killing curvatures in the normal section e_{γ} on the surface M. The length of the quadratic form $II(\theta)$ in the direction $n = n(\theta)$ is defined as

$$\begin{aligned} S^2(\theta) &= \sum_{i,j}\left(\sum_{\gamma} h^{\gamma}_{ij}(\theta)\right)^2 \\ &= \sum_{\gamma} S^2_{\gamma}\cos^2\theta_{\gamma} + 2\sum_{i,j} h^3_{ij}h^4_{ij}\cos\theta_3\cos\theta_4 \end{aligned} \tag{17}$$

where $S^2_{\gamma} = \sum_{i,j} h^{\gamma}_{ij}h^{\gamma}_{ij}$ is the length of the quadratic forms $II^{\gamma} = h^{\gamma}_{ij}\omega^i\omega^j$ in the normal section e_{γ}.
From (15) we have

$$4H^2(\theta) = \sum_{i,j} h^{\gamma}_{ij}h^{\gamma}_{ij}\cos^2\theta_{\gamma} + 2\sum_{i,j} h^3_{ii}h^4_{jj}\cos\theta_3\cos\theta_4. \tag{18}$$

The scalar curvature $R(\theta)$ in the normal direction $n(\theta)$ is defined as

$$R(\theta) = 4H^2(\theta) - S^2(\theta).$$

From (17) and (18) we have

$$R(\theta) = \sum_{\gamma} R^{\gamma}(\theta) + 2\left(\sum_{i,j} h^3_{ii}h^4_{jj} - \sum_{i\neq j} h^3_{ij}h^4_{ij}\right)\cos\theta_3\cos\theta_4 \tag{19}$$

where

$$R^\gamma(\theta) = R^\gamma \cos^2\theta_\gamma,\ R^\gamma = 2G^\gamma,\ \gamma = 3,4. \tag{20}$$

For any differentiable function $\phi = \phi(u^1, u^2), \phi \varepsilon \wedge^\circ (M)$ (the space of differentiable function on M), $d\phi = A_i\omega^i \varepsilon \wedge^1 (M)$ (the space of one forms on M), we define the gradient $\nabla\phi$ of ϕ as

$$\nabla\phi = \sum_i d\phi(e_i)e_i = \sum_i A_i e_i \tag{21}$$

where A_i is the $1\underline{st}$ covariant derivative of the function ϕ with the following properties [8]

$$\left.\begin{aligned} &\langle\nabla\phi, \nabla\phi\rangle = |\nabla\phi|^2 = \sum_i A_i^2 \\ &\langle\nabla\phi, \nabla f\rangle = \sum_i A_i B_i, \nabla\phi = \sum_i B_i e_i \\ &\nabla(\phi f) = \phi\nabla f + f\nabla\phi. \end{aligned}\right\} \tag{22}$$

The Laplace operator $\Delta\phi$ of the function ϕ is defined as

$$\Delta\phi = *^{-1} d * d\phi \tag{23}$$

where $*$ is the star operator defined as

$$*\omega^1 = -\omega^2, *\omega^2 = \omega^1 \text{ and } *1 = \omega^1 \wedge \omega^2. \tag{24}$$

From (23), (24) and direct computations, one can obtain [9]

$$\Delta\phi = A_{ij}\delta^{ij} \tag{25}$$

where A_{ij} is the $2\underline{nd}$ order covariant derivative of ϕ which is given from [4]

$$A_i\Omega^i + A_{ij}\omega^i\omega^j = 0. \tag{26}$$

The Laplacian satisfies the following property

$$\Delta(\phi f) = \phi\Delta f + f\Delta\phi + 2\langle\nabla f, \nabla\phi\rangle, \forall\phi, f \varepsilon \wedge^\circ (M). \tag{27}$$

2. Normal Variation of the 1st Fundamental Quantities

This section modifies the results on the variational problem which have been introduced in [3], [10]. Let M be a compact submanifold with piecewise smooth boundary ∂M, the function $\phi = 0$ on ∂M and $\int_M \phi * 1 = 0, \forall \phi \varepsilon \wedge^\circ (M)$.
A variation of x is a differentiable map

$$x : (-\varepsilon, \varepsilon) \times M \to E^4, x_t(P) = x(t,P), \forall P \varepsilon M$$

is an immersion, $x_o = x$ and $x_t|\partial M = x|\partial M, \forall t\varepsilon(-\varepsilon, \varepsilon)$. The image $x_t(P)$ is represented by the regular parametrization

$$\bar{x}(t, u^i) = x(u^i) + t\phi(u^i)n(u^i, \theta). \tag{28}$$

This representation defines a normal variation of M in E^4, associated with ϕ, in the normal direction $n(\theta)$. We define the operator δ_t as $\delta_t = \frac{\partial}{\partial t}$ given at $t = 0$. Thus, the variation vector field $\delta_t x = \phi n(\theta)$ where n is given by (8). Exterior differentiation of (28) and using (10) we have the infinitesimal displacement on the variation $\bar{x}$ as

$$\left.\begin{aligned} d\bar{x} &= \bar{\omega}^i e_i + \bar{\omega}^\gamma e_\gamma \\ \bar{\omega}^i &= (\delta_{ij} - t\phi h^\gamma_{ij} \cos\theta_\gamma)\omega^j \\ \bar{\omega}^\gamma &= t\left(A_i \cos\theta_\gamma + (-1)^\gamma \phi(k_i + \ell^4_{3i}) \cos\psi_\gamma\right)\omega^i \end{aligned}\right\} \tag{29}$$

where $k_i = d\theta(e_i)$.
The vector valued one-form $d\bar{x}$ can be written as

$$d\bar{x} = \omega^i \bar{e}_i \tag{30}$$

where

$$\begin{aligned} \bar{e}_i = &\sum_j (\delta_{ij} - t\phi h^\gamma_{ij} \cos\theta_\gamma) e_j \\ &+ t\sum_\gamma \left(A_i \cos\theta_\gamma + (-1)^\gamma \phi(k_i + \ell^\gamma_{3i}) \cos\psi_\gamma\right) e_\gamma \end{aligned}$$

are the tangent vectors on the variation $\bar{x}$.

The $1\underline{st}$ fundamental form of the variation $\bar{x}$ is

$$\bar{I} = \langle d\bar{x}, d\bar{x}\rangle = \bar{g}_{ij}\omega^i\omega^j,$$

where

$$\begin{aligned}\bar{g}_{ij}(t) = (\delta_{ij} - 2t\phi h^{\gamma}_{ij}\cos\theta_{\gamma}) + t^2\big(\phi^2\sum_k h^{\gamma}_{ik}h^{\beta}_{kj}\cos\theta_{\gamma}\cos\theta_{\beta} \\ + \phi^2(k_i + \ell^4_{3i})(k_j + \ell^4_{3j}) + A_iA_j\big), \bar{g}_{ij}(0) = \delta_{ij}.\end{aligned} \tag{31}$$

Thus, the $1\underline{st}$ variation of the metric quantities g_{ij} is defined as $\partial_t g_{ij} = \frac{\partial}{\partial t}\bar{g}_{ij}(t) \underset{t=0}{\rfloor}$
Thus,

$$\partial_t g_{ij} = -2\phi h^{\gamma}_{ij}\cos\theta_{\gamma}. \tag{32}$$

Using the relation

$$\bar{g}_{ij}\bar{g}^{ik} = \delta^k_j, \bar{g}^{ik}(0) = \delta^{ik}$$

it follows

$$\partial_t g^{ik} = 2\phi h^{\gamma}_{ik}\cos\theta_{\gamma}. \tag{33}$$

The $2\underline{nd}$ variations of g_{ij} and g^{ij} are defined as

$$\partial^2_t g_{ij} = \frac{\partial^2}{\partial t^2}\bar{g}_{ij}(t) \underset{t=0}{\rfloor}.$$

Using (31) and (14) we get

$$\begin{aligned}\partial^2_t g_{ij} =& 2\{\phi^2\big(\bar{\gamma}_{ij}(\theta) + \sum_{\gamma\neq\beta}\sum_k h^{\gamma}_{ik}h^{\beta}_{kj}\cos\theta_{\gamma}\cos\theta_{\beta} \\ &+ (k_i + \ell^4_{3i})(k_j + \ell^4_{3j})\big) + A_iA_j\}.\end{aligned} \tag{34}$$

Similarly

$$\begin{aligned}\partial^2_t g^{jm} =& 2\{\phi^2\big(3(\bar{\gamma}_{jm}(\theta) + \sum_{\gamma\neq\beta}\sum_i h^{\gamma}_{ij}h^{\beta}_{im}\cos\theta_{\gamma}\cos\theta_{\beta} \\ &- (k_j + \ell^4_{3j})(k_m + \ell^4_{3m})\big) - A_jA_m\}.\end{aligned} \tag{35}$$

From (30), the variation of the tangent vector dx is

$$\begin{aligned} &\partial_t dx = (\partial_t e_i)\omega^i \\ (36) \qquad &\partial_t e_i = \phi \sum_j h^\gamma_{ij} \cos\theta_\gamma e_j + A^\gamma_i e_\gamma \end{aligned}$$

where

$$A^\gamma_i = A_i \cos\,\theta_\gamma + (-1)^\gamma(k_i + \ell^4_{3i})\phi \cos\,\psi_\gamma.$$

From the relation

$$\langle dx, \partial_t e_\gamma\rangle = -\langle \partial_t dx, e_\gamma\rangle$$

we have

$$(37) \qquad \partial_t e_\gamma \equiv -\sum_i A^\gamma_i e_i + B^\gamma e_\beta,\ (\text{mod } e_\gamma),\ \gamma \neq \beta$$

where

$$B^\gamma = B^\gamma(u^1, u^2, \theta, \phi).$$

The variation $\partial_t x_{,i}$ of the tangent vectors $x_{,i}$ is obvious from (28) and is given by

$$\partial_t x_{,i} = \phi_{,i} n + n_{,i}\phi = d\phi(x_{,i})n + \phi dn(x_{,i});\ \ i = \frac{\partial}{\partial u^i}.$$

Using the 1st equation of (9) and 2nd equation of (1) it follows that

$$(38) \qquad \begin{aligned} \partial_t x_{,k} = &-\phi \sum_i h^\gamma_{ij}\lambda^j_k \cos\,\theta_\gamma e_i \\ &+ \sum_\gamma \left(\phi_{,k} \cos\,\theta_\gamma - \phi \sum_{\beta\neq\gamma} \ell^\beta_{\gamma i}\lambda^i_k \cos\,\theta_\beta\right) e_\gamma - \theta_{,k}\phi n^\perp(\theta). \end{aligned}$$

From (15) and (32) one can obtain $\partial_t\psi = 0$ and thus we have

$$(39) \qquad \partial_t W = -2\phi H(\theta) W.$$

The volume element of the variation $\bar{x}$ is $*\bar{1} = \bar{W} * 1$ and from (39) we obtain

$$(40) \qquad \partial_t * 1 = \frac{\partial}{\partial t}(\bar{W} * 1)|_{t=0} - 2\phi H(\theta) * 1.$$

3. The Normal Variation of the Quantities h_{ij}^{γ} and $\ell_{\gamma i}^{\beta}$

From the 2nd formula of (8) and (37) one can obtain the variation of the normal direction $n(\theta)$ as

(41)
$$\partial_t n(\theta) \equiv -\sum_i A_i e_i + (B^3 \cos\theta_3 - B^4 \tan\theta_3 \cos\theta_4) e_4, (\operatorname{mod} n(\theta)).$$

Using the 1st formula of (8) and (1), (4), (38), (39) and straightforward computations we get

$$\partial_t n(\theta) \equiv -\sum_i A_i e_i + \Phi e_4, \ (\operatorname{mod} n(\theta)) \tag{42}$$

where

$$\Phi = \frac{\sqrt{g}}{W \cos\theta_3} (\langle \nabla\phi, x\rangle - \phi).$$

Comparing (41) and (42), yields

$$\sqrt{g}(\langle \nabla\phi, x\rangle - \phi) = W(B^3 \cos^2\theta_3 - B^4 \cos^2\theta_4). \tag{43}$$

The variation of the quantities ℓ_{3i}^4 is given from (36) and (37) as

(44)
$$\partial_t \ell_{3i}^4 = -\sum_j h_{ji}^{\gamma}((-1)^{\gamma} A_j \, sin\,\theta_{\gamma} - \phi(k_j + \ell_{3j}^4)\cos\theta_{\gamma}) - (-1)^{\gamma}\sum_{\gamma} B_i^{\gamma}.$$

Using $\partial_t(\ell_{3i}^4 + \ell_{4i}^3) = 0$, one can see that

$$B^3 + B^4 = C, \qquad (C \text{ is constant})$$

and from (43) it follows

$$B^{\gamma} = \Phi + (-1)^{\gamma+1} \cos^2\theta_3. \tag{45}$$

Exterior differentiation of (29) and using (11), (7) we have

$$d^2\bar{x} = d^2 x + t\partial_t d^2 x,$$

where

$$\partial_t d^2 x \equiv \phi d^2 n + (-1)^\gamma \big(2d\phi(d\theta + \omega_3^4) + \phi(k_i + \ell_{3i}^4) d\omega^i\big) \cos\psi_\gamma e_\gamma, \ (\text{mod } e_i). \tag{46}$$

Taking the variation on both sides of (13), using (42), (46) and equating the coefficient of $\cos\theta_\gamma$ we have

$$\partial_t h_{ij}^\gamma = A_{ij}\cos\theta_\gamma - \phi\Big\{\sum_k \big(h_{ki}^\gamma h_{kj}^\gamma \cos\theta_\gamma + h_{ki}^3 h_{kj}^4 \cos\psi_\gamma\big) + (k_i + \ell_{3i}^4)(k_j + \ell_{3j}^4)\cos\theta_\gamma + (-1)^\gamma h_{ij}^\beta\Big\} + h_{ij}^4 \Phi \cos\theta_\gamma. \tag{47}$$

As revealed from the foregoing results we give

LEMMA 1. *The variations of the 2nd and the 3rd fundamental quantities are given by* (47) *and* (44) *respectively.*

The mean curvature function $\bar{H}(\theta, t)$ of the immersion $\bar{x}$ in the normal direction n is defined from

$$4\bar{H}^2(\theta,t) = \bar{g}^{ij}(t)\bar{g}^{\ell k}(t)\bar{h}_{ij}^\gamma(t)\bar{h}_{\ell k}^\beta \cos\theta_\gamma \cos\theta_\beta, \ \bar{H}(\theta,0) = H(\theta).$$

The variation on both sides and using (33), (15) and (17) yields

$$\partial_t H^2(\theta) = \sum_\gamma H^\gamma(\theta)\left\{2\phi S^2(\theta) + \sum_\ell \delta h_{\ell\ell}^\gamma \cos\theta_\gamma\right\}. \tag{48}$$

From (47) and (25) we have

$$\sum_i \partial_t h_{ii}^\gamma(\theta) = \Delta\phi - \phi\{S^2(\theta) + \sum_i (k_i + \ell_{3i}^4)^2 - 2(-1)^\gamma(\psi)\} + 2\Phi\, H^4 \tag{49}$$

where

$$H^\gamma(\psi) = H^\gamma \cos\psi_\gamma.$$

From (48)

$$\partial_t H^2(\theta) = \sum_\gamma H^\gamma(\theta)\{\Delta\phi - \phi\big(S^2(\theta) - \sum_i (k_i + \ell_{3i}^4)^2 + 2(-1)^\gamma H^\gamma(\psi)\big) + 2\Phi\, H^4\}. \tag{50}$$

Thus, we have the following

LEMMA 2. *The variation $\partial_t H^2(\theta)$ is an extension to that obtained in* [4] *and* [11].

4. Stability with respect to the integral $\int_M H^2(\theta) * 1$

The variation of the integral $\int_M H^2(\theta) * 1$ is given from (40) and (50) as

$$\begin{aligned}
&\partial_t \int_M H^2(\theta) * 1 \\
(51) \qquad &= \int_M \sum_\gamma \{(\Delta\phi) H^\gamma(\theta) + \phi H^\gamma(\theta)\big(S^2(\theta) - \sum_i (k_i + \ell_{3i}^4)^2 \\
&\quad + 2(-1)^\gamma H^\gamma(\psi) - 2H^2(\theta)\big) + 2\Phi\, H^4\} * 1.
\end{aligned}$$

Since the immersion x is compact, we use the Green theorem as the following.

$$\begin{aligned}
&\partial_t \int_M H^2(\theta) * 1 \\
(52) \qquad &= \int_M \sum_\gamma \{\phi(\Delta(H^\gamma(\theta)) + H^\gamma(\theta)\big(S^2(\theta) - \sum_i (k_i + \ell_{3i}^4)^2 \\
&\quad + 2(-1)^\gamma H^\gamma(\psi) - 2H^2(\theta)\big)\big) + 2\Phi\, H^4\} * 1.
\end{aligned}$$

From (21), (22), (23) and (27) one can see

$$(53) \qquad \left.\begin{aligned}
\Delta \cos\theta_\gamma &= \big((-1)^\gamma \Delta\theta - |\nabla\theta|^2 \cos\theta_\gamma \\
\Delta \cos\theta_\gamma &= (-1)^\gamma \nabla\theta \cos\psi_\gamma
\end{aligned}\right\}$$

and this leads to

$$\begin{aligned}
\Delta H^\gamma(\theta) =& H^\gamma\big((-1)^\gamma \Delta\theta - |\nabla\theta|^2\big) \cos\theta_\gamma + (\Delta H^\gamma) \cos\theta_\gamma \\
&+ 2\big((-1)^\gamma \langle \nabla\theta, \nabla H^\gamma\rangle \cos\psi_\gamma\big).
\end{aligned}$$

Therefore (52) takes the form

(54)

$$\begin{aligned}
\partial_t \int_M H^2(\theta) * 1 = \int_M &\{\phi\big(\big((-1)^\gamma \Delta\theta - |\nabla\theta|^2 H^\gamma(\theta) \\
&+ \Delta H^\gamma \cos\theta_\gamma + 2\big((-1)^\gamma \langle\nabla\theta, \nabla H^\gamma\rangle \cos\psi_\gamma\big) \\
&+ H^\gamma(\theta)\big(S^2(\theta) - \sum_i (k_i + \ell_{3i}^4)^2 + 2(-1)^\gamma H^\gamma(\psi) \\
&- 2H^2(\theta)\big)\big) + 2\Phi H^4\} * 1.
\end{aligned}$$

The immersion x is called stable with respect to $\int_M H^2(\theta) * 1$ if and only if (iff) the right-hand side of (54) is identically zero, that is [11]

$$S : \phi\Big\{ \sum_\gamma H^\gamma(\theta)\big((-1)^\gamma \Delta\theta - |\nabla\theta|^2\big) + \Delta H^\gamma \cos\theta_\gamma + 2(-1)^\gamma \langle \nabla\theta, \nabla H^\gamma \rangle \cos\psi_\gamma + \sum_\gamma H^\gamma(\theta)\big(S^2(\theta) - \sum_i (k_i + \ell^4_{3i})^2 + 2(-1)^\gamma H^\gamma(\psi) - 2H^2(\theta)\big)\Big\} + 2\Phi H^4 = 0.$$

The integral condition S is an extension to that obtain in [3], [4]. Thus we reach to the proof of the main theorem

THEOREM 1. *The oriented compact immersion $x : M \to E^4$ is stable with respect to the integral $\int_M H^2(\theta) * 1$ iff the condition S is valid.*

Now, consider the following cases:

1. Surface for which $\Phi = 0$.

Putting $\Phi = 0$ in (43) we have an equivalent condition $B^\gamma = (-1)^{\gamma+1}\cos^2\theta_3$, then the condition S takes the form

$$S_1 : \sum_\gamma H^\gamma(\theta)\{\big((-1)^\gamma\Delta\theta - |\nabla\theta|^2\big) + \big(S^2(\theta) - \sum_i (k_i + \ell^4_{3i})^2 + 2(-1)^\gamma H^\gamma(\psi) - -2H^2(\theta)\big)\} + (\Delta H^\gamma)\cos\theta_\gamma + 2(-1)^\gamma\langle\nabla\theta, \nabla H^\gamma\rangle \cos\psi_\gamma = 0.$$

So, we obtain

LEMMA 3. *The immersion $x : M \to E^4$ is stable iff the differential equation S_1 is satisfied.*

2. The frames are constructed as

$$G(\theta) = G^\gamma \cos^2\theta_\gamma.$$

From (16), (17) and (18) we have

$$S^2(\theta) = \sum_\gamma S^2_\gamma \cos^2\theta_\gamma + 2H^3H^4\cos\theta_3\cos\theta_4. \tag{55}$$

Substituting in S_1 we have

$$S_{11}:\sum_\gamma H^\gamma(\theta)\{((-1)^\gamma\Delta\theta-|\nabla\theta|^2)+\sum_\gamma S_\gamma^2\cos^2\theta_\gamma$$
$$+2H^3H^4\cos\theta_3\cos\theta_4-\sum_i(k_i+\ell_{3i}^4)^2+2\sum_\gamma(-1)^\gamma H^\gamma(\psi)$$
$$-2h^2(\theta)\}+2(-1)^\gamma\langle\nabla\theta,\nabla H^\gamma\rangle\cos\psi_\gamma=0.$$

3. The frames having the property

$$G(\theta)=G^\gamma\cos^2\theta_\gamma\ \text{ and } S^2(\theta)=\sum_\gamma S_\gamma^2\cos^2\theta_\gamma.$$

From (55) we have $H^3H^4=0$, thus the condition of stability for the surface, for which the functions $G(\theta)$ and $S(\theta)$ are positive definite, is divided into the following conditions

$$S_{113}:H^4(\theta)\{(\Delta\theta-|\nabla\theta|^2)+S_4^2\cos^2\theta_4-\sum_i(k_i+\ell_{3i}^4)^2+2H^4(\psi)$$
$$-2(H^4)^2\}+(\Delta H^4)\cos\theta_4+2\langle\nabla\theta,\nabla H^4\rangle\cos\theta_3=0$$

and

$$S_{114}:H^3(\theta)\{-(\Delta\theta+|\nabla\theta|^2)+S_3^2\cos^2\theta_3-\sum_i(k_i+\ell_{3i}^4)^2-2H^3(\psi)$$
$$-2(H^3)^2\}+(\Delta H^3)\cos\theta_3-2\langle\nabla\theta,\nabla H^3\rangle\cos\theta_4=0$$

according as $H^3=0$ and $H^4=0$ respectively.

Thus we have

LEMMA 4. *The types of stability S_{113}, S_{114} of the oriented compact surface $x: M\to E^4$ characterize the stability in the minimal sections $e_\gamma(H^\gamma=0,\gamma=3,4)$ respectively.*

The variation in the normal direction $n=e_3(\theta=0)$ is given from S_1 as

$$S_{10}:\Delta H^3+H^3\left(S_3^2-\sum_i(\ell_{3i}^4)^2-2(H^3)^2\right)=0.$$

This result is coincident with that obtained by Chen [2].

If the considered surface is a hypersurface that is $\theta = 0, h^4_{ij} = 0$, $\omega^4_3 = \omega^3_4 = 0$, the general stability condition S become

$$S_0 : \Delta H^3 + H^3 \left(S^2_3 - 2(H^3)^2\right) = 0$$

which was obtained in [1].

Therefore, we reach to the following.

LEMMA 5. *The classes of stability S_0 and S_{10} are a continuation to the result in* [1] *and* [2] *respectively.*

LEMMA 6. *The classes of stability of the immersion $x : M \to E^4$ are related by the inclusion $S_0 \subset S, S_{113}, S_{114} \subset S_{11} \subset S_1 \subset S$.*

From the foregoing results, we have the classification.

THEOREM 2. *The immersion $x : M \to E^4$ is stable with respect to the integral $\int_M H^2(\theta) * 1$ in six different cases, say, $S_1, S_{11}, S_{113}, S_{114}, S_{10}, S_0$.*

REMARK. The classification of the preceding theorem contains all the types which were obtained in [1], [2], [3] and [4].

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