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ON CONDITIONAL WEAK POSITIVE DEPENDENCE

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ABSTRACT. A random vector $\underline{X} = (X_1, \cdots, X_n)$ is conditionally weakly associated if and only if for every pair of partitions $\underline{X_1} = (X_{\pi(1)}, \cdots, X_{\pi(k)})$, $\underline{X_2} = (X_{\pi(k+1)}, \cdots, X_{\pi(n)})$ of $\underline{X}$ $P(\underline{X_1} \in A | \underline{X_2} \in B, \theta \in I) \geq P(\underline{X_1} \in A | \theta \in I)$ whenever A and B are open upper sets and π is any permutation of $\{1, \cdots, n\}$. In this note we develop some concepts of conditional positive dependence, which are weaker than conditional weak association but stronger than conditional positive orthant dependence, by requiring the above inequality to hold only for some upper sets and applying the arguments in Shaked (1982).

1. Introduction

An important principle of probability theory is that the notions of dependence and independence are conditional, the conditioning being done on some observable or unobservable quantity, say Θ. It is common to think of Θ as a parameter and this is the point of view that we adopt. Brady and Singpurwalla (1990) introduced some concepts of conditional dependence between random variables. Two random variables are not unconditionally dependent or independent but are probably dependent or independent, depending on the disposition of the conditioning variable: Let $\underline{X}$ and $\underline{Y}$ be two vector valued random variables, of dimension p and q respectively.

DEFINITION 1.1. (Brady et al., 1990). The random vector $\underline{X}$ is $\theta \in I_1$ conditionally independent of $\underline{Y}$, and is $\theta \in I_2(I_3)$ conditionally positively (negatively) dependent on $\underline{Y}$, denoted by $\{(\underline{X} \coprod \underline{Y}) | (\theta \in I_1, > \theta \in I_2, <$

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$\theta \in I_3)\}$, if

$$(a)\quad P\{X \in A|Y \in B, \theta \in I_1\} = P\{X \in A|\theta \in I_1\},$$
$$(b)\quad P\{X \in A|Y \in B, \theta \in I_2\} \geq P\{X \in A|\theta \in I_2\},$$
$$(c)\quad P\{X \in A|Y \in B, \theta \in I_3\} \leq P\{X \in A|\theta \in I_3\}, \forall A, B, \theta,$$

where A, B are open upper sets (U is an upper set if $\underline{a} \in U$, and $\underline{a} \leq \underline{b}$ implies $\underline{b} \in U$ (Shaked, 1982)).

Assume that $p = q = 1$. Then Definition 1.1 can be stated in terms of the joint and marginal distribution functions of X and Y. Let

$$H(x, y|\theta) = P(X \leq x, Y \leq y \mid \theta),$$
$$F(x|\theta) = P(X \leq x \mid \theta),$$
$$G(y|\theta) = P(Y \leq y \mid \theta).$$

DEFINITION 1.2. The random variable X is $\theta \in I_1$ conditionally independent of Y and is $\theta \in I_2(I_3)$ conditionally positively (negatively) quadrant dependent (CPQD) on Y, denoted by $\{(X \coprod Y)|\theta \in I_1, > \theta \in I_2, < \theta \in I_3)\}$, if

(1)
$$(a)\quad H(x, y|\theta \in I_1) = F(x|\theta \in I_1)G(y|\theta \in I_1),$$
$$(b)\quad H(x, y|\theta \in I_2) \geq F(x|\theta \in I_2)G(y|\theta \in I_2),$$
$$(c)\quad H(x, y|\theta \in I_3) \leq F(x|\theta \in I_3)G(y|\theta \in I_3).$$

As an example, suppose that X and Y have a bivariate normal distribution with mean $\underline{\mu} = (\mu_1, \mu_2)$ and covariance

$$\sum = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

where ρ is the coefficient of correlation. Then ρ is our conditioning variable and as is well known $\{(X \coprod Y)|\rho = 0\}$, and X and Y have positive (negative) dependence when $\rho > (<)0$. The survival function of Marshall and Olkin's (1967) bivariate exponential distribution is given as

$$\bar{F}(x, y) = e^{-\lambda_1 x - \lambda_2 y - \lambda \max(x,y)},$$

where we suppose λ_1 and λ_2 known but λ is unknown. Then $\{X \coprod Y)|\lambda = 0, > \lambda \in (0, \infty)\}$; i.e., X and Y are independent if $\lambda = 0$, and otherwise always positive dependent. By way of some motivation, we divide an n components system into two disjoint subsystems consisting of k and $n-k$ components, and have to consider conditional dependence between

them. In other words, we desire to obtain various notions of weak positive dependence by splitting a random vector into two subvectors and investigate the conditional positive dependence between them with the arguments similar to those in Shaked (1982).
The purpose of this note is to introduce some conditional positive dependence concepts which are weaker than conditional weak association but stronger than conditional positive orthant dependence and to derive some relations among them. The general propositions and definitions are given in Section 2. In Section 3 the illustrative special case are introduced. Based on the propositions some properties are proven in Section 4. Before concluding this section we introduce conditional weak association of the bivariate random variable introduced by Brady and Singpurwalla [2]: The random variables X and Y are conditionally $\theta \in I_2(I_3)$ weakly (negatively) associated if for any nondecreasing functions f, g

$$Cov(f(X), g(Y)|\theta \in I_2(I_3)) \geq (\leq) 0. \tag{2}$$

2. Propositions and Definitions

We start from extending the concepts of conditional quadrant dependence.

DEFINITION 2.1. A random vector $\underline{X} = (X_1, \cdots, X_n)$ is $\theta \in I_1$, conditionally independent (CI) and is $\theta \in I_2$, conditionally positively upper orthant dependent (CPUOD), if

$$(a) \quad P(X_1 > x_1, \cdots, X_n > x_n|\theta \in I_1) = \prod_{i=1}^{n} P(X_i > x_i|\theta \in I_1),$$

$$(b) \quad P(X_1 > x_1, \cdots, X_n > x_n|\theta \in I_2) \geq \prod_{i=1}^{n} P(X_i > x_i|\theta \in I_2).$$

A random vector $\underline{X}$ is $\theta \in I_2$, conditionally positively lower orthant dependent (CPLOD), if

$$(b)' \quad P(X_1 \leq x_1, \cdots, X_n \leq x_n|\theta \in I_2) \geq \prod_{i=1}^{n} P(X_i \leq x_i|\theta \in I_2).$$

A random vector $\underline{X}$ is $\theta \in I_2$ conditionally positively orthant dependent (CPOD) if $\underline{X}$ is CPUOD and CPLOD.

DEFINITION 2.2. A random vector $\underline{X} = (X_1, \cdots, X_n)$ is conditionally $\theta \in I_2$ weakly associated if and only if for every pair of partitions $\underline{X}_1 = (X_{\pi(1)}, \cdots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \cdots, X_{\pi(n)})$ of $\underline{X}$, and every pair of nondecreasing functions f on R^k and g on $R^{(n-k)}$

$$Cov(f(\underline{X}_1), g(\underline{X}_2)|\theta \in I_2) \geq 0 \tag{3}$$

whenever π is any permutation of $\{1, 2, \cdots, n\}$ and $1 \leq k \leq n-1$.

PROPOSITION 2.1. *A random vector $\underline{X} = (X_1, \cdots, X_n)$ is conditionally $\theta \in I_2$ weakly associated (WA) if for every pair of partitions $\underline{X_1} = (X_{\pi(1)}, \cdots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \cdots, X_{\pi(n)})$ of $\underline{X}$*

$$P(\underline{X}_1 \in A | X_2 \in B, \theta \in I_2) \geq P(\underline{X}_1 \in A | \theta \in I_2) \tag{4}$$

whenever A and B are open upper sets, $1 \leq k \leq n-1$, and π is any permutation of $\{1, 2, \cdots, n\}$.

Proof. We only show the converse. Let π denote any permutation of $\{1, 2, \cdots, n\}$, $\underline{X}_1 = (X_{\pi(1)}, \cdots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \cdots, X_{\pi(n)})$ be arbitrary partitions of $\underline{X}$ and f, g be arbitrary nondecreasing functions on R^k, R^{n-k} respectively. Then for every real numbers s and t, $A = \{f(\underline{X}_1) > s\}$ and $B = \{g(\underline{X}_2) > t\}$ are open upper sets and

$$\begin{aligned} & P(f(\underline{X}_1) > s | g(\underline{X}_2) > t,\ \theta \in I_2) \\ & = P(\underline{X}_1 \in A | \underline{X}_2 \in B,\ \theta \in I_2) \\ & \geq P(\underline{X}_1 \in A | \theta \in I_2) \\ & = P(f(\underline{X}_1) > s | \theta \in I_2). \end{aligned} \tag{5}$$

Define

$$X_f(s) = \begin{cases} 1, & \text{if } f(\underline{X}_1) > s, \\ 0, & \text{otherwise,} \end{cases}$$

$$X_g(t) = \begin{cases} 1, & \text{if } g(\underline{X}_2) > t, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$(6)\qquad \begin{aligned} & Cov(f(X_1), g(X_2)|\theta \in I_2) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Cov(X_f(s), X_g(t)|\theta \in I_2)dsdt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{E(f(\underline{X}_1)g(\underline{X}_2)|\theta \in I_2) \\ & -E(f(\underline{X}_1)|\theta \in I_2)E(g(\underline{X}_2)|\theta \in I_2)\}dsdt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{P(f(\underline{X}_1) > s, g(\underline{X}_2) > t|\theta \in I_2) \\ & -P(f(\underline{X}_1) > s|\theta \in I_2)P(g(\underline{X}_2) > t|\theta \in I_2)\}dsdt \geq 0. \end{aligned}$$

The nonnegativity of (6) follows from (5). So by (3), $\underline{X}$ is conditionally weakly associated.

A possible way of weakening the condition of conditional weak association is to require that (4) holds for all A and B which belong to subcollections of the collections of all upper sets in R^k and R^{n-k}, respectively. This will be the approach in this paper. Let $A^{(k)}$ be a collection of sets in R^k, $k = 1, \cdots, n-1$. Usually the sets in $A^{(k)}$ are upper sets in $R^{(k)}$. □

DEFINITION 2.3. A random vector $\underline{X} = (X_1, \cdots, X_n)$ is $\theta \in I_1$ conditionally independent relative to $A^{(k)}$ and $A^{(n-k)}$ denoted by CI $(A^{(k)}, A^{(n-k)}|\theta \in I_1)$ and is $\theta \in I_2$ conditionally weakly positive dependent relative to $A^{(k)}$ and $A^{(n-k)}$ denoted by CWPD$(A^{(k)}, A^{(n-k)}|\theta \in I_1)$ if for a pair of partitions $\underline{X}_1 = (X_{\pi(1)}, \cdots, X_{\pi(k)})$, $\underline{X}_2 = (X_{\pi(k+1)}, \cdots, X_{\pi(n)})$ of $\underline{X}$

$$(a)P(\underline{X}_1 \in A|\underline{X}_2 \in B,\ \theta \in I_1) = P(\underline{X}_1 \in A|\theta \in I_1)$$
$$(7)\qquad (b)P(\underline{X}_1 \in A|\underline{X}_2 \in B,\ \theta \in I_2) \geq P(\underline{X}_1 \in A|\theta \in I_2)$$

whenever $A \in A^{(k)}$ and $B \in A^{(n-k)}$ and π is any permutation of $\{1, 2, \cdots, n\}$ and a random vector $\underline{X}$ is $\theta \in I_1$ conditionally independent relative to $A^{(n)}$ denoted by CI$(A^{(n)}|\theta_1)$ and is $\theta \in I_2$ conditionally weakly positive dependent relative to $A^{(n)}$ denoted by CWPD$(A^{(n)}|\theta \in I_2)$ if (7) holds for every $k(k = 1, \cdots, n-1)$.

PROPOSITION 2.2. *If $A^{(k)} \subset \tilde{A}^{(k)}$ and $A^{(n-k)} \subset \tilde{A}^{(n-k)}$ then CWPD $(\tilde{A}^{(k)}, \tilde{A}^{(n-k)}|\theta \in I_2)$ implies CWPD$(A^{(k)}, A^{(n-k)}|\theta \in I_2)$ and if for every $k(k = 1, 2, \cdots, n-1)$, $A^{(k)} \subset \tilde{A}^{(k)}$ then CWPD$(\tilde{A}^{(n)}|\theta \in I_2)$ implies*

CWPD$(A^{(n)}|\theta \in I_2)$. *Put* $\bar{A}^{(k)} = \{\bar{A} : A \in A^{(k)}\}$ ($\bar{A}$ *denoted the complement of* A *in* R^k) *and* $-A^{(k)} = \{A : -A \in A^{(k)}\}$ *where* $-A$ *denoted* $\{\underline{x} : -\underline{x} \in A\}$.

PROPOSITION 2.3. *The random vector* $\underline{X}$ *is* *CWPD*$(A^{(k)}, A^{(n-k)}|\theta \in I_2)$ *if and only if* $\underline{X}$ *is* *CWPD*$(\bar{A}^{(k)}, \bar{A}^{(n-k)}|\theta \in I_2)$. *Moreover, the random vector* $\underline{X}$ *is* *CWPD*$(A^{(n)}|\theta \in I_2)$ *if and only if* $\underline{X}$ *is* *CWPD*$(\bar{A}^{(n)}|\theta \in I_2)$.

Proof. For fixed k, assume that $\underline{X}$ is CWPD$(A^{(k)}, A^{(n-k)}|\theta \in I_2)$. Let π be any permutation of $\{1, \cdots, n\}$ and $\underline{X_1} = (X_{\pi(1)}, \cdots, X_{\pi(k)})$, $\underline{X_2} = (X_{\pi(k+1)}, \cdots, X_{\pi(n)})$ be any pair of partitions of $\underline{X}$. Then we have for every $A \in A^{(k)}$ and $B \in A^{(n-k)}$

$$\begin{aligned} (8) \qquad & P(\underline{X_1} \in A, \underline{X_2} \in B|\theta \in I_2) \\ & \geq P(\underline{X_1} \in A|\theta \in I_2)P(\underline{X_2} \in B|\theta \in I_2) \\ & = (1 - P(\underline{X_1} \in \bar{A}|\theta \in I_2))(1 - P(\underline{X_2} \in \bar{B}|\theta \in I_2)) \\ & = 1 - P(\underline{X_1} \in \bar{A}|\theta \in I_2) - P(\underline{X_2} \in \bar{B}|\theta \in I_2) \\ & \quad + P(\underline{X_1} \in \bar{A}|\theta \in I_2)P(\underline{X_2} \in \bar{B}|\theta \in I_2). \end{aligned}$$

Since in general,

$$\begin{aligned} & P(\underline{X_1} \in A, \underline{X_2} \in B|\theta \in I_2) \\ & = 1 - P(\underline{X_1} \in \bar{A}|\theta \in I_2) - P(\underline{X_2} \in \bar{B}|\theta \in I_2) \\ & \quad + P(\underline{X_1} \in \bar{A}, \underline{X_2} \in \bar{B}|\theta \in I_2) \end{aligned}$$

it follows from (8) that

$$P(\underline{X_1} \in \bar{A}, \underline{X_2} \in \bar{B}|\theta \in I_2) \geq P(\underline{X_1} \in \bar{A}|\theta \in I_2)P(\underline{X_2} \in \bar{B}|\theta \in I_2)$$

which yields

$$P(\underline{X_1} \in \bar{A}|\underline{X_2} \in \bar{B}, \theta \in I_2) \geq P(\underline{X_1} \in \bar{A}|\theta \in I_2). \qquad \square$$

The converse can be proved in the same way as above. Next, assume that $\underline{X}$ is CWPD$(A^{(n)}|\theta \in I_2)$. Then (8) holds for every integer $k (1 \leq k \leq n-1)$ that is, if $\underline{X}$ is CWPD$(A^{(n)}|\theta \in I_2)$ then $\underline{X}$ is CWPD$(\bar{A}^{(n)}|\theta \in I_2)$.
The following result shows that, as can be expected, if $\underline{X}$ is conditionally weakly positive dependent (CWPD) then also $-\underline{X}$ is conditionally weakly positive dependent in an appropriate sense.

PROPOSITION 2.4. *The random vector $\underline{X}$ is CWPD($A^{(k)}, A^{(n-k)}|\theta \in I_2$) if and only if $-\underline{X}$ is CWPD($-A^{(k)}, -A^{(n-k)}|\theta \in I_2$) and $\underline{X}$ is CWPD ($A^{(n)}|\theta \in I_2$) if and only if $-\underline{X}$ is CWPD($-A^{(n)}|\theta \in I_2$).*

REMARK. From Propositions 2.3 and 2.4 it follows that if $\bar{A}^{(k)} = -A^{(k)}$ for every $k (1 \le k \le n)$ then $\underline{X}$ is CWPD($A^{(n)}|\theta \in I_2$) if and only if $-\underline{X}$ is CWPD($A^{(n)}|\theta \in I_2$).

3. Conditional Weak Positive Dependence

The following notions of $A_j^{(n)}, j = 1, 2, 3, 4, 5$, collections of upper sets in R^n, are already discussed by Shaked (See Section 3 of Shaked (1982):
(1) Let $A_1^{(n)}$ be the collection of all open upper orthants in $R^n, A \in A_1^{(n)}$, that is, if and only if

(9) $$A = \{\underline{x} : x_i > a_i, i = 1, \cdots, n\}$$

for some $a_i \in (-\infty\ \infty]$, $i = 1, \cdots, n$.
(2) Let $A_2^{(n)}$ be the collection of all open upper half spaces, that is, $A \in A_2^{(n)}$ if and only if

(10) $$A = \left\{\underline{x} : \sum_{i=1}^{n} a_i x_i > a_0\right\}$$

for some $a_0 \in (-\infty\ \infty]$ and $a_i \in [0,\ \infty)$, $i = 1, \cdots, n$.
(3) Let $A_3^{(n)}$ be the collection of all sets of the form

(11) $$A = \bigcap_{1 \le \beta \le \gamma} \bigcup_{\alpha \in C_\beta} \{\underline{x} : x_\alpha > a_\alpha\}$$

for some $a_i \in [-\infty\ \infty]$, $i = 1, \cdots, n$ or of the form

(12) $$A = \bigcup_{1 \le \beta \le \delta} \bigcap_{\alpha \in P_\beta} \{\underline{x} : x_\alpha > a_\alpha\}$$

for some $a_i \in [-\infty\ \infty]$, $i = 1, \cdots, n$, where for some positive integers γ and δ, $C_\beta \in \{1, \cdots, n\}, \beta = 1, \cdots, \gamma$, and $P_\beta \in \{1, \cdots, n\}, \beta = 1, \cdots, \delta$.
(4) Let $A_4^{(n)}$ be the collection of all convex open upper sets in R^n.
(5) Let $A_5^{(n)}$ be the collection of all open upper sets in R^n.

Note that from Proposition 2.1 and (5) $\underline{X} = (X_1, \cdots, X_n)$ is conditionally weakly associated (CWA) if and only if CWPOD($A_5^{(n)}$) and that from Definition 2.3 and (1) by induction we have

$$P(X_1 > x_1, \cdots, X_n > x_n | \theta \in I_2) \geq \prod_{i=1}^{n} P(X_i > x_i | \theta \in I_2).$$

Since conditional weak association represents a strong conditional positive dependence, some weaker concepts will be considered in this section. Many of these can be viewed as variations of the class from which upper sets A, B are chosen and (4) imposed.

THEOREM 3.1.

$$\begin{array}{ccc} & CWA = CWPD(A_5^{(n)} | \theta \in I_2) & \\ \swarrow & & \searrow \\ CWPD(A_4^{(n)} | \theta \in I_2) & & CWPD(A_3^{(n)} | \theta \in I_2) \\ \downarrow & \searrow & \downarrow \\ CWPD(A_2^{(n)} | \theta \in I_2) & & CWPD(A_1^{(n)}) \Rightarrow CPUOD \end{array}$$

REMARK. (1) From Proposition 2.1 it follows that a random vector $\underline{X}$ is CWA if and only if $\underline{X}$ is CWPD($A_5^{(n)}$).
(2) From Proposition 2.2 it follows that relationships of CWPD($A_j^{(n)}$) for $j = 1, \cdots, 5$ are obtained.
(3) Note that it will be seen later (Theorem 3.3) that CWPD($A_1^{(n)}$) implies CPUOD, thus, for $j = 1, 3, 4, 5$, and then CWPD($A_j^{(n)}$) and then the conditional positive upper orthant dependence.

Some of the results of Section 2 can be specialized now to the notions of this section as follows: Since for every positive integer k, $A_j^{(k)} = -A_j^{(k)} j = 2, 3, 5$ we obtain the following theorem from Remark in section 2.

THEOREM 3.2. *For $j = 2, 3, 5$, $\underline{X}$ is CWPD($A_j^{(n)}$) if and only if $-\underline{X}$ is CWPD($A_j^{(n)}$).*

THEOREM 3.3. (a) *If $\underline{X}$ is CWPD($A_j^{(n)}$) then $\underline{X}$ is CPUOD, $j = 1, 3, 4, 5$.*
(b) *If $\underline{X}$ is CWPD($-A_j^{(n)}$) then $\underline{X}$ is CPUOD, $j = 1, 3, 4, 5$.*

Proof. (a) By Theorem 3.1 it is enough to prove (a) for $j = 1$. Let $\underline{X} = (X_1, \cdots, X_n)$. If CWPD$(A_1^{(n)})$ then $\underline{X}$ is CWPD$(A_1^{(k)}, A_1^{(n-k)})$ for every $k(1 \leq k \leq n-1)$. When $k = 1$ take $\underline{X} = (X_1)$ and $\underline{X}_2 = (X_2, \cdots, X_n)$ as partition of $\underline{X}$ and take $\underline{a}_1 = (a_1)$ and $\underline{a}_2 = (a_2, \cdots, a_n)$ as a partition of $\underline{a} = (a_1, \cdots, a_n)$, respectively, then

$$\begin{aligned}(13)\qquad & P(X_1 > a_1, \cdots, X_n > a_n | \theta \in I_2) \\ &= P(\underline{X} > \underline{a} | \theta \in I_2) \\ &= P(\underline{X}_1 > \underline{a}_1, \underline{X}_2 \geq \underline{a}_2 | \theta \in I_2) \\ &= P(\underline{X}_1 > \underline{a}_1 | \underline{X}_2 > \underline{a}_2, \theta \in I_2) P(\underline{X}_2 \geq \underline{a}_2 | \theta \in I_2) \\ &\geq P(\underline{X}_1 > \underline{a}_1 | \theta \in I_2) P(\underline{X}_2 \geq \underline{a}_2 | \theta \in I_2) \\ &= P(X_1 > a_1 | \theta \in I_2) P(X_2 > a_2, \cdots, X_n > a_n | \theta \in I_2).\end{aligned}$$

When $k = 2$ take $\underline{X}_1 = (X_1, X_2)$ and $\underline{X}_2 = (X_3, \cdots, X_n)$ as a partition of $\underline{X}$ and take $\underline{a}_1 = (a_1, a_2)$ and $\underline{a}_2 = (a_3, \cdots, a_n)$ as partition of $\underline{a}$ then

$$\begin{aligned}(14)\qquad & P(X_1 > a_1, \cdots, X_n > a_n | \theta \in I_2) \\ &= P(\underline{X}_1 > \underline{a}_1, \underline{X}_2 > \underline{a}_2 | \theta \in I_2) \\ &\geq P(\underline{X}_1 > \underline{a}_1 | \theta \in I_2) P(\underline{X}_2 > \underline{a}_2 | \theta \in I_2) \\ &= P(X_1 > a_1, X_2 > a_2 | \theta \in I_2) P(X_3 > a_3, \cdots, X_n > a_n | \theta \in I_2).\end{aligned}$$

By choosing $a_1 = -\infty$ we obtain

$$\begin{aligned}(15)\qquad & P(X_2 > a_2, \cdots, X_n > a_n | \theta \in I_2) \\ &= P(X_2 > a_2, \underline{X}_2 > \underline{a}_2 | \theta \in I_2) \\ &\geq P(X_2 > a_2 | \theta \in I_2) P(\underline{X}_2 > \underline{a}_2 | \theta \in I_2) \\ &= P(X_2 > a_2 \mid \theta \in I_2) P(X_3 > a_3, \cdots, X_n > a_n | \theta \in I_2).\end{aligned}$$

By induction we also obtain

$$\begin{aligned}(16)\qquad & P(X_{n-1} > a_{n-1}, X_n > a_n | \theta \in I_2) \\ &\geq P(X_{n-1} > a_{n-1} | \theta \in I_2) P(X_n > a_n | \theta \in I_2).\end{aligned}$$

Finally, it follow from (13), (15) and (16) that

$$P(X_1 > a_1, \cdots, X_n > a_n | \theta \in I_2) \geq \prod_{i=1}^{n} P(X_i > a_i | \theta \in I_2).$$

(b) Since for every $k, -A_j^{(k)} \supset -A_1^{(k)}$ $j = 3, 4, 5$, it is enough to prove (b) for $j = 1$, $\underline{X}$ is CWPD$(-A_1^{(n)}) \Rightarrow -\underline{X}$ is CWPD$(A_1^{(n)}) \Rightarrow -\underline{X}$ is CPUOD. Thus $\underline{X}$ is CPUOD.

THEOREM 3.4. *For $j = 1, \cdots, 5$, if $\underline{X}$ is CWPD($A_j^{(n)}$) then ($X_{\alpha_1}, \cdots, X_{\alpha_m}$) is CWPD($A_j^{(m)}$) whenever $\{\alpha_1, \cdots, \alpha_m\} \subset \{1, \cdots, n\}$.*

Proof. Clearly if $\underline{X}$ is CPUOD then $(X_{\alpha_1}, \cdots, X_{\alpha_m})$ is CPUOD whenever $\{\alpha_1, \cdots, \alpha_m\} \subset \{1, \cdots, n\}$. Thus from Theorem 3.1 for $j = 1, 3, 4, 5$, the result follows.
It remains to prove the result for $j = 2$. Since for $m(\leq n)$ $A_2^{(m)} \subset A_2^{(n)}$ if $\underline{X}$ is CWPD($A_2^{(n)}$) then $(X_{\alpha_1}, \cdots, X_{\alpha_m})$ is CWPD($A_2^{(m)}$) and the proof is complete. □

4. Conditional Functional Weak Positive Dependence

Let $F^{(k)}$ be a family of real k-variate functions, $k = 1, 2, \cdots, n-1$.

DEFINITION 4.1. We will say that $\underline{X} = (X_1, \cdots, X_n)$ is conditionally functionally weakly positive dependent relative to $F^{(k)}$ and $F^{(n-k)}$ (denoted by $CFWPD(F^{(k)}, F^{(n-k)})$ if for any partitions $\underline{X}_1 = (X_{\pi(1)}, \cdots X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \cdots X_{\pi(n)})$ of $\underline{X} = (X_1, \cdots X_n)$

$$Cov(f(\underline{X}_1),\ g(\underline{X}_2) \mid \theta \in I_2) \ \geq 0 \tag{17}$$

whenever $f \in F^{(k)}$, $g \in F^{(n-k)}$ and π is a permutation of $\{1, \cdots, n\}$, provided the expectations exist and that $\underline{X}$ is conditionally functionally weakly positive dependent relative to $F^{(n)}$ (denoted by $CFWPD(F^{(n)})$) if $\underline{X}$ satisfies for every $k(1 \leq k \leq n-1)$.

REMARK. Shaked (1982) introduced that the random vector $\underline{X} = (X_1, \cdots, X_n)$ is functionally positive dependent relative to $F^{(n)}$ (denoted by $FPD(F^{(n)})$) if for any $f,\ g \in F^{(n)}$

$$Cov(f(\underline{X}),\ g(\underline{X})) \ \geq 0. \tag{18}$$

The following concepts of conditional functional weak positive dependence are obtained by the arguments similar to those of Shaked (1982).

PROPOSITION 4.1. *If $F^{(k)} \subset \tilde{F}^{(k)}$ then $CFWPD(F^{(k)}, F^{(n-k)})$ implies CFWPD ($\tilde{F}^{(n)}$, $\tilde{F}^{(n-k)}$) and if for every $k(1 \leq k \leq n)$ $F^{(k)} \subset \tilde{F}^{(k)}$ then $CFWPD(F^{(n)})$ implies $CFWPD(\tilde{F}^{(n)})$.*

The notion of $F_j^{(n)}$, $j = 1, 2, 3, 4, 5$, the collections of functions, is already discussed in Section 4 of Shaked (1982). From Proposition 4.1 and Definition 4.1 we obtain the following theorem.

THEOREM 4.1.

$$\begin{array}{ccc} CFWPD(F_5^{(n)}) = CWPD(A_5^{(n)}) & & \\ \downarrow \qquad \searrow & & \\ CFWPD(F_4^{(n)}) \quad CFWPD(F_3^{(n)}) & & \\ \downarrow \qquad \searrow \quad \downarrow & & \\ CFWPD(F_2^{(n)}) \quad CFWPD(F_1^{(n)}) & & \end{array}$$

We are going to show now that for $j = 1, \cdots, 5$, the notion of $CWPD(A_j^{(n)})$ essentially implies the notion of $CFWPD(F_j^{(n)})$. First the following lemma which characterizes $CWPD(A_j^{(n)})$ is proven.

LEMMA 4.1. *For $j = 1, \cdots, 5$, is $CWPD(A_j^{(n)})$ if and only if for every partitions $\underline{X}_1 = (X_{\pi(1)}, \cdots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \cdots, X_{\pi(n)})$ of $\underline{X}$, and every $k(1 \le k \le n-1)$,*

(19) $\quad (f(\underline{X}_1),\ g(\underline{X}_2))$ *is CPQD whenever* $f \in F_j^{(k)}, g \in F_j^{(n-k)}$

provided $\underline{X}$ is nonnegative. If $\underline{X}$ is not nonnegative then the above equivalence is true for $j = 2, 4, 5$.

Proof. Note that we will apply the arguments in the proof of Lemma 4.1 in Shaked (1982). When $j = 1$ and $P(\underline{X} \ge \underline{0}) = 1$ then the $\underline{X}$'s in $A \in A_1^{(n)}$ are nonnegative. Since for every $k(1 \le k \le n-1)$ $A \in A_1^{(n)}$, $B \in A_1^{(n-k)}$ if and only if $A = \{\underline{x}_1 : \min_{1 \le i \le k} b_i x_{\pi(i)} > s\}$ for some $s \in [-\infty, \infty]$ and $b_i \ge 0, i = 1, \cdots, k$, and $B = \{\underline{x}_2 : \min_{k+1 \le i \le n} c_i x_{\pi(i)} > t\}$ for some $t \in [-\infty, \infty]$ and $c_i \ge 0, i = k+1, \cdots, n$ then the result follows from the definition of $CWPD(A_1^{(n)})$. When $j = 2$ then the result follows directly from the definition of $CWPD(A_2^{(n)})$; see (3.2) of Shaked (1982). when $j = 3$ and $P(\underline{X} \ge 0) = 1$ then, to construct sets in $A_3^{(k)}$ and $A_3^{(n-k)}$, we can consider (3.3) or (3.4) in Shaked (1982) only the sets for every $k(1 \le k \le n-1)$

(20) $\{\underline{x}_1 : b_i x_{\pi(i)} > 1\}$ for some $i \in \{1, \cdots, n\}$ and $b_i \in [0, s_{\pi(i)}]$

and

(21) $\{\underline{x}_2 : c_i x_{(\pi(i)} > 1\}$ for some $i \in \{k+1, \cdots, n\}$ and $c_i \in [0, \infty]$.

By taking unions and intersections of the from (4.4) and (4.5), respectively we obtain sets of the form

(22) $\qquad \{\underline{x}_1 : f(\underline{x}_1) > 1\}$ for some f

$$(23) \qquad \{\underline{x}_2 : g(\underline{x}_2) > 1\} \text{ for some } g$$

of the form (4.1) or (4.2) in Shaked (1982) that is, for every $k(1 \le k \le n-1)$ $A \in A_3^{(k)}$, $B \in A_3^{(n-k)}$ if and only if A is of the form (22) and B is of the form (23). Using the homogeneity and nonnegativity of (22) and (23) for every $k(1 \le k \le n-1)$ $A \in A_3^{(k)}$ and $B \in A_3^{(n-k)}$ if and only if $A = \{\underline{x}_1 : f(\underline{x}_1) > b\}$ for some $f \in F_3^{(k)}$ and some $b \in [0, \infty]$ and $B = \{\underline{x}_2 : g(\underline{x}_2) > c\}$ for some $g \in F_3^{(n-k)}$ and $c \in [0, \infty]$. The result then follows from definition of $CWPD(A_3^{(n)})$. When $j = 5$ we deal with conditionally weakly associated random variables. Since (19) and (3) are equivalent, this yields the result. Finally let $j = 4$. First assume $\underline{X}$ satisfies (19). Let π be any permutation of $\{1, 2, \cdots, n\}$, $\underline{X}_1 = (X_{\pi(1)}, \cdots, X_{\pi(k)})$ and $\underline{X}_2 = (X_{\pi(k+1)}, \cdots, X_{\pi(n)})$ be arbitrary partitions of $\underline{X} = (X_1, \cdots, X_n)$ and A and B be in $A_4^{(k)}$, $A_4^{(n-k)}$ respectively. Since A and B are open, convex, and upper sets, they can be approximated by intersections of sets the form

$$\left\{\underline{x}_1 : \sum_{i=1}^{k} a_i x_{\pi(i)}\right\}, \quad \text{where } a_i \ge 0,\ i = 1, \cdots, k$$

and

$$\left\{\underline{x}_2 : \sum_{i=1}^{k} b_i x_{\pi(i)}\right\}, \quad \text{where } b_i \ge 0,\ i = k+1, \cdots, n$$

respectively. Explicitly, for every $\varepsilon > 0$ there exist K_1 and K_2 such that

$$\left| P(\underline{X}_1 \in A | \theta \in I_2) - P\left(\min_{1 \le l \le K_1} \sum_{i=1}^{k} a_i^{(l)} X_{\pi(i)} > 1 | \theta \in I_2 \right) \right| < \varepsilon$$

where $a_i^{(l)} \ge 0$, $i = 1, \cdots, k$, $l = 1, \cdots K_1$ and

$$\left| P(\underline{X}_2 \in B | \theta \in I_2) - P\left(\min_{1 \le l \le K_2} \sum_{i=k+1}^{n} b_i^{(l)} X_{\pi(i)} > 1 | \theta \in I_2 \right) \right| < \varepsilon$$

where $b_i^{(l)} \ge 0$, $i = k+1, \cdots, n$, $l = 1, \cdots K_2$. Denoting $f_{K_1}(\underline{x}_1) = \min_{1 \le l \le K_1} \sum_{i=1}^{k} a_i^{(l)} x_{\pi(i)}$ and $g_{K_2}(\underline{x}_2) = \min_{1 \le l \le K_2} \sum_{i=k+1}^{n} b_i^{(l)} x_{\pi(i)}$ we can also assume that

$$|P(\underline{X}_1 \in A, \underline{X}_2 \in B | \theta \in I_2) - P(f_{K_1}(\underline{X}_1) > 1, g_{K_2}(\underline{X}_2) > 1 | \theta \in I_2)| < \varepsilon.$$

Since for $k(1 \leq k \leq n-1)$ $f_{K_1} \in F_4^{(k)}$ and $g_{K_2} \in F_4^{(n-k)}$, thus, it follows from (19) that

$$
\begin{aligned}
&P(\underline{X}_1 \in A, \underline{X}_2 \in B|\theta \in I_2) - \varepsilon \\
&\leq P(f_{K_1}(\underline{X}_1) > 1, g_{K_2}(\underline{X}_2) > 1|\theta \in I_2) \\
&\leq P(f_{K_1}(\underline{X}_1) > 1|\theta \in I_2)P(g_{K_2}(\underline{X}_2) > 1|\theta \in I_2) \\
&\leq [P(\underline{X} \in A|\theta \in I_2) + \varepsilon][P(\underline{X}_2) \in B|\theta \in I_2) + \varepsilon].
\end{aligned}
$$

Letting $\varepsilon \longrightarrow 0$ we obtain

$$P(\underline{X}_1 \in A, \underline{X}_2 \in B|\theta \in I_2) \leq P(\underline{X}_1 \in A|\theta \in I_2)P(\underline{X}_2|\theta \in I_2),$$

that is, $\underline{X}$ is $CWPD(A^{(n)})$. To show the converse assume that $\underline{X}$ is $CWPD(A_4^{(n)})$. Let $f \in F_4^{(k)}$ and $g \in F_4^{(n-k)}$ for every $k(1 \leq k \leq n-1)$. Then for every a and b the sets $A = \{\underline{x}_1 : f(\underline{x}_1) > a\}$ and $B = \{\underline{x}_2 : g(\underline{x}_2) > b\}$ are in $A_4(k), A_4^{(n-k)}$ respectively. Thus, since $\underline{X}$ is $CWPD(A_4^{(n)})$,

$$
\begin{aligned}
&P(f(\underline{X}_1) > a, g(\underline{X}_2) > b|\theta \in I_2) \\
&= P(\underline{X}_1 \in A, \underline{X}_2 \in B|\theta \in I_2) \\
&\geq P(\underline{X}_1 \in A|\theta \in I_2)P(\underline{X}_2|\theta \in I_2) \\
&= P(f(\underline{X}_1) > a|\theta \in I_2)P(g(\underline{X}_2) > b|\theta \in I_2),
\end{aligned}
$$

that is, (19) holds. □

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