J. Korean Math. Soc. **36** (1999), No. 3, pp. 581–592

INCOMPLETENESS OF SPACE-TIME SUBMANIFOLD

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ABSTRACT. Let M be a properly immersed timelike hypersurface of $\overline{M}$. If M is a diagonal type, M satisfies the generic condition under the certain conditions of the eigenvalues of the shape operator. Moreover, applying them to Raychaudhuri equation, we can show that M satisfies the generic condition. Thus, by these results, we establish the singularity theorem for M in $\overline{M}$.

1. Preliminaries

Let $\overline{M}$ be an $(n+1)$-dimensional Lorentzian manifold with the signature $(-,+,\cdots,+)$, and M be a properly immersed hypersurface of $\overline{M}$ which means that the pull back metric tensor is Lorentzian. The immersion is actually assumed as an isometric immersion. We shall identify local vector fields on M with local vector fields on $\overline{M}$ by the suitable extension process, and use the same notations for them. Thus, we may use the metric tensor $\langle\ ,\ \rangle$ on M and $\overline{M}$ as the same notation.

In this paper, we assumed that the M has a non-vanishing second fundamental form S_N and a continuous spacelike unit normal vector field N on M. If X and Y are the local vector fields on M and their local extensions to $\overline{M}$, the connection ∇ on M is defined by

$$\nabla_X Y = (\overline{\nabla}_X Y)^T,$$

where $\overline{\nabla}$ is the connection defined by on $\overline{M}$ and T indicates the projection to the tangent part of M. We know that this connection ∇ on M is a well defined connection relative to the tensor induced on M via the pull back metric tensor.

Received November 30, 1998. Revised April 13, 1999.

1991 Mathematics Subject Classification: 53C50, 83C75.

Key words and phrases: timelike hypersurface, diagonal point, generic condition, singularity, Raychaudhuri equation, Lagrange tensor, expansion, timelike convergence condition, strong energy condition.

Research partially supported by Yeungnam University Research fund, 1997.

The second fundamental form B_N is represented as

$$B_N(X,Y) = \langle \nabla_X Y, N \rangle$$

and the shape operator S_N is defined by

$$B_N(X,Y) = \langle S_N(X), Y \rangle \; for \; X, Y \in T_pM,$$

where T_pM is a tangent space at p in M. This satisfies $S_N(X) = -(\overline{\nabla}_X N)^T$ as well as $\langle S_N(X), Y \rangle = \langle X, S_N(Y) \rangle$, that is, S_N is self-adjoint linear operator on T_pM.

It is well-known property that, in the case of Riemann manifolds, there exists an orthonormal basis of T_pM at each point p in M such that this basis consists of eigenvectors for the shape operator S_N. However, in the case of Lorentzian manifolds, it is, in general, not true [2]. That is, eigenvectors of S_N need not span T_pM [10]. A point in M at which the (real) eigenvectors of S_N span T_pM will be called a diagonal point. At such a point, there is an orthonormal basis $e_1, e_2, \ldots, e_n$ of T_pM with e_1 a unit future-directed timelike vector and $S_N(e_i) = k_i e_i$ for each $i = 1, \ldots, n$. Thus, $e_2, e_3, \ldots, e_n$ are spacelike [2]. We will also assume that the sets of the form $M \cap \{p \in \overline{M} |$ the first component of p on a neighborhood of p is constant $\}$ are Cauchy surfaces for M. We will also consider that the connections are torsion free, and, by these connections, we have the relations between the curvatures R and $\overline{R}$ of M and $\overline{M}$ respectively in terms of the shape operator as follows:

$$R(X,Y)Z = \overline{R}(X,Y)Z + [\langle S_N(Y), Z \rangle S_N X - \langle S_N(X), Z \rangle S_N Y]$$

for the tangent vector fields X, Y, Z on M and their local extensions to $\overline{M}$ [11].

2. The main results

Let K and $\overline{K}$ be the sectional curvature on M and $\overline{M}$ respectively, σ the plane section of T_pM with the basis X, Y in T_pM, and

$$Q(\sigma) = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2.$$

Then, using the self-adjoint shape operator S_N,

$$\begin{aligned}
&K(\sigma)\\
&=\frac{1}{Q(\sigma)}\langle R(X,Y)Y,X\rangle\\
&=\frac{1}{Q(\sigma)}\{\langle\overline{R}(X,Y)Y+[\langle S_N(Y),Y\rangle S_N(X)-\langle S_N(X),Y\rangle S_N(Y)],X\rangle\}\\
&=\overline{K}(\sigma)+\frac{1}{Q(\sigma)}\{\langle S_N(Y),Y\rangle\langle S_N(X),X\rangle-\langle S_N(X),Y\rangle^2\}.
\end{aligned}$$

If $\{e_i\}$ is an orthonormal vectors consisting of eigenvectors for S_N in T_pM and $\{k_i\}$ is the corresponding eigenvalues, then letting e_1 be a future-directed timelike vector,

$$\begin{aligned}
K(\sigma_{ij})&=\overline{K}(\sigma_{ij})+\frac{1}{Q(\sigma_{ij})}\{\langle k_ie_i,e_i\rangle\langle k_je_j,e_j\rangle\}\\
&=\overline{K}(\sigma_{ij})+k_ik_j,
\end{aligned}$$

where σ_{ij} is a plane section with the basis e_i and e_j with $i\neq j$.

Thus we have the following Lemma.

LEMMA 1. *The difference of $K(\sigma_{ij})$ and $\overline{K}(\sigma_{ij})$ at p in M is positive if and only if the eigenvalues of S_N are all negative or all positive.*

At a diagonal point p, take an orthonormal basis $\{e_i\}$ of T_pM consisting of eigenvectors for S_N with a future-directed timelike unit vector e_1, and let $S_N(e_i)=k_ie_i$. Thus for X in T_pM, letting $X=\sum x_ie_i$,

$$\begin{aligned}
B_N(X,X)=\langle S_N(X),X\rangle&=\left\langle S_N\left(\sum_{i=1}^n x_ie_i\right),\sum_{j=1}^n x_je_j\right\rangle\\
&=\sum_{i=1}^n x_i\left\langle S_N(e_i),\sum_{j=1}^n x_je_j\right\rangle\\
&=\sum_{i=1}^n x_i\left\langle k_ie_i,\sum_{j=1}^n x_je_j\right\rangle\\
&=\sum_{i=1}^n k_ix_i{}^2\langle e_i,e_i\rangle=-k_1x_1^2+\sum_{i=2}^n k_ix_i^2.
\end{aligned}$$

Thus we have the following Lemma.

LEMMA 2. *If $B_N(X,X)$ is positive for all nonzero vector X in T_pM at a diagonal point p if and only if k_1 is negative and k_i's $(i \geq 2)$ are positive. This implies that $K(\sigma_p) > \overline{K}(\sigma_p)$.*

Let X be a vector at p in M and X^a, X_b contravariant and covariant components respectively. In general, the generic condition for X is defined by

$$\sum_{i,j=1}^{n} X^i X_j X_{[a} R_{b]ij[c} X_{d]} \neq 0.$$

Let γ be a unit speed timelike geodesic curve and $\gamma'(t_0) = X$. Let $V^{\perp}(\gamma(t_0))$ be the set of vectors orthogonal to X, that is,

$$V^{\perp}(\gamma(t_0)) = \{Y \in T_{\gamma(t_0)}M \,|\, \langle Y, X\rangle = 0\}.$$

Then, we know that the curvature tensor R induces a linear map from $V^{\perp}(\gamma(t_0))$ to itself:

$$R(\ ,X)X : V^{\perp}(\gamma(t_0)) \to V^{\perp}(\gamma(t_0)).$$

The generic condition can be interpreted by this map. A timelike curve γ satisfies the generic condition means that there exists a point t_0 in the domain of γ such that $\gamma'(t_0)$ satisfies the generic condition, which is equivalent to the nontriviality of the linear map $R(\ ,\gamma'(t_0))\gamma'(t_0)$ [1].

Now, we will consider the generic condition on a timelike hypersurface M of $\overline{M}$. Let γ be a unit speed timelike curve on M, and suppose that γ has a point t_0 in a domain of γ such that $\gamma'(t_0) = e_1$ is an eigenvector for S_N and the corresponding eigenvalue is k_1. For any vector X_0 in $V^{\perp}(\gamma(t_0))$, the following holds:

$$\begin{aligned} &R(X_0, e_1)e_1 \\ &= \overline{R}(X_0, e_1)e_1 + \{\langle S_N(e_1), e_1\rangle S_N(X_0) - \langle S_N(X_0), e_1\rangle S_N(e_1)\} \\ &= \overline{R}(X_0, e_1)e_1 + \{\langle k_1 e_1, e_1\rangle S_N(X_0) - \langle X_0, k_1 e_1\rangle k_1 e_1\} \\ &= \overline{R}(X_0, e_1)e_1 - k_1 S_N(X_0). \end{aligned}$$

Thus, we have the following result.

THEOREM 3. *If a unit timelike curve γ on M has a point t_0 in the domain of γ such that $\gamma'(t_0)$ is an eigenvector for S_N, and there exists a non-zero vector X_0 in $V^{\perp}(\gamma(t_0))$ such that the vector $S_N(X_0)$ and the vector $\overline{R}(X_0, \gamma'(t_0))\gamma'(t_0)$ are independent, then γ satisfies the generic condition.*

We recall that, if $\overline{M}$ has the constant curvature C, then $\overline{R}(X,Y)Z = C\{\langle X,Z\rangle X - \langle X,Z\rangle Y\}$. Suppose that a point p in M is a diagonal point. Let us compute the Ricci curvature $Ric(X,X)$. If X is a timelike vector in T_pM, $X = \sum_{i=1}^{n} x_i e_i$ for the orthonormal basis $\{e_i\}$ as before, $\langle e_i, e_i\rangle = \epsilon_i$ and k_i's are the eigenvalues of S_N corresponding to the eigenvector e_i, then

$$\begin{aligned}
Ric(X,X) &= \sum_{i=1}^{n}\langle e_i, e_i\rangle\langle R(e_i,X)X, e_i\rangle \\
&= -\sum_{i=1}^{n}\langle e_i, e_i\rangle\langle R(e_i,X)e_i, X\rangle \\
&= -\sum_{i=1}^{n}\epsilon_i\langle \overline{R}(e_i,X)e_i + \langle S_N(X), e_i\rangle S_N(e_i) \\
&\qquad\qquad - \langle S_N(e_i), e_i\rangle S_N(X), X\rangle \\
&= -\sum_{i=1}^{n}\epsilon_i\langle C(\langle X, e_i\rangle e_i - \langle e_i, e_i\rangle X) + \langle S_N(X), e_i\rangle S_N(e_i) \\
&\qquad\qquad - \langle S_N(e_i), e_i\rangle S_N(X), X\rangle \\
&= -\sum_{i=1}^{n}\epsilon_i\{C(\langle X, e_i\rangle^2 - \epsilon_i\langle X, X\rangle) + \langle X, k_i e_i\rangle^2 \\
&\qquad\qquad - \langle k_i e_i, e_i\rangle\langle S_N(X), X\rangle\} \\
&= C(n-1)\langle X, X\rangle + \sum_{i=1}^{n} k_i\langle S_N(X), X\rangle - \langle S_N(X), S_N(X)\rangle.
\end{aligned}$$

Suppose that $\max\{|k_2|, |k_3|, \ldots, |k_n|\} = |k_1|$. Let $X = \sum_{i=1}^{n} x_i e_i$ in

T_pM. Since $\sum_{i=1}^{n} k_i = H$, letting $H = 0$,

$$\begin{aligned} Ric(X,X) &= C(n-1)\langle X,X\rangle + H\langle S_N(X),X\rangle - (-k_1^2x_1^2 + \cdots + k_n^2x_n^2) \\ &\geq C(n-1)\langle X,X\rangle - k_1^2\langle X,X\rangle \end{aligned}$$

Thus, we have the following result by Proposition 2.8 [1].

THEOREM 4. *If p is a diagonal point in M, $H = 0$, and $\overline{M}$ has constant curvature C which is less than or equal to $\frac{k_1^2}{n-1}$, then $Ric(X,X) \geq 0$ for the timelike vector X in T_pM. Moreover, if the curvature is less than $\frac{k_1^2}{n-1}$, then X satisfies the generic condition.*

Now, if p is a diagonal point in M, we have the Ricci curvature for a timelike vector X in T_pM as follows:

$$\begin{aligned} &Ric(X,X) \\ &= C(n-1)\langle X,X\rangle + \sum_{i=1}^{n} k_i\langle S_N(X),X\rangle - \langle S_N(X),S_N(X)\rangle \\ &= C(n-1)\langle X,X\rangle - k_1(k_2+\cdots+k_n)x_1^2 + k_2(k_1+k_3+\cdots+k_n)x_2^2 \\ &\qquad + \cdots + k_n(k_1+k_2+\cdots+k_{n-1})x_n^2 \\ &= \{-C(n-1) - k_1(k_2+\cdots+k_n)\}x_1^2 \\ &\qquad + \{C(n-1) + k_2(k_1+k_3+\cdots+k_n)\}x_2^2 \\ &\qquad + \cdots + \{C(n-1) + k_n(k_1+\cdots+k_{n-1})\}x_n^2. \end{aligned}$$

If $\min_{2\leq i\leq n}\{k_i(k_1+k_2+\cdots+k_{i-1}+k_{i+1}+\cdots+k_n)\}$
$= k_1(k_2+\cdots+k_n) = a$,
then

$$\begin{aligned} Ric(X,X) &\geq \{C(n-1)+a\}(-x_1^2+x_2^2+\cdots+x_n^2) \\ &= \{C(n-1)+a\},\langle X,X\rangle. \end{aligned}$$

Thus, we have the following result by Proposition 2.8 [1].

THEOREM 5. *Let p be a diagonal point in M and $\overline{M}$ has constant curvature C. If $C \leq -\frac{a}{n-1}$, then $Ric(X,X) \geq 0$ for a timelike vector X in T_pM. Moreover, if $C < -\frac{a}{n-1}$, then X satisfies the generic condition.*

In the case that $\overline{M}$ has arbitrary curvature and p is a diagonal point in M, the Ricci curvature of M for a timelike vector X in T_pM can be represented as follows; for $X = \sum_{i=1}^{n} x_i e_i$,

$$\begin{aligned}
&Ric(X,X)\\
&= \overline{Ric}(X,X) - \sum_{i=1}^{n} \epsilon_i \langle X, S_N(e_i)\rangle^2\\
&\qquad + \sum_{i=1}^{n} \epsilon_i \langle S_N(e_i), e_i\rangle \langle S_N(X), X\rangle\\
&= \overline{Ric}(X,X) - \sum_{i=1}^{n} \epsilon_i \langle X, k_i e_i\rangle^2 + \sum_{i=1}^{n} \epsilon_i \langle k_i e_i, e_i\rangle \langle S_N(X), X\rangle\\
&= \overline{Ric}(X,X) - \sum_{i=1}^{n} \epsilon_i k_i^2 \langle X, e_i\rangle^2 + \sum_{i=1}^{n} \epsilon_i^2 k_i \langle S_N(X), X\rangle\\
&= \overline{Ric}(X,X) - \sum_{i=1}^{n} \epsilon_i k_i^2 \langle X, e_i\rangle^2 + \sum_{i=1}^{n} k_i \langle S_N(X), X\rangle\\
&= \overline{Ric}(X,X) - (-k_1^2 x_1^2 + k_2^2 x_2^2 + \cdots + k_n^2 x_n^2) + \sum_{i=1}^{n} k_i \langle S_N(X), X\rangle\\
&= \overline{Ric}(X,X) - k_1(k_2 + k_3 + \cdots + k_n) x_1^2 + k_2(k_1 + k_3 + \cdots + k_n) x_2^2\\
&\qquad + k_3(k_1 + k_2 + k_4 + \cdots + k_n) x_3^2 + \cdots\\
&\qquad + k_n(k_1 + k_2 + \cdots + k_{n-1}) x_n^2.
\end{aligned}$$

If $K(\sigma_{ij}) - \overline{K}(\sigma_{ij}) > 0$, then, by Lemma 1, we have

$$Ric(X,X) \geq \overline{R}(X,X) - k_1(k_2 + k_3 + \cdots + k_n) x_1^2.$$

Let k_0 be the maximum of $\{k_1k_2, k_1k_3, \dots, k_1k_n\}$. This implies

$$\begin{aligned} Ric(X,X) &\geq \overline{R}(X,X) - k_0 x_1^2 \\ &= \overline{R}(X,X) - k_0\langle X, e_1\rangle^2. \end{aligned}$$

Thus, we have the following result by Proposition 2.8 [1].

THEOREM 6. *Let M be a timelike hypersurface of $\overline{M}$ and p be a diagonal point in M. If the relation of the sectional curvatures $K(\sigma_{ij}) - \overline{K}(\sigma_{ij}) > 0$ and $\overline{R}ic(X,X) \geq k_0\langle X, e_1\rangle^2$ for timelike vector X in T_pM, then $Ric(X,X) \geq 0$, and X satisfies the generic condition if $\overline{R}ic(X,X) > k_0\langle X, e_1\rangle^2$.*

Now, we will consider Raychaudhuri equation. Given a timelike geodesic segment $\gamma : [a,b] \to M$ which may be extended to $\overline{M}$, let $N(\gamma(t))$ be the n-dimensional subspace of $T_{\gamma(t)}\overline{M}$ consisting of tangent vectors orthogonal to $\gamma'(t)$, $\overline{A}(t)$ $(1,1)$ tensor field on $V^\perp(\gamma)$ as a linear map from $N(\gamma(t))$ to itself for each t, and $\overline{R}$ the curvature tensor of $\overline{M}$. Then a smooth tensor field $\overline{A}(t)$ satisfying

$$\overline{A}'' + \overline{R}\,\overline{A} = 0 \quad \text{and} \quad \ker(\overline{A}(t)) \cap \ker(\overline{A}'(t)) = \{0\}$$

is called a Jacobi tensor field, where $\ker(\overline{A}(t))$ is kernel of $\overline{A}(t)$. A Jacobi tensor $\overline{A}$ satisfying

$$(\overline{A}')^*\overline{A} - \overline{A}^*\overline{A}' = 0 \text{ for all } t \in [a,b]$$

is called a Lagrange tensor field, where $(\overline{A}')^*$ is adjoint of $\overline{A}'$.

If $\overline{A}$ is a Jacobi tensor field along a timelike geodesic, and $\overline{B} = \bar{A}'\,\bar{A}^{-1}$ at points where $\bar{A}^{-1}$ is defined, then the expansion θ, the vorticity ω and the shear tensor σ are defined by $\theta = tr\overline{B}$, $\omega = \frac{1}{2}(\overline{B} - \overline{B}^*)$ and $\sigma = \frac{1}{2}(\overline{B} + \overline{B}^*) - \frac{\theta}{n}E$ respectively, where E is identity transformation of $N(\gamma(t))$ and $tr\overline{B}$ is trace of $\overline{B}$. If Lagrange tensor field $\overline{A}$ satisfies with $tr\overline{B}^2 = (tr\overline{B})^2$, then $\overline{A}$ is called regular Lagrange tensor field. As we know, the Raychaudhuri equation for Jacobi tensor fields along timelike geodesic γ is

$$\theta' = -\overline{R}ic(\gamma', \gamma') - tr(\omega^2) - tr(\sigma^2) - \frac{\theta^2}{n}.$$

If Lagrange tensor field $\overline{A}$ is regular, then the Raychaudhuri equation reduces as

$$\theta' = -\overline{Ric}(\gamma', \gamma') - \theta^2,$$

because B is self-adjoint and the vorticity tensor vanishes along γ. Thus,

$$\overline{Ric}(\gamma', \gamma') = -\{\theta' + \theta^2\}.$$

Now, let us consider the inequality

$$\theta' \leq -\theta^2$$

in the right hand term of the above equation. Integrating this inequality in the suitable domain (actually $\theta' = -\theta^2$ is the Bernoulli differential equation), we have

$$\theta \leq -\frac{1}{t},$$

where t is parameter of θ in suitable domain. Thus, if

$$\theta \leq -\frac{1}{t},$$

then

$$\overline{Ric}(\gamma', \gamma') \geq 0.$$

On the other hand, if $\gamma(t_0) = p$ is a diagonal point, letting $\gamma'(t_0) = X = \sum_{i=1}^{n} x_i e_i$ in T_pM, where $\{e_i\}$ is an orthonormal basis of T_pM which are eigenvectors for S_N as before,

$$\begin{aligned}
&Ric(X, X) \\
&= \overline{Ric}(X, X) - \sum_{i=1}^{n} \epsilon_i \langle X, S_N(e_i)\rangle^2 \\
&\qquad + \sum_{i=1}^{n} \epsilon_i \langle S_N(e_i), e_i\rangle \langle S_N(X), X\rangle \\
&= \overline{Ric}(X, X) - \sum_{i=1}^{n} \epsilon_i {k_i}^2 \langle X, e_i\rangle^2 + \sum_{i=1}^{n} {\epsilon_i}^2 k_i \langle S_N(X), X\rangle \\
&= \overline{Ric}(X, X) - (-k_1^2 x_1^2 + k_2^2 x_2^2 + \cdots + k_n^2 x_n^2) + \sum_{i=1}^{n} k_i \langle S_N(X), X\rangle.
\end{aligned}$$

If $H = \sum_{i=1}^{n} k_i = 0$ and $\max\{|k_i| \mid i = 1, 2, \dots, n\} = |k_1|$, then

$$Ric(X,X) \geq \overline{R}ic(X,X) - k_1^2\langle X,X\rangle.$$

Thus, we have the following result by Proposition 2.8 [1].

THEOREM 7. *If a timelike geodesic segment γ in M has a diagonal point at t_0 which may be extended to $\overline{M}$, $H = 0$, Lagrange tensor field introduced by γ is regular, the maximum of eigenvalues $|k_i|(i \geq 2)$ is $|k_1|$ at $\gamma(t_0)$ and its expansion θ in $\overline{M}$ is bounded above by $\frac{1}{t}$ (parameter of γ is t), then $Ric(\gamma'(t_0), \gamma'(t_0)) > 0$ and $\gamma'(t_0)$ satisfies the generic condition.*

A space-time is said to be causally disconnected if there is some compact set A and two infinite sequences $\{p_n\}$ and $\{q_n\}$ diverging to infinite such that for each n, $p_n \leq q_n$, and $p_n \neq q_n$ and all future-directed nonspacelike curves from p_n to q_n meet A. $p \leq q$ means that either $p = q$ or there is a smooth future-directed nonspacelike curve from p to q. A space-time is said to be chronological if it contains no closed timelike curve. If a space-time is an inextendible space-time which has an inextendible incomplete nonspacelike geodesic, then the space-time is said to have a singularity.

A space-time satisfies the timelike convergence condition if $Ric(X,X) \geq 0$ for all nonspacelike tangent vectors in TM. By continuity, the curvature condition of it is equivalent to the definition of Hawking and Ellis [7]. In Hawking and Ellis [7], a space-time with energy momentum tensor T is said to satisfy the weak energy condition if $T(X,X) \geq 0$ for all timelike vectors X in TM. If Einstein equations hold for the space-time and T with cosmological constant Λ, then the condition $Ric(X,X) \geq 0$ for all timelike vectors X in TM implies that

$$T(X,X) \geq \left(\frac{trT}{2} - \frac{\Lambda}{8\pi}\right)\langle X,X\rangle$$

for all timelike vectors X in TM. If $T\langle X,X\rangle \geq \left(\frac{trT}{2}\right)\langle X,X\rangle$ for all timelike vectors X in TM, the space-time and T are said to satisfy the strong energy condition. In the case of $\Lambda = 0$, this equivalent to the condition $Ric(X, X) \geq 0$ for all timelike vectors X in TM. Thus, if the space-time satisfies the timelike convergence condition or the strong energy condition, then the space-time satisfies the null convergence condition.

We will now apply these to the timelike hypersurface of a given space, and the following theorem shows how the relation between the two curvatures of M and $\overline{M}$ acts on the local universe to have a singularity. This space-time will contain at least one nonspacelike geodesic which is both inextendible and incomplete. Such a geodesic has an end point $\overline{p}$ in causal boundary of M which may be thought of as being outside the universe but not at infinity.

We have shown that, if a timelike hypersurface M of $\overline{M}$ is diagonal type, M satisfies the generic condition under the additional conditions on the relation between the curvature K of M and $\overline{K}$ of $\overline{M}$. It may be interpreted how the given universe can physically influence a local spacetime in it. We can also deduce the strong energy condition without the difficult computations in our procedure. Thus, applying these results with the extra conditions to Penrose's, Hawking's and others singularity theorems, we can see that M has singularities under the different circumstances from the known singularity theorems so far.

THEOREM 8. *Let a timelike hypersurface M of $\overline{M}$ be chronological and satisfy the timelike convergence condition.*

(1) *If M with a smooth boundary point satisfies the hypothesis in* Theorem $3, 4, 5, 6$ *or* 7, *then M is nonspacelike incomplete.*

(2) *If causally disconnected M satisfies the hypotheses in* Theorem $3, 4, 5, 6$ *or* 7, *then M is nonspacelike incomplete.*

These are obtained from an extrinsic geometric condition on the timelike hypersurface M of $\overline{M}$ which may imply these important physical conditions, that is, the generic condition or the strong energy condition, which come from cosmology theory and general relativity. Thus, it may be indicated to suggest that a certain universe in the given space (if such a case is happened) can physically be studied with extrinsic and intrinsic circumstance as we have shown in this paper.

ACKNOWLEDGEMENTS. The author wishes to thank the referee for the helpful comments.

References

[1] J. K. Beem , P. E. Ehrlich and K. L. Easley, *Global Lorenzian Geometry* (2, ed.), Marcel Dekker, 1996.

[2] J. K. Beem and P. E. Ehrlich, *Incompleteness of timelike submanifolds with nonvanishing second fundamental form*, GRG **17** (1985), no. 3.

[3] C. J. S. Clarke, *On the global isometric embedding of pseudo-Riemannian manifolds*, Proc. Roy. Soc. A **314** (1970).

[4] ______, *Space-time singularities*, Math. Phys. **49** (1976).

[5] ______, *The analysis of space-time singularities*, Cambridge Lecture Notes in Physics, Cambridge University Press **1** (1993).

[6] R. E. Greene, *Isometric embeddings of Riemannian and Pseudo-Riemannian Manifolds*, A.M.S. Memoir **97** (1970).

[7] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Camb. Univ. Press, 1973.

[8] S. W. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. R. Soc. Lond. Ser. A **314** (1970).

[9] D. E. Lerner, *Techniques of topology and differential geometry in general relativity*, Springer Lectures Notes in Phys. **14** (1972).

[10] M. A. Magid, *Shape operators of Einstein hypersurfaces in indefinite space forms*, Proc. A.M.S. **84** (1982).

[11] B. O'Neil, *Semi-Riemannian Geometry with application to relativity*, Pure and Applied Ser. Acad. Press **103** (1983).

[12] R. K. Sachs and H. Wu, *General Relativity for mathematician*, vol. 48, Grad. Texts in Math., Springer-Verlag, New York, 1977.

[13] F. Tipler, *Singularities and causality violation*, Ann. of Phys. **108** (1977), 1-36.

[14] ______, *Singularities and universes with negative cosmological constant*, Astrophys. J. **209** (1977), 12-15.

[15] F. Tipler, C. J. S. Clarke, and G. F. R. Ellis, *Singuarities and horizons- a review article, in General Relativity and Gravitation*, vol. 2 (A. Held, ed.), Plenum Press, New York, 1980, pp. 72-206.

[16] R. M. Wald, *General Relativity*, University of Chicago Press, 1984.

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