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BIFURCATION ANALYSIS ON AN UNFOLDING OF THE TAKENS-BOGDANOV SINGULARITY

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ABSTRACT. A complete analysis of the equation $x' = y$, $y' = \beta y - \alpha x^2 + \alpha^2 x + \delta xy$, where α and β small, describing a particular unfolding of the Takens-Bogdanov singularity is presented.

1. Introduction

In this paper we provide a complete bifurcation analysis of a nilpotent singularity of the Takens-Bogdanov type with a two-parameter unfolding given by

$$
\begin{aligned}
x' &= y + O(3) \\
y' &= \beta y - \alpha x^2 + \alpha^2 x + \delta xy + O(3),
\end{aligned}
\tag{1.1}
$$

subject to the nondegeneracy hypotheses $\delta \neq 0$. When $\alpha = \beta = 0$, the vector field (1.1) has a double zero eigenvalue at the origin and has a line of fixed points. By a simple rescaling, we can set $\delta = \pm 1$. In addition, the $O(3)$ terms in (1.1) may be made arbitrary small relative to the terms retained, and the normal form truncated at second order. Although we do not prove it, the phase portrait we obtain are structurally stable ([1]) and the dynamics of (1.1) are not qualitatively changed by the higher order terms in the normal form. The unfolding system

$$
\begin{aligned}
x' &= y + O(3) \\
y' &= \beta + \alpha x + ax^2 + bxy + O(3),
\end{aligned}
$$

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has been extensively studied ([1,8]). Also the unfolding system

$$\begin{aligned} x' &= y + O(3) \\ y' &= \beta x + \alpha y + ax^2 + bxy + O(3), \end{aligned}$$

is also understood ([5]). The problem with Z_2 symmetry, described by

$$\begin{aligned} x' &= y + O(5) \\ y' &= \beta x + \alpha y + ax^3 + bx^2 y + O(5), \end{aligned}$$

is studied by Carr [2], Knobloch and Proctor [7]. The unfoldings can be produced by center manifold reduction and normal form calculation from a larger, even infinite dimensional set of ordinary differential equations.

2. The Hopf bifurcation

Fix $\delta = -1$. (There is a similar analysis for the case of $\delta = 1$). Consider

$$\begin{aligned} x' &= y \\ y' &= \beta y - \alpha x^2 + \alpha^2 x - xy. \end{aligned} \tag{2.1}$$

Then (0,0) and $(\alpha, 0)$ are the only fixed points for the system (2.1). The Jacobian matrix is the following:

$$J = \begin{pmatrix} 0 & 1 \\ -2\alpha x + \alpha^2 - y & \beta - x \end{pmatrix}.$$

So

$$J \mid_{(0,0)} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & \beta \end{pmatrix} \text{ and } J \mid_{(\alpha,0)} = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & \beta - \alpha \end{pmatrix}.$$

Thus

$$\det J \mid_{(0,0)} = -\alpha^2, \ \mathrm{tr} J \mid_{(0,0)} = \beta,$$

and

$$\det J \mid_{(\alpha,0)} = \alpha^2, \ \mathrm{tr} J \mid_{(\alpha,0)} = \beta - \alpha.$$

Therefore the origin is always a saddle and the nontrivial fixed point $(\alpha, 0)$ may undergo a Hopf bifurcation on $\beta = \alpha$. Actually, the eigenvalues associated with the linearization of $(\alpha, 0)$ are given by

$$\lambda_{1,2} = \frac{(\beta-\alpha) \pm \sqrt{(\beta-\alpha)^2 - 4\alpha^2}}{2}$$

and so those on the curve $\beta = \alpha$ are given by $\lambda_{1,2} = \pm i\alpha$. Thus we have $\frac{d}{d\beta} Re\lambda_{1,2} \mid_{\beta=\alpha} = \frac{1}{2} \neq 0$. Therefore the fixed point $(\alpha, 0)$ undergoes a Hopf bifurcation on $\beta = \alpha$.

Next, we check the stability of bifurcating periodic orbit.

THEOREM 2.1. ([2, 9]) *Consider the system*

$$x' = F(x, \mu), \quad x \in R^n, \ \mu \in R. \tag{2.2}$$

Assume that at the bifurcation point (i.e., $\mu = 0$), the reduction to the center manifold takes the form:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} G^1(x, y, 0) \\ G^2(x, y, 0) \end{pmatrix},$$

where ω is the imaginary part of the eigenvalues of $D_xF(0,0)$ and $(x, y) \in R^2$. The terms $G^i(x, y, 0)$ are nonlinear in x and y. Then the stability of the bifurcating periodic orbit is determined by

$$\gamma = \frac{1}{16}[G^1_{xxx} + G^1_{xyy} + G^2_{xxy} + G^2_{yyy}] + \frac{1}{16\omega}[G^1_{xy}(G^1_{xx} + G^1_{yy})$$

$$-G^2_{xy}(G^2_{xx} + G^2_{yy}) - G^1_{xx}G^2_{xx} + G^1_{yy}G^2_{yy}], \tag{2.3}$$

where all partial derivatives are evaluated at the bifurcation point $(x, y, \mu) = (0, 0, 0)$. If $\gamma > 0$, then the bifurcating periodic solution is unstable and if $\gamma < 0$, then the bifurcating periodic orbit is asymptotically stable.

By some transformations and by using the above theorem, we get $\gamma = \frac{1}{8\alpha}$. Therefore if $\alpha > 0$, then $\gamma > 0$ and so the correspoding bifurcation is a subcritical to an unstable periodic orbit. On the other hand, if $\alpha < 0$, then $\gamma < 0$ and so the corresponding bifurcation is a supercritical to a stable periodic orbit. Note that in the case of $\delta = +1$, we can analyze the system similarily. The local bifurcation diagrams are shown in Figure 1. and Figure 2.

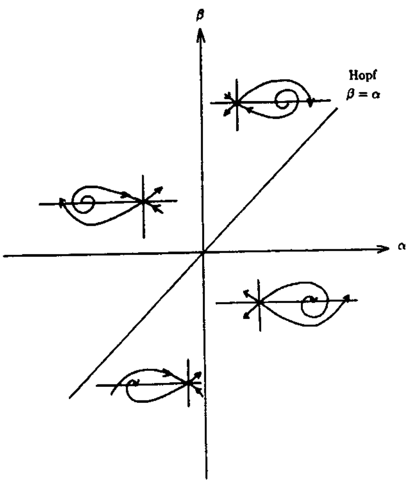

Figure 1.

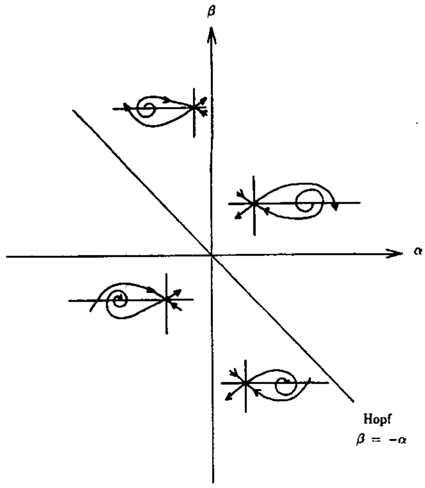

Figure 2.

3. The Existence of homoclinic bifurcation

In section 2, we have analyzed all possible local bifurcations for the system

$$(3.1) \qquad \begin{aligned} x' &= y \\ y' &= \beta y - \alpha x^2 + \alpha^2 x \pm xy, \quad \alpha,\, \beta \in O(\epsilon). \end{aligned}$$

However, a careful study of Figure 1 and Figure 2 reveals that there must be additional bifurcations. This remark is based on the behavior of the stable and unstable manifolds of the saddle point. For instance, for $\beta > \alpha$ and $\alpha > 0$, the positions of the stable and unstable manifolds are interchanged relative to the situation for $\beta < 0$, $\alpha > 0$ and for $\beta < \alpha$ and $\alpha < 0$, the positions of the stable and unstable manifolds are interchanged relative to $\beta > 0$, $\alpha < 0$. (See Figure 1). For the case of Figure 2, in region $\beta < -\alpha$ and $\alpha > 0$, the stable and the unstable manifolds have opposite direction compared with the case $\beta > 0$, $\alpha > 0$.

In region $\beta > -\alpha$ and $\alpha < 0$, the stable and the unstable manifolds have opposite direction compared with the case $\beta < 0$, $\alpha < 0$. (See Figure 2). In all cases, a likely candidate for the global bifurcation which will complete the bifurcation diagram is a homoclinic bifurcation. We now introduce the following theorem.

THEOREM 3.1. ([3]) *Consider the following system:*

$$\begin{aligned} x' &= y \\ y' &= \mu_1 + \mu_2 y + x^2 \pm xy. \end{aligned} \tag{3.2}$$

Then, for any μ_1 and μ_2, there exists a homoclinic bifurcation in region

$$\begin{aligned} 0 < \mu_2 < \sqrt{-\mu_1} \quad &\text{for } +xy \\ -\sqrt{-\mu_1} < \mu_2 < 0 \quad &\text{for } -xy. \end{aligned}$$

The local bifurcation analysis for system (3.2) via Melnikov integrals indicate that for small values of μ_1 and μ_2, there exists a homoclinic bifurcation which is given by $\mu_1 = -\frac{49}{25}\mu_2^2 + O(\mu_2^{\frac{5}{2}})$ ([4]). However, for the system (3.1), we cannot find any rescaling transformation for using Melnikov method ([4]). Now, for the system (3.1), we show existence of homoclinic bifurcation and find the curve numerically by calculation with DSTOOL ([6]). We begin by rescaling the dependent variables and parameters of (3.1) as follows:

$$x = \epsilon u, \quad y = \epsilon^2 v, \quad \alpha = \epsilon\mu, \quad \beta = \epsilon\nu,$$

and we rescale the independent variable time as follows:

$$t \longmapsto \frac{t}{\epsilon}$$

so that (3.1) becomes

$$\begin{aligned} u' &= v \\ v' &= \nu v - \mu u^2 + \mu^2 u \pm uv, \qquad \nu,\, \mu \in O(1). \end{aligned} \tag{3.3}$$

The system (3.3) transforms into

$$\begin{aligned} \bar{u}' &= \bar{v} \\ \bar{v}' &= \frac{\mu^3}{4} + (\nu \pm \frac{\mu}{2})\bar{v} - \mu\bar{u}^2 \pm \bar{u}\bar{v} \end{aligned} \tag{3.4}$$

by the transformation $u = \bar{u} + \frac{\mu}{2}$ and $v = \bar{v}$. We rescale the dependent variables as

$$\bar{u} = -\mu p \quad \text{and} \quad \bar{v} = -\mu^2 q$$

and the independent variable as

$$t \longmapsto \mu t$$

so that the system (3.4) becomes

$$(3.5) \qquad \begin{aligned} p' &= q \\ q' &= -\frac{1}{4} + (\frac{\nu}{\mu} \pm \frac{1}{2})q + p^2 \mp pq, \quad \nu,\, \mu \in O(1). \end{aligned}$$

Therefore the system (3.1) is equivalent to the system (3.5). Now the system (3.5) is of the form of the system (3.2), where $\mu_1 = -\frac{1}{4}$ and $\mu_{2\pm} = \frac{\nu}{\mu} \pm \frac{1}{2}$.

Consider first the system

$$(3.6) \qquad \begin{aligned} p' &= q \\ q' &= -\frac{1}{4} + (\frac{\nu}{\mu} + \frac{1}{2})q + p^2 - pq, \quad \nu,\, \mu \in O(1), \end{aligned}$$

which is equivalent to the system

$$(3.7) \qquad \begin{aligned} x' &= y \\ y' &= \beta y - \alpha x^2 + \alpha^2 x + xy, \quad \alpha,\, \beta \in O(\epsilon). \end{aligned}$$

The local bifurcation theory has shown that the system (3.7) undergoes a Hopf bifurcation on $\beta = -\alpha$. Also from the Theorem 3.1, the system (3.6) has a homoclinic bifurcation which occurs in region $-\frac{1}{2} < \mu_{2+} < 0$. The statement $-\frac{1}{2} < \mu_{2+} < 0$ is equivalent to the statement $-1 < \frac{\beta}{\alpha} < -\frac{1}{2}$ for the system (3.7). Therefore by Theorem 3.1, we can conclude that the system (3.7) has a homoclinic bifurcation and the curve exists in region $-1 < \frac{\beta}{\alpha} < -\frac{1}{2}$. Calculations with DSTOOL show that the stable and the unstable manifolds intersect at $\frac{\beta}{\alpha} \approx -0.864546$ (See Figure 4). The calculation with DSTOOL also provide that there exists an unstable periodic orbit in region $-1 < \frac{\beta}{\alpha} < -0.864546$, $\alpha < 0$, and a stable periodic orbit in region $-1 < \frac{\beta}{\alpha} < -0.864546$, $\alpha > 0$. Therefore we can say that numerically, the system (3.7) has a homoclinic bifurcation on $\beta = -0.864546\alpha$ for all α.

Now consider

$$(3.8) \qquad \begin{aligned} p' &= q \\ q' &= -\frac{1}{4} + (\frac{\nu}{\mu} - \frac{1}{2})q + p^2 + pq, \quad \nu, \mu \in O(1) \end{aligned}$$

which is equivalent to the system

$$(3.9) \qquad \begin{aligned} x' &= y \\ y' &= \beta y - \alpha x^2 + \alpha^2 x - xy, \quad \alpha, \beta \in O(\epsilon). \end{aligned}$$

A similar analysis shows that the system (3.9) has a Hopf bifurcation on $\beta = \alpha$ and a homoclinic bifurcation which occurs in region $\frac{1}{2} < \frac{\beta}{\alpha} < 1$. Calculations with DSTOOL show that the stable and the unstable manifolds for the system (3.9) intersect at $\frac{\beta}{\alpha} \approx 0.864546$ (See Figure 3). The calculation with DSTOOL also provide that there exists an unstable periodic orbit in region $0.864546 < \frac{\beta}{\alpha} < 1$, $\alpha > 0$, and a stable periodic orbit in region $0.864546 < \frac{\beta}{\alpha} < 1$, $\alpha < 0$. Therefore, numerically, the system (3.9) has a homoclinic bifurcation on $\beta = 0.864546\alpha$ for all α.

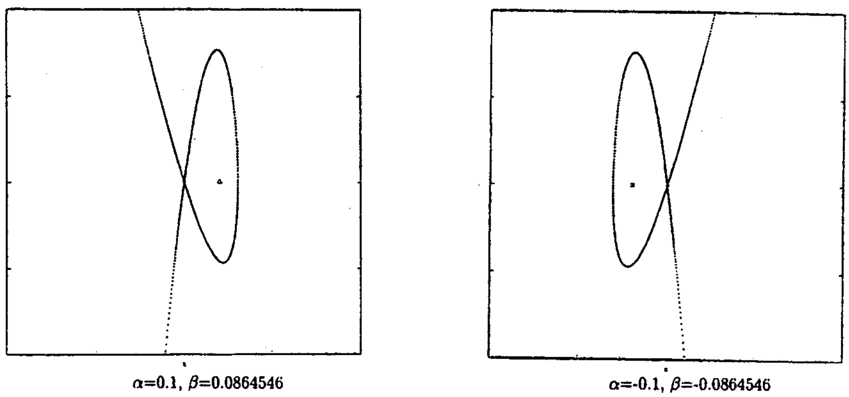

Figure 3. Saddle Connection for (3.9).

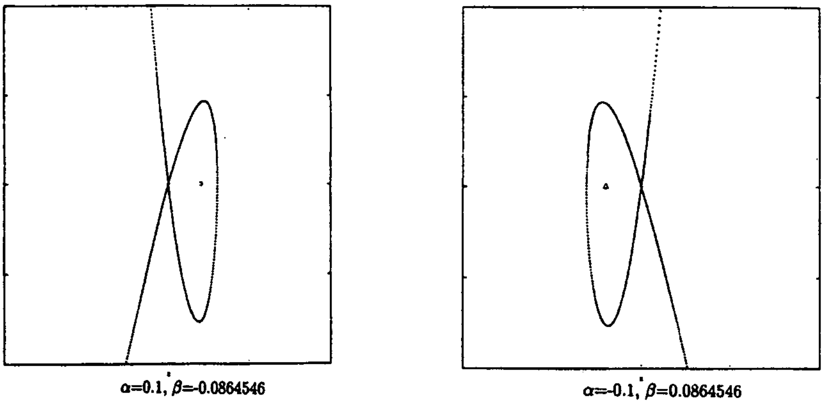

Figure 4. Saddle Connection for (3.7).

We complete the bifurcation diagrams in Figure 5 and Figure 6 for the systems (3.9) and (3.7) respectively.

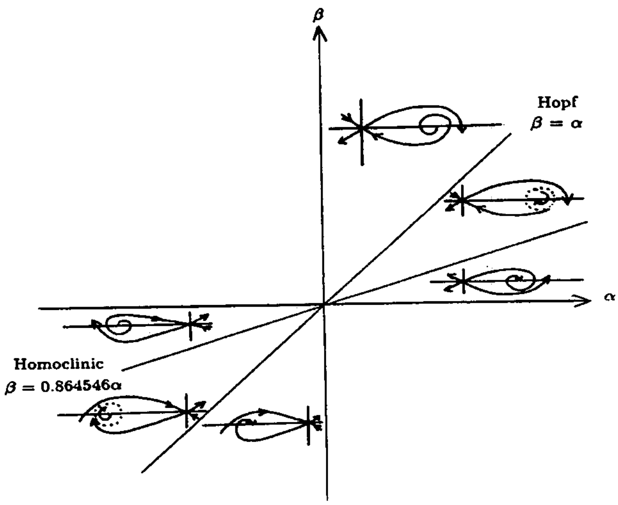

Figure 5. Bifurcation Diagram for (3.9).

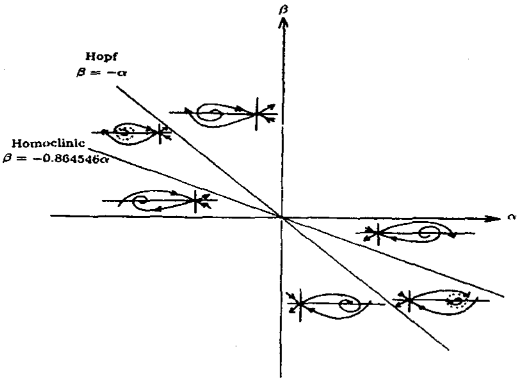

Figure 6. Bifurcation Diagram for (3.7).

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