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ON THE CRYSTALLIZATION OF 3-MANIFOLDS ASSOCIATED WITH POLYHEDRAL SCHEMATA

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ABSTRACT. In this paper we introduce a method of presenting 3-manifolds by polyhedral schemata with 2 vertices so that they may be naturally associated with given crystallizations of 3-manifolds. As applications, we explicitly present crystallizations of n-fold cyclic branched coverings of S^3 over 2-bridge knots.

1. Introduction

It is well known that any closed 3-manifolds M^3 can be presented by a 4-regular properly edge colored graph $\Gamma = (V, E)$ with coloring $r : E \longrightarrow \triangle_3 = \{0, 1, 2, 3\}$ in such a way that a ball complex $B(\Gamma)$ associated with Γ yields the underlying space $|B(\Gamma)|$ homeomorpic to M^3. We briefly sketch what $B(\Gamma)$ is alike (see [7]). Take Γ as the 1-skelecton of $B(\Gamma)$. For each pair $\triangle_1^i = \{\alpha, \beta\}$ of colors of $\triangle_3$, we can determine $b_{\triangle_1^i}$ many mutually disconnected 1-cycles $\Gamma^j_{\triangle_1^i}$ in Γ whose edges are colored by α and β alternatively, i.e., $\Gamma^j_{\triangle_1^i}$ are 2-regular proper edge colored graphs for $1 \le j \le b_{\triangle_1^i}$ and $1 \le i \le 6$. Attaching 2 discs D^2 to Γ so that $\partial D^2 = \Gamma^j_{\triangle_1^i}$, we get the 2-skelecton of $B(\Gamma)$ with $b = \sum_{1 \le i \le 6} b_{\triangle_1^i}$ many 2 cells. Similarly for each triple $\triangle_2^i = \{\alpha, \beta, \gamma\}$ of colors of $\triangle_3$, we have $t_{\triangle_2^i}$ connected components of 3-regular properly edge colored graphs $\Gamma^j_{\triangle_2^i}$ colored by $\triangle_2^i$ for $1 \le j \le t_{\triangle_2^i}$ and $1 \le i \le 4$. Γ presents a closed 3-manifold if and only if $\Gamma^j_{\triangle_2^i}$ presents S^2, the 2-sphere, i.e., $\Gamma^j_{\triangle_2^i}$ together with 2 cells determined by all pairs of colors in $\triangle_2^i$ forms a closed

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surface homeomorpic to S^2. Finally, we get $B(\Gamma)$ with $t = \sum_{1\leq i\leq 4} t_{\triangle_2^i}$ many 3 cells by attaching 3-discs D^3 to the 2-skelecton of $B(\Gamma)$ for each $\Gamma^j_{\triangle_2^i}$. In the above discussion, we say $\Gamma^j_{\triangle_2^i}$ is a triball and Γ is a 3-gem (3-dimensional graph encoded 3-manifold). In particular, a 3-gem Γ is said to be a crystallization of M^3 if $t_{\triangle_2^i} = 1$ for each $1 \leq i \leq 4$, i.e., Γ has exactly 4 triballs obtained by deleting i-colored edges of Γ for each $i \in \triangle_3$.

In the sequel, we assume that M^3 is a closed connected orientable 3-manifold. Let P be a polyhedron, $P/\sim$ be the polyhedron after pairwise identifications of the faces of P and $\pi : P \longrightarrow P/\sim$ be the associated natural projection. Then $P/\sim$ is said to be a polyhedral schema of M^3 if the underlying space $|P/\sim|$ is homeomorphic to M^3. Now let us talk about main results of this paper. Gagliardi[4] showed how to get a crystallization of M^3 from a given Heegaard diagram of M^3. On the other hand, Mandel and Lins[7] provide a general method of getting a 3-gem from a given polyhedral schema of M^3. This 3-gem can be brought into a crystallization by a succession of 1-dipole cancellations,kinds of moves corresponding to the Alexander moves in simplicial complexes. In this paper, we discuss a method of getting crystallization directly from a given polyhedral schema of M^3 based on Gagliardl's work on the Heegaard diagram. What we claim is that if a polyhedral schema of M^3 has 2 vertices and its 1-skeleton is loopless, then we can get an associated crystallization in a fairly straightforward manner. For instance, we can show how to get a crystallization $\varphi(3,3,2,1)$(c.f. Fig. 4 in [7]) for the guaternion space S^3/Q_8 from the polyhedral schema of S^3/Q_8 described in [7] (c.f. Fig. 19 in [6]) without going through 1-dipole cancellations. Moreover we see that any crystallization $(\Gamma,\ r)$ of M^3 with a simple graph Γ can be associated with a polyhedral schema of the form discussed above. For more rigorous treatments, see theorem 1. As applications, we explicitly present crystallizations of n-fold cyclic branched coverings $M_n(k,\ h)$ of 2-bridge knots $b(k,\ h)$ in theorem 2 through Minkus' polyhedral schemata of $M_n(k,\ h)$[5], which may be thought of as partial extensions of Ferri's work[3] on crystallizations of 2-fold branched coverings of S^3.

2. Heegaard diagrams and crystallizations

We introduce a useful concept to explain succinctly how Gagliardi[4] obtained crystallizations from given Heegaard diagrams. Let $\mathcal{M} = \{m_i : 1 \leq i \leq g\}$ be a set of meridians for the handlebody H_g of genus g. By cutting ∂H_g along the meridians, we have a surface S_g homeomorphic to $S^2\backslash 2g$-holes with boundaries m_i^+ and m_i^- corresponding to each $m_i \in \mathcal{M}$. Let c be a simple closed curve on S_g disjoint from $\partial S_g = \{m_i^+, m_i^- : 1 \leq i \leq g\}$ and S_g^i, $1 \leq i \leq 2$ be two pieces of S_g obtained by cutting S_g along c. Then c is said to be a cutting meridian of H_g with respect to $\mathcal{M}$ if for each pair $\{m_i^+, m_i^-\}_{1\leq i\leq g}$, S_g^1 must contain only one of the pair (and hence the other must belong to S_g^2). If m_{g+1} is a cutting meridian of H_g with respect to $\mathcal{M} = \{m_i : 1 \leq i \leq g\}$ then it is easy to see that for each $1 \leq i \leq g$, m_i is a cutting meridian with respect to $\mathcal{M}_i = \mathcal{M}\backslash\{m_i\} \cup \{m_{g+1}\}$. This observation may justify a following definition. We say $\mathcal{M}_E = \{m_i : 1 \leq i \leq g\}$ is a set of extended meridians of H_g iff one of $\mathcal{M}_E$ is a cutting meridian with respect to the rest. And $(\mathcal{M}_E = \{m_i\}, \mathcal{M'}_E = \{m_i'\})_{1\leq i\leq g+1}$ is said to be an extended Heegaard diagram associated with a Heegaard splitting $M^3 = H_g \cup H_g'$ iff $\mathcal{M}$ (resp $\mathcal{M}_E'$) is a set of extended meridians of H_g (resp. H_g').

With those concepts, we may rephrase Gagliardi's theorems relating Heegaard diagrams to crystallizations as follows.

Lemma 1. (Gagliardi [4]) *Let (Γ, r)be a crystallization of M^3. For each pair $\Delta_1 = \{\alpha, \beta\}$ (resp. $\overline{\Delta}_1 = \Delta_3 - \Delta_1$) of colors of Δ_3, let $\{\Gamma^i_{\Delta_1} : 1 \leq i \leq g+1\}$ (resp. $\{\Gamma^i_{\overline{\Delta}_1} : 1 \leq i \leq g+1\}$) be mutually disconnected 1-cycles in Γ whose edges are colored by Δ_1 (resp. $\overline{\Delta}_1$) then there exist a handlebody H_g and a regular imbedding $i : |\Gamma| \longrightarrow \partial H_g$ such that $(\mathcal{M}_E, \mathcal{M}_E')$ is an extended Heegaard diagram for M^3, where $\mathcal{M}_E = \{i(\Gamma^i_{\Delta_1})\}_{1\leq i\leq g+1}$ and $\mathcal{M'}_E = \{i(\Gamma^i_{\overline{\Delta}_1})\}_{1\leq i\leq g+1}$.*

Lemma 2. (Gagliardi [4]) *Given a Heegaard diagram $(\mathcal{M}, \mathcal{M}')$ of M^3 associated with genus g Heegaard splitting, let $(\mathcal{M}_E, \mathcal{M}_E')$ be an extended Heegaard diagram obtained by adding cutting meridians to $\mathcal{M}$, $\mathcal{M}'$ respectively. Then $(\mathcal{M}_E,\ \mathcal{M}_E')$ yields a crystallization of $(\Gamma,\ r)$ of M^3.*

For graphical presentations of Heegaard diagrams or extended Heegaard diagrams, we make use of the following notations. Let $P(\mathcal{M}, \mathcal{M}')$ be a planar Heegaard diagram, i.e., $2g$ circles $m_i^{\pm}$ representing meridians in $\mathcal{M}$ and mutually disjoint arcs with endpoints on $m_i^{\pm}$ representing meridians in $\mathcal{M}'$. Similarly we denote $P(\mathcal{M}, \ \mathcal{M}'_E)$ (resp. $P(\mathcal{M}', \ \mathcal{M}_E)$) by a planar Heegaard diagram $P(\mathcal{M}, \mathcal{M}')$ (resp. $P(\mathcal{M}', \mathcal{M})$) with arcs representing a cutting meridian with respect to $\mathcal{M}'$ (resp. $\mathcal{M}$). Finally we mean $P(\mathcal{M}_E, \mathcal{M}'_E)$ by a planar presentation $P(\mathcal{M}, \mathcal{M}'_E)$ of $(\mathcal{M}, \mathcal{M}'_E)$ with a cutting meridian with respect to $\mathcal{M}$ which is depicted by a simply closed curve.

REMARK 1. Gagliardi's construction process of crystallization from a Heegaard diagram may be interpreted in our terms as follows. First step of crystallization is to find a cutting meridian m_{g+1} with respect to $\mathcal{M}$ from $P(\mathcal{M}, \mathcal{M}')$ so that m_{g+1} meets the arcs representing meridians in $\mathcal{M}'$ transversely (c.f. (a), (b), and (c) in [4, lemma 4]). Interchanging the roles of $\mathcal{M}$ and $\mathcal{M}'$, we can find a cutting meridian m'_{g+1} with respect to $\mathcal{M}'$ which meets the arcs representing meridians in $\mathcal{M}_E$ transversely. Then we have a planar presentation $P(\mathcal{M}_E, \mathcal{M}'_E)$ of an extended Heegaard diagram. Next step for coloring of edges in Γ can be carried out as follows. For fixed colors $\alpha, \beta \in \triangle_3$, let $\mathcal{M}_E^{\alpha}$ (resp. $\mathcal{M}_E^{\beta}$) be a set of the arcs representing meridians in $\mathcal{M}_E$ and lying in side of S_g^{α} (resp. S_g^{β}), where S_g^{α} and S_g^{β} are two regions in the plane model $P(\mathcal{M}', \ \mathcal{M}_E)$ determined by m'_{g+1}. Then we assign the color α (resp. β) to arcs in $\mathcal{M}_E^{\alpha}$ (resp. $\mathcal{M}_E^{\beta}$). Note that a cutting meridian m'_{g+1} plays a role of cyclic assignments of colors α, β to arcs representing each meridians in $\mathcal{M}_E$. Now we assign remained two colors $\gamma, \delta \in \triangle_3$ to the arcs representing meridians in $\mathcal{M}'_E$ by taking the same step as the above with $P(\mathcal{M}, \ \mathcal{M}'_E)$ and m_{g+1}. Finally, we get a desired crystallization $(\Gamma, \ r)$ by identifying each pair of circles $m_i^{\pm}$, $1 \le i \le g$ in $P(\mathcal{M}_E, \ \mathcal{M}'_E)$.

Let us talk about a graphical method of finding cutting meridians. Given a planar Heegaard diagram $P(\mathcal{M}, \ \mathcal{M}')$, let K be the graph obtained by collapsing each circle $m_i^{\pm}$ to a vertex $v_i^{\pm}$, $1 \le i \le g$ so that arcs with end points on $m_i^{\pm}$ are incident with $v_i^{\pm}$. Here K is said to be the graph of $P(\mathcal{M}, \ \mathcal{M}')$.

A partition $\{V_1, V_2\}$ of $V = \{v_i^{\pm} : 1 \le i \le g\}$ is said to be admissible if for each $1 \le i \le g$, V_1 contains the only one of the pair $\{v_i^+, v_i^-\}$. An admissible partition $\{V_1, V_2\}$ of V determines a set C, say a cut of K, of all edges joining one vertex in V_1 and the other in V_2. By isotopic moves, we may locate all vertices in V_1 (resp. V_2) above (resp. below) the x-axis so that any edge with both end points in V_1 or V_2 is disjoint from the x-axis. Then we can make all edges in C transversely meet the x-axis once. Here the x-axis together with the ideal point may be thought of as a cutting meridian with respect to $P(\mathcal{M}, \mathcal{M}')$. Conversely for a cutting meridian c with respect to $\mathcal{M}$ in $P(\mathcal{M}, \mathcal{M}')$, there exists an admissible pair $\{V_1, V_2\}$ of V and a cut C of K associated with c.

The above graphical arguments are still valid if $\mathcal{M}'$ is replaced by an extended meridian $\mathcal{M}'_E$ in $P(\mathcal{M}, \mathcal{M}')$.

3. Polyhedral Schemata and crystallization

Suppose M^3 has a polyhedral schema $P/\sim$, then by considering a regular neighborhood $RN(K_1)$ of 1-skelecton K_1 of $P/\sim$ in M^3, we can get a Heegaard decomposition

$$M^3 = |P/\sim| \simeq \overline{M^3 \backslash RN(K_1)} \cup RN(K_1),$$

where $H_g = \overline{M^3 \backslash RN(K_1)}$ is a handlebody of genus $g \equiv$ the number of pairs of identified faces in P and so is $H'_g = RN(K_1)$ by Poincaré duality. For each edge e of K_1, we consider a disc D^2 properly imbedded in H'_g and meeting transversely at the midpoint of e. Then the boundary of D^2 is said to be a pre-meridian (of H'_g) corresponding to e. Suppose K_1 has k vertices and hence $g+k-1$ edges. Then by considering a maximal tree T with $k-1$ edges, we see that pre-meridians corresponding to g edges in $K_1 \backslash T$ contribute meridians of H'_g. Hence we have.

LEMMA 3. *Take $m_i = RN(K_1) \cap$ ($i-th$ faces of $P/\sim$), $1 \leq i \leq g$ as the standard meridians of H_g. Then by reading pre-meridians corresponding to each edge in $K_1 \backslash T$ in terms of simply closed curves on ∂H_g, we get the associated Heegaard diagram.*

From lemma 3, we see that any polyhedral schema can be crystallized by applying Gagliardis' method to the associated Heegaard diagram. But it normally requires two steps in order to get two cutting meridians one from $P(\mathcal{M}, \mathcal{M}')$ and the other from $P(\mathcal{M}', \mathcal{M})$. Suppose $P(\mathcal{M}, \mathcal{M}')$ is given by reading meridians $\mathcal{M}'$ of $H'_g = RN(K_1)$ in the framework that meridians $\mathcal{M}$ of H_g corresponding to identified faces are thought of as standard meridians. Then we need a step for getting $P(\mathcal{M}', \mathcal{M})$ dual Heegaard diagram to $P(\mathcal{M}, \mathcal{M}')$ which amounts to reading meridians $\mathcal{M}$ in the framework that meridians $\mathcal{M}'$ are thought of as standard ones. After we get a cutting meridian with respect to $\mathcal{M}$ from $P(\mathcal{M}, \mathcal{M}')$, we need one more step for getting a cutting meridian with respect to $\mathcal{M}'$ from $P(\mathcal{M}', \mathcal{M}_E)$. Even in polyhedral schemata with a single vertex, the latter step is cumbersome compared with the former. For the process of getting $P(\mathcal{M}', \mathcal{M})$, see for instance [9].

One of main results of this paper is to provide a family of polyhedral schemata that saves the entire process of obtaining cutting meridians from $P(\mathcal{M}, \mathcal{M}')$. Moreover, we show that any crystallization (Γ, r) of M^3 with a simple graph Γ can be naturally associated with such a polyhedral schema. We say a polyhedral schema of M^3 is loopless if its 1-skelecton K_1 does not contain any loop as a graph.

PROPOSITION 1. *Let $P/\sim$ be a loopless polyhedral schema of M^3 with 2 vertices. With the notations in* lemma 3 *and reading pre-meridians corresponding to all edges in K_1 in terms of a simply closed curve on ∂H_g, we immediately arrive at $P(\mathcal{M}, \mathcal{M}'_E)$.*

Proof. Since K_1 is loopless, any edge e in K_1 contributes a tree of K_1. Hence pre-meridians corresponding to the rest of edges form meridians of $H'_g = RN(K_1)$. Furthermore it is obvious that the pre-meridian corresponding to e plays a role of the cutting meridian. Thus all pre-meridians form extended meridians $\mathcal{M}'_E$ of $H'_g = RN(K_1)$. □

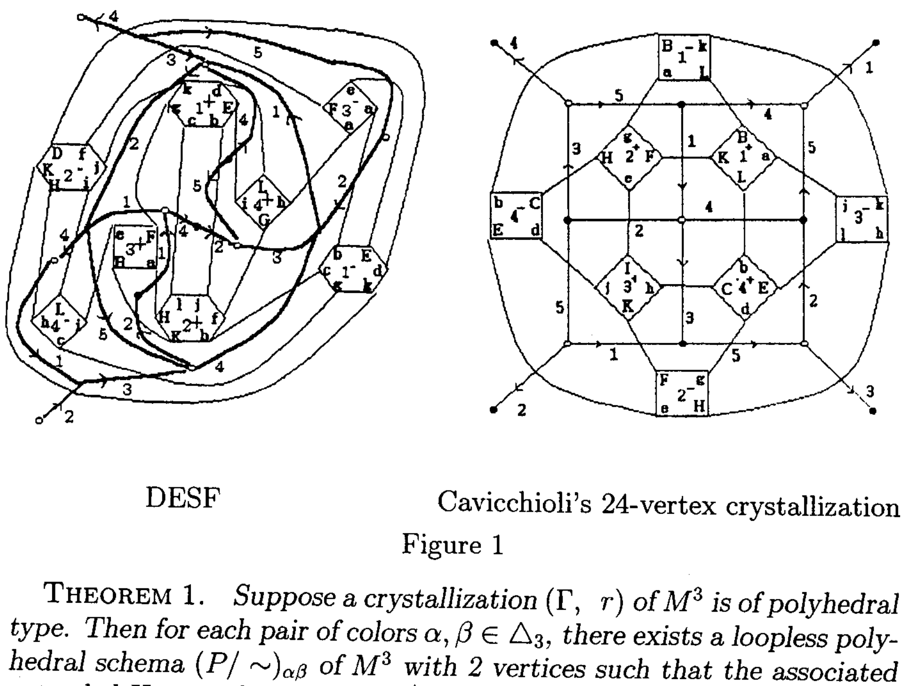
Conversely we have

PROPOSITION 2. *Suppose the graph K of $P(\mathcal{M}, \mathcal{M}'_E)$ is loopless. Then there exists a loopless polyhedral schema with 2 vertices associated with $P(\mathcal{M}, \mathcal{M}'_E)$ by* Proposition 1.

Proof. Let K be the graph of $P(\mathcal{M}, \mathcal{M}'_E)$. Then K is connected due to the role of the cutting meridian even if the graph of $P(\mathcal{M}, \mathcal{M}')$ has multi connected components. Moreover since K is loopless, we may think of K as the 1-skelecton of a polyhedron P. Let P^* be the dual polyhedron. We identify a face of P^* dual to the vertex of K corresponding to m_i^+ with that of P^* dual to the vertex of K corresponding to m_i^- so that it may induce the identification of edges of P^* dual to those of K representing oriented meridians in $\mathcal{M}'_E$. Then it is easy to see that $P^*/\sim$ is a loopless polyhedral schema with 2 vertices yielding $P(\mathcal{M},\ \mathcal{M}'_E)$. □

Conversely, we see that any crystallization (Γ, r) of M^3 with a simple graph Γ can be naturally associated with a loopless polyhedral schema with 2 vertices. A crystallization $(\Gamma,\ r)$ is said to be of polyhedral type if for each pair $\{\alpha,\ \beta\}$ of colors in Δ_3, $P_{\alpha\beta}(\mathcal{M},\ \mathcal{M}'_E)$ is loopless. For instance, a crystallization $(\Gamma,\ r)$ is of polyhedral type if Γ is a simple graph. By lemma 1 and proposition 2, we have

REMARK 2. Not all crystallizations are of polyhedral type. For example, see a crystallization of $S^1 \times S^2$ in [7, p. 267]. Figure 1 shows loopless polyhedral schemata with 2 vertices recovered from a crystallization DESF of the Poincaré homology sphere and Cavicchioli's 24-vertex crystallization of the binary tetrahedral space in [7, p. 280].



DESF Cavicchioli's 24-vertex crystallization

Figure 1

THEOREM 1. *Suppose a crystallization $(\Gamma,\ r)$ of M^3 is of polyhedral type. Then for each pair of colors $\alpha, \beta \in \Delta_3$, there exists a loopless polyhedral schema $(P/\sim)_{\alpha\beta}$ of M^3 with 2 vertices such that the associated extended Heegaard diagram yields Γ.*

4. Crystallizations of n-fold cyclic branched coverings of S^3 over 2-bridge knots

It would be nice to get explicit crystallizations of n-fold cyclic branched coverings of S^3 over some knots or links. To authors' knowledges, Ferri's paper[3] is only available source in this line of works. He suggested a general method of constructing crystallizations of 2-fold branched coverings of S^3 through bridge presentations of knots or links. But it seems rather difficult to see how to extend his construction for n-fold cyclic branched coverings even in simplest 2-bridge knots. On the other hand, Minkus[5] has shown that n-fold cyclic branched coverings $M_n(k,\ h)$ of S^3 over 2-bridge knots or links $b(k,\ h)$ admit polyhedral schemata presentations similar to those for lens space $L(k,\ h)$.

For polyhedral schemata of $M_n(k,\ h)$, we consider n equally spaced great semicircle S_i joining the north pole N and the south pole S of S^2. We insert $k-1$ vertices $v_1^i,\ v_2^i, \cdots, v_{k-1}^i$ on S_i so that v_1^i (resp. v_{k-1}^i) is the nearest vertex to N (resp. S), for each $1 \leq i \leq n$. Then S^2 may be thought of as a polyhedron P with $2n$ $(k+1)$-gons $R_i,\ \overline{R_i}$ by inserting arcs c_i from v_h^i to v_{k-h}^{i+1} (i is reduced mod n), where subscripts i are assigned so that $V_i = \{v_h^i,\ v_{h+1}^i, \cdots, N, v_1^{i+1},\ v_2^{i+1}, \cdots, v_{k-h}^{i+1}\}$ (resp. $\overline{V_i} = \{v_h^{i-1},\ v_{h+1}^{i-1}, \cdots, v_{k-1}^{i-1},\ S, v_{k-1}^i,\ v_{k-1}^i, \cdots, v_{k-h}^i\}$) forms vertices of R_i (resp. $\overline{R_i}$) and hence c_i is a common edge of R_i and $\overline{R_{i+1}}$. Now we identify R_i with $\overline{R_i}$ so that $c_i = \overline{v_{k-h}^{i+1} v_h^i}$ is identified with $c_{i-1} = \overline{v_{k-h}^i v_h^{i-1}}$ while keeping the order of vertices listed in V_i and $\overline{V_i}$. Then by utilizing combinatorial covering space theory and splitting complexes of Neuwirth [8], Minkus [5] showed that if $(k,\ h) = 1$, $h \equiv 1 \mod 2$ and $1 \leq h < k$, then $M_n(k,\ h)$ is the n-fold cyclic branched covering of the two bridge knot or link $b(k,\ h)$. Here the branched set in $M_n(k,\ h)$ is the fixed point set of a homeomorphism $r : M_n(k,\ h) \to M_n(k,\ h)$ of period n induced by a rotation $r' : P \to P$ by $2\pi/n$ around the north-south pole axis $NS \subset P$. Let $a_i = \pi(\overline{v_h^i v_{h-1}^i})$ and $c = \pi(c_i) = \cdots = \pi(c_n)$. Then the edges $\{a_i : 1 \leq i \leq n\}$ must be distinct because $\pi(NS)$ and c is fixed point set of r. Hence the 2-skelecton of $M_n(k,\ h)$ has 2 vertices represented by $\{N,\ S\}$, $n+1$-edges $\{a_i : 1 \leq i \leq n\} \cup \{c\}$ and n faces $\pi(R_i) = \pi(\overline{R_i})$. Moreover $M_n(2l+1,\ h)$ is loopless because c has different endpoints whereas c has the same endpoint in $M_n(2l,\ h)$.

REMARK. If $(k,\ h)=1$, $k\equiv 1 \mod 2$, $h\equiv 0\ mod\ 2$ and $1\leq h<k$. Then by schubert's classification theorem [2], $b(k,\ h)=b(k,\ k-h)$ and $k-h\equiv 1 \mod 2$. Hence for $h\equiv 0 \mod 2$, $M_n(k,\ k-h)$ may be thought of as the n-fold cyclic branched coverings of S^3 over $b(k,\ h)$.

Except $M_n(k,\ 1)$, it requires knowledges on the sign $e_j(k,\ h)$ of crossings of $b(k,\ h)$ to figure out how the edges arrange themselves in the 2-skelecton of $M_n(k,\ h)$. For details, see figure 4 for $M_n(k,\ 1)$ and theorem 8 or lemma 9.1 for $e_j(k,\ h)$ in [5]. Furthermore, general description of edge configurations such as $M_n(k,\ 1)$ is not available for $h>1$ (see Table 1 in [5]). But as far as crystallizations are concerned two classes of 4-colored 4-regular graphs are required that admit following general descriptions.

For $2h<k$ (resp. $2h>k$), let P_i (resp. P'_i) be a $k+1$-gon with vertices v_{ij} (resp. v'_{ij}) for $1\leq i\leq n$ and Q (resp. Q') be a $n(k+1-2h)$-gon (resp. $n(2h-k+1)$-gon) with vertices w_{ij} (resp. w'_{ij}) for $1\leq i\leq n$ and $2h+1\leq j\leq k+1$ (resp. $2k-2h+1\leq j\leq k+1$). Assume that P_i (resp. P'_i) and Q (resp. Q') is embedded in S^2. Now we tesselate $S\equiv S^2\backslash\bigcup_{1\leq i\leq n}P_i\cup Q$ (resp. $S'\equiv S^2\backslash\bigcup_{1\leq i\leq n}P'_i\cup Q'$) with 1 $2n$-gon, n hexagons and $n(k-h-1)$ (resp. $n(h-1)$) rectangles by inserting arcs joining a pair of vertices of P_i (resp. P'_i) or Q (resp. Q'). For this purpose, we make use of a planar presentation of S^2 such that P_i (resp. P'_i) are counter-clockwisely (resp. clockwisely) located inside Q (resp. Q') for the increasing order of i where subscripts i are reduced mod n. Furthermore we assume that w_{ij} (resp. w'_{ij}) are clockwisely located on Q (resp. Q') for the lexicographic order of i and j. Followings are lists of arcs inserted for the tesselation of S (resp. S') and hence S^2.

(i) For $1\leq i\leq n$ and $1\leq j\leq h$ (resp. $1\leq j\leq k-h$)
v_{ij} (resp. v'_{ij}) is connected to $v_{i-1,2h+1-j}$ (resp. $v'_{i+1,2k-2h+1-j}$).

(ii) For $1\leq i\leq n$ and $2h+1\leq j\leq k+1$ (resp. $2k-2h+1\leq j\leq k+1$)
v_{ij} (resp. v'_{ij}) is connected to w_{ij} (resp. w'_{ij}).

Up to this construction, we have a 3-regular graph $\mathcal{K}$ (resp. $\mathcal{K}'$) embedded in S^2 with $2n(k-h+1)$ (resp. $n(2h+2)$) vertices and hence $3n(k-h+1)$ (resp. $3n(h+1)$) edges. Now by adding $n(k-h+1)$ (resp. $n(h+1)$) edges to $\mathcal{K}$ (resp. $\mathcal{K}'$) we will get a 4-regular graph $\Gamma(n,k,h)$ (resp. $\Gamma'(n,k,h)$). Followings are list of edges added.

(iii) For $1\leq i\leq n$ and $k-2h+1\leq j\leq k-h$ (resp. $2h-k+1\leq j\leq h$)
v_{ij} (resp. v'_{ij}) is connected to $v_{i-1,2k-2h+1-j}$ (resp. $v'_{i+1,2h+1-j}$).

(iv) For $1\leq i\leq n$ and $1\leq j\leq k-2h$ (resp. $1\leq j\leq 2h-k$)

v_{ij} (resp. v'_{ij}) is connected to $w_{i+1,2h+j}$ (resp. $w_{i,2k-2h+j}$).

(v) For $1 \leq i \leq n$

$v_{i,k+1}$ (resp. v'_{ij}) is connected to $w_{i+1,k+1}$ (resp. $w'_{i-1,k+1}$). Edge colorings of Γ and Γ' are done as follows.

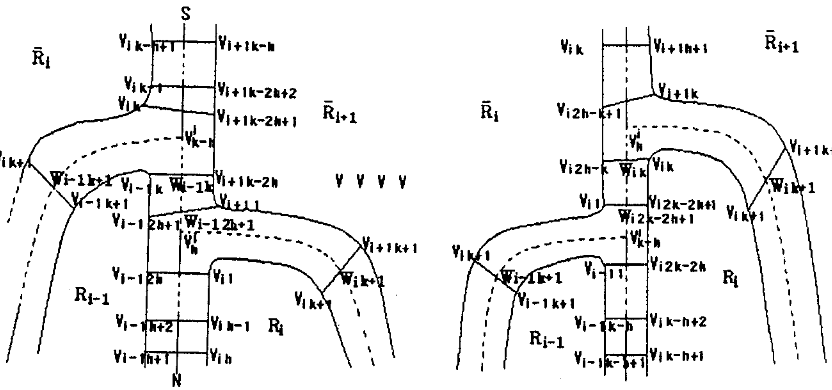
A. Edges of P_i (resp. P'_i) and Q (resp. Q') are alternatively colored by α and β

B. Edges constructed by (i) and (ii) are colored by γ

C. Edges constructed by (iii), (iv) and (v) are colored by δ.

THEOREM 2. *$\Gamma(n, k, h)$ (resp. $\Gamma'(n, k, h)$) is a crystallization of $M_n(k, h)$ if $2h < k$ (resp. $2h > k$), where $(k, h) = 1$, $k, h \equiv 1 \mod 2$ and $1 \leq h < k$.*

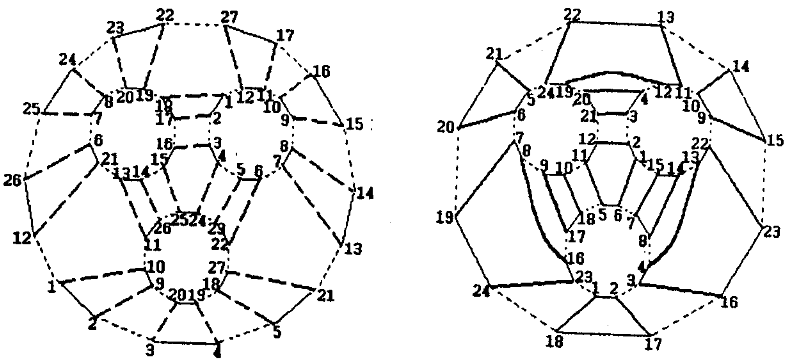
Proof. We prove the theorem for $2h < k$ since proof for the other case is analogous. By lemma 3 and proposition 1, we have n meridians $m_i = RN(K_1) \cap \pi(R_i)$ of $H_n = \overline{M_n(k, h) \backslash RN(K_1)}$ $1 \leq i \leq n$ and $n+1$ extended meridians m'_i (resp. m'_{i+1}) of $H'_n = RN(K_1)$ corresponding to edge a_i (resp. c) $1 \leq i \leq n$, where $RN(K_1)$is the regular neighborhood of the 1-skelecton K_1 of $M_n(k, h)$. Then for each $1 \leq i \leq n$, there exist k edges $e_{i1} = \overline{v^i_h v^i_{h-1}}$, $e_{i2}, \cdots, e_{ik}$ of P such that $a_i = \pi(e_{i1}) = \pi(e_{i2}) = \cdots = \pi(e_{ik})$ Take a neighborhood $N_{i,j}$ of $e_{i,j}$ in P so that $\pi(N_{i,j})$ contributes one of k-pieces of regular neighborhood of a_i for $1 \leq i \leq n$ and consider m'_{ij} one of k-pieces of m'_i lies on $\partial\pi(N_{ij})$ in such a way that $m'_i \cap \mathcal{M} = \{m_j | 1 \leq j \leq n\}$ consists of k endpoints of m'_{ij}. In particular, let v_{i1} be one of endpoints of m'_{i1} lying on $m_i \subset \pi(R_i) = \pi(\overline{R_i})$ for $1 \leq i \leq n$. On the other hand, the meridian m'_{n+1} corresponding to c meets with m_i only once, say at v_{ik+1} for each $1 \leq i \leq n$. Then kn endpoints of m'_{ij} including v_{i1} and v_{ik+1} $(1 \leq i \leq n,\ 1 \leq j \leq k)$ are located on m_i so that each m_i contains $k+1$ points $v_{i1}, v_{i2}, \cdots, v_{ik+1}$ arranged counterclockwisely on R_i and hence arranged clockwisely on $\overline{R_i}$ where we utilize the same notation for v_{ij} and $\pi^{-1}(v_{ij})$. For each edge e of P, there is a pair $\{v_{pq}, v_{rs}\}$ uniquely determined by e such that a segment $\overline{v_{pq}v_{rs}}$ meeting e transversely presents a part of the meridian corresponding to $\pi(e) \in \{a_i | 1 \leq i \leq n\} \cup \{c\}$. Hence all $(k+1)n$ number of such segments constitutes extended meridians $\mathcal{M}'_E = \{m'_i | 1 \leq i \leq n+1\}$. □



$M_n(k,\ h)(k > 2h)$ $M_n(k,\ h)(k < 2h)$

Figure 2

Now we show how to get a cutting meridian with respect to $\mathcal{M}$ in $P(\mathcal{M},\ \mathcal{M}'_E)$. The polyhedron P is divided by two regions $R^+ = \bigcup_{1\leq i\leq n} R_i$ and $R^- = \bigcup_{1\leq i\leq n} \overline{R_i}$. Let W be the common boundary of R^+ and R^-. Since all m_i^+ (resp. m_i^-) are located in the interior of R^+ (resp. R^-) and each edge e of W determines a unique segment $\overline{v_{pq}v_{rs}}$ meeting transversely with e in our presentation of $P(\mathcal{M},\ \mathcal{M}'_E)$, W plays a role of a cutting meridian with respect to $\mathcal{M}$ in $P(\mathcal{M},\ \mathcal{M}'_E)$. We denote w_{pq} by the transversely intersecting point of e and $\overline{v_{pq}v_{rs}}$, where $v_{pq} \in R^+$ and $v_{rs} \in R^-$. Figure 3 are examples of crystallizations of $M_n(k,\ h)$, where we take a convention that each pair of vertices with a same number is joined by the last colored edge.



Crystallization of $M_3(11,\ 3)$ Crystallization of $M_3(11,\ 7)$

Figure 3

COROLLARY. *$\varphi(n,3,2,1)$ is a crystallization of n-fold cyclic branched covering of S^3 over the trefoil knot, i.e., $b(3,1)$.*

Proof. It is easy to see that $\varphi(n,3,2,1)$ is isomorphic to $\Gamma(n,3,1)$. □

Mandel and Lins pointed out a family of homology spheres in crystallizations $\varphi(n,3,2,1)$. For details, see theorem 11 in [7]. This can be deduced from the above corollary and first homology groups of $M_n(3,1)$ in [11, p. 304].

When $b(k, h)$ is a proper link, i.e., $k \equiv 0 \mod 2$ then the 1-skelecton of the associated polyhedral schema $M_n(k, h)$ contains a loop, hence the method in theorem 2 can not be applied.

PROBLEM. Find explicit presentation of a crystallization of $M_n(k, h)$ as theorem 2 for $k \equiv 0 \mod 2$.

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