J. Korean Math. Soc. **36** (1999), No. 1, pp. 159–171

A STABILITY RESULT FOR THE COMPRESSIBLE STOKES EQUATIONS USING DISCONTINUOUS PRESSURE

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Abstract. We formulate and study a finite element method for a linearized steady state, compressible, viscous Navier-Stokes equations in 2D, based on the discontinuous Galerkin method. Dislike the standard discontinuous Galerkin method, we do not assume that the triangle sides be bounded away from the characteristic direction. The unique stability follows from the inf-sup condition established on the finite dimensional spaces for the (incompressible) Stokes problem. An error analysis having a jump discontinuity for pressure is shown.

1. Introduction

We consider a linearized stationary, barotropic, compressible, viscous Navier-Stokes system which consists of the momentum equations having an elliptic character in velocity and the continuity one having a hyperbolic nature in pressure. Hence the system is neither elliptic nor hyperbolic and so of mixed type which has features of each class of equations. We study a finite element method for the equations. Our purpose is to apply to a weak formulation of the compressible Stokes problem the finite element spaces used in searching for the continuous velocity and discontinuous pressure solution pairs of the incompressible Stokes problem and to establish a unique existence and some error estimate for

Received June 15, 1998.

1991 Mathematics Subject Classification: 65N15, 35M10.

Key words and phrases: compressible viscous Stokes flows, continuity equation, finite element method.

This work is partially supported by Silla University and Com^{2}MaC in POSTECH.

a proposed method. The equations to be considered are

$$(1.1)\qquad \begin{cases} -\mu\Delta \mathbf{u} - \nu\nabla \mathrm{div}\, \mathbf{u} + \rho(\mathbf{w}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \ \text{ in } \Omega, \\ \mathrm{div}\, \mathbf{u} + \kappa \mathbf{w}\cdot\nabla p = \rho^{-1} g \ \text{ in } \Omega, \\ \mathbf{u} = 0 \ \text{ on } \Gamma, \end{cases}$$

where $\Omega \subset \mathbf{R}^2$ is an open bounded domain with a smooth boundary Γ. Here $\mathbf{u} = [u, v]$ is a velocity vector, p is the pressure; $\mathbf{w} = [U, V]$ with is an ambient flow vector function with zero boundary value $\mathbf{w}|_\Gamma = 0$ and P is a pressure of ambient fluid, $\rho \equiv \rho(P)$ is a given positive increasing function that is the density function of pressure, and $\kappa = \rho'(P)/\rho(P)$, $\rho'(P) = d\rho/dP$. The functions $\mathbf{f}$ and g are given functions, and the numbers μ and ν are the viscous constants with $\mu > 0$ and $\mu > -\nu$.

It is assumed that the coefficients κ and ρ are $\rho = \kappa = 1$ for simplicity. Since the ambient velocity vector $\mathbf{w}$ is assumed to be zero on the boundary Γ, no boundary condition for pressure is imposed on a portion of Γ even though the second equation in (1.1) is the first order partial differential equation in pressure p. Later the solution pressure p is required to satisfy the condition of pressure mean zero ($p \in L_0^2(\Omega)$.) If $U \geq C_0 > 0$ for a constant C_0, then flows move from left to right and so specified values for pressure can be assigned on those portions of the boundary where the ambient velocity vector $\mathbf{w}$ points into the region (see [4, 5]). If the first component U of $\mathbf{w}$ is assumed to be nonnegative on the interior portion of the domain Ω, then the directions of flows can not be reversed in the region of Ω and may oscillate up and down, depending on the second component V.

Let $[\mathbf{u}, p]$ be the solution of (1.1) and $[\mathbf{u}_h, p_h]$ the finite element solution of (2.7). In this note we obtain the following main result: Assume that the condition (2.9) holds. Then there exists a constant K, not depending on h such that

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 + \Big(\sum_{K\in\mathcal{T}_h} \int_{\partial K_-} (p_h^+ - p_h^-)^2 |\mathbf{w}\cdot\mathbf{n}|\Big)^{1/2} \\ &\leq K \inf\Big\{\|\nabla(\mathbf{u} - \hat{\mathbf{u}})\|_0 + \|p - \hat{p}\|_0 + \Big(\sum_{K\in\mathcal{T}_h} \int_{\partial K_-} (p - \hat{p}^-)^2\Big)^{1/2}\Big\} \\ &\quad + Ch^{l-1}\|p\|_{l,\Omega}, \ (l \geq 1), \end{aligned}$$

where the infimum is taken over all $\hat{\mathbf{u}} \in \mathbf{V}_h$ and $\hat{p} \in Q_h$.

It is well-known that as the choices of the velocity-pressure spaces for the finite element approximation of the Stokes problem, one scheme is to use continuous piecewise polynomial approximations for both velocity and pressure, and the other scheme is to use a continuous piecewise polynomial approximation to the velocity and a discontinuous piecewise polynomial approximation to the pressure. The stabilities for the schemes can be shown by establishing the inf-sup condition (see [7]). For the former velocity-pressure spaces, it has been shown in [3] that when the finite dimensional spaces for velocity and pressure satisfy the inf-sup condition associated with the (incompressible) Stokes system, the approximate method proposed gives a unique existence of finite element solution and also gives an error estimate. However discontinuous pressure finite element spaces were not included there. In our analysis the continuous finite dimensional space for velocity is applied to the compressible Stokes problem while the discontinuous finite dimensional space for pressure is. In doing so, on the basis of the discontinuous Galerkin method for the neutron transport equation, an integral containing a jump for pressure on the incoming portion (if it exists) of triangle is added to a usual formulation of the convective derivative of pressure in the continuity equation (see (2.6)), but it is not assumed that the triangle sides for a given triangulation be bounded away from the characteristic direction. The unique existence of the finite element solution follows from the inf-sup condition shown for the incompressible Stokes problem.

In §2 we introduce a weak formulation for (1.1) and also a discrete form corresponding to it. A unique stability of the finite element solution is shown and an error analysis is given.

In this note the Sobolev spaces and norms to be used are given below: the space $H^k(\Omega)$ is the space of real-valued L^2 functions on Ω so that all their derivatives up to order k belong to $L^2(\Omega)$. We denote by $\|u\|_{0,\Omega}$ the L^2 norm on Ω and $\|u\|_{k,\Omega}$ the norm of $H^k(\Omega)$. Also $H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}$ and $H_0^k(\Omega) = H^k(\Omega) \cap H_0^1(\Omega)$. The sup norm is defined by $|u|_\infty = \max_{\mathbf{x}\in\Omega} |u(\mathbf{x})|$ and the space $L^\infty(\Omega)$ is defined likewise. The Sobolev imbedding theorems to be used occasionally are $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ or $H^1(\Omega) \hookrightarrow L^4(\Omega)$, and the notation $\chi_\beta = \mathbf{w}\cdot\nabla\chi$ and norm $\|\chi\|_Q = \sqrt{\|\chi\|_{0,\Omega}^2 + \|\chi_\beta\|_{0,\Omega}^2}$ are used and the space $Q = \{q \in L_0^2(\Omega) : \|q\|_Q < \infty\}$ is defined and also

$$(u,\, v)_\Omega = \int_\Omega u\, v\, \mathrm{d}\mathbf{x}.$$

2. Finite element method

In this section our purpose is to formulate and study a finite element method for (1.1) on the finite element spaces used in searching for the continuous velocity and discontinuous pressure finite element solutions for the (incompressible) Stokes problem, and to show a unique existence of the discrete solution for a proposed finite element method and establish some error estimates for it. Several bilinear forms are considered in order to formulate problem (1.1) into a discrete version. We let $\mathbf{V} = H_0^1(\Omega)^N$, and $M = L_0^2(\Omega)$, and $Q = \{\chi \in M : \|\chi\|_Q < \infty\}$. Note that $\|\chi\|_0 \le C\|\nabla\chi\|_{-1}$ for $\chi \in M$ (see [3]). The momentum equation in (1.1) can be formulated into a weak form by considering two bilinear forms a and b

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \int_\Omega \mu\nabla\mathbf{u}\cdot\nabla\mathbf{v} + \nu\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v} + \mathbf{w}\cdot\nabla\mathbf{u}\,\mathbf{v}\,d\mathbf{x},\ \mathbf{u},\mathbf{v}\in\mathbf{V},\\ b(\mathbf{v},\chi) &= \int_\Omega \chi\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x},\quad \mathbf{v}\in\mathbf{V},\chi\in M, \end{aligned}$$

and the continuity equation in (1.1) by considering a bilinear form c

$$c(p,\chi) = \int_\Omega \mathbf{w}\cdot\nabla p\,\chi\,d\mathbf{x},\ p\in Q,\ \chi\in M$$

and also define

$$\begin{aligned} \langle\mathbf{f},\mathbf{v}\rangle &= \int_\Omega \mathbf{f}\,\mathbf{v}\,d\mathbf{x},\\ \langle g,\chi\rangle &= \int_\Omega g\,\chi\,d\mathbf{x}. \end{aligned}$$

Hence the resulting weak formulation is to find $[\mathbf{u},p]\in V\times Q$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) &= \langle\mathbf{f},\mathbf{v}\rangle,\quad \forall\mathbf{v}\in\mathbf{V},\\ c(p,\chi) + b(\mathbf{u},\chi) &= \langle g,\chi\rangle,\quad \forall\chi\in M. \end{aligned} \tag{2.1}$$

Letting $\gamma_0 = \frac{1}{2}|\mathrm{div}\,\mathbf{w}|_\infty$ and $\mu_0 = \min\{\mu,\mu+\nu\}$ and $\mu_1 = \mu+|\nu|$, we have:

LEMMA 2.1. Assume that $\mu_0 > (\bar{C}^2+\mu_1)\gamma_0$, where $\bar{C}$ is the *Poincaré* constant. Then problem (2.1) has a unique solution $[\mathbf{u},p]\in\mathbf{V}\times M$ satisfying $k\|\mathbf{u}\|_1 + \|p\|_Q \le C(\|\mathbf{f}\|_0 + \|g\|_0)$ where $k = \mu_0 - (\bar{C}^2+\mu_1)\gamma_0$.

Proof. Taking $\mathbf{v} = \mathbf{u}$ and $\chi = p$ in (2.1), and using the integration by parts, and using the fact that $\|\chi\|_0 \leq C\|\nabla\chi\|_{-1}$, $\chi \in M$, we easily have $k\|\mathbf{u}\|_1 + \|p\|_0 \leq C(\|\mathbf{f}\|_0 + \|g\|_0)$. Next, from the second equation in (1.1) we have $\|p_\beta\|_0 \leq (\|\nabla\mathbf{u}\|_0 + \|g\|_0)$. A standard argument by Lax-Milgram lemma show a unique existence of the solution of (2.1). □

For a more regularity result of the solution of (2.1) see [10, 11]. For convenience we let $\mathcal{H} = \mathbf{V} \times M$, and define a bilinear form $\mathcal{B}$ on $\mathcal{H} \times \mathcal{H}$ and a linear functional Λ on $\mathcal{H}$ respectively:

$$\begin{aligned} \mathcal{B}([\mathbf{u},p];[\mathbf{v},\chi]) &= a(\mathbf{u},\varphi) - b(\mathbf{v},p) + c(\mathbf{w};p,\chi) + b(\mathbf{u},\chi), \\ \Lambda[\mathbf{v},\chi] &= \langle \mathbf{f},\mathbf{v}\rangle + \langle g,\chi\rangle. \end{aligned} \tag{2.2}$$

We now try to approximate the solution $[\mathbf{u},p] \in \mathbf{V} \times Q$ of (1.1) by the finite element method using continuous piecewise elements for velocity and discontinuous piecewise elements for pressure. We let $\mathcal{T}_h$ be a family of triangles K of a regular triangulation of Ω with a mesh size h and $P_l(K)$ the space of all polynomials on K of degree $\leq l$. We also denote by λ_i (i=1, 3) the barycentric co-ordinates of $K \in \mathcal{T}_h$. We now take the velocities $\mathbf{v}$ in the polynomial subspace of $(P_3(K))^2$

$$\mathcal{P}_2(K) = \left[P_2(K) \oplus \operatorname{span}\{\lambda_1\lambda_2\lambda_3\}\right]^2, \tag{2.3}$$

and the pressure q in $P_1(K)$. This leads to the following choice of spaces:

$$\begin{aligned} X_h &= \{\mathbf{v}_h \in (C(\Omega))^2 : \mathbf{v}_h|_K \in \mathcal{P}_2(K),\ \forall K \in \mathcal{T}_h\}, \\ Y_h &= \{q_h \in L^2(\Omega) : q_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}, \end{aligned}$$

and $\mathbf{V}_h = X_h \cap \mathbf{V}$, $M_h = Y_h \cap M$, and $Q_h = Y_h \cap Q$. Note that $\mathbf{V}_h \subset \mathbf{V}$ and $M_h \subset M$ and $Q_h \subset Q$. It is shown in [7, p. 144] that the pair $(\mathbf{V}_h, M_h)$ satisfies the uniform inf-sup condition: there exists a constant β_* such that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \left\{ \left(\int_\Omega q_h \operatorname{div} \mathbf{v}_h \, d\mathbf{x} \right) / |\mathbf{v}_h|_{1,\Omega} \right\} \geq \beta_* \|q_h\|_{0,\Omega},\ \forall q_h \in M_h. \tag{2.4}$$

We are now going to define a discrete weak formulation corresponding to (2.1) using the continuous finite dimensional space $\mathbf{V}_h$ for velocity and the discontinuous finite dimensional space Q_h for pressure. On each triangle $K \in \mathcal{T}_h$ the discrete formulation of the momentum equation is usually defined by two bilinear forms

$$\begin{aligned} &a_K(\mathbf{w};\mathbf{u}_h,\mathbf{v}_h) \\ &= \mu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h)_K + \nu(\operatorname{div}\mathbf{u}_h,\operatorname{div}\mathbf{v}_h)_K + (\mathbf{w}\cdot\nabla\mathbf{u}_h,\mathbf{v}_h)_K, \end{aligned} \tag{2.5}$$

and $b_K(p_h, \mathbf{v}_h) = (p_h, \operatorname{div} \mathbf{v}_h)_K$. For defining a weak formulation of continuity (hyperbolic) equation, we consider for each $K \in \mathcal{T}_h$

$$\partial K_- = \{(x,y) \in \partial K : (\mathbf{w} \cdot \mathbf{n})(x,y) < 0\},$$
$$\partial K_+ = \{(x,y) \in \partial K : (\mathbf{w} \cdot \mathbf{n})(x,y) \geq 0\},$$

and define the left and right hand limits u^- and u^+ by

$$u^-(\mathbf{x}) = \lim_{s \to 0^-} u(\mathbf{x} + s\beta) \text{ and } u^+(\mathbf{x}) = \lim_{s \to 0^+} u(\mathbf{x} + s\beta).$$

For any $K \in \mathcal{T}_h$ we denote by $\mathcal{E}(K)$ the set of its edges and set

$$\mathcal{E}(\Omega) = \bigcup_{K \in \mathcal{T}_h} \mathcal{E}(K).$$

Now for edge $E \in \mathcal{E}(K)$ we have $E = E_- \cup E_+$ where $E_- = E \cap \partial K_-$ and $E_+ = E \cap \partial K_+$. Hence the integral on E is given by

$$\begin{aligned} \int_E \chi_h \mathbf{w} \cdot \mathbf{n} &= \int_{E_-} \chi_h^+ \mathbf{w} \cdot \mathbf{n} + \int_{E_+} \chi_h^- \mathbf{w} \cdot \mathbf{n} \\ &= -\int_{E_-} \chi_h^+ |\mathbf{w} \cdot \mathbf{n}| + \int_{E_+} \chi_h^- |\mathbf{w} \cdot \mathbf{n}|. \end{aligned}$$

In order to avoid such a situation as above one may assume that the ambient vector field $\mathbf{w}$ be only one sign on each edge $E \in \mathcal{E}(\Omega)$. Furthermore we known that the ordering of triangles for solving the discrete problem of the neutron transport equation based on the standard discontinuous Galerkin method can be arranged along the directions of the vector field $\mathbf{w}$ on the whole domain Ω, and so the discrete problem can be solved from triangle to triangle (see [2]). Thus, if a suitable condition on $\mathbf{w}$ is not imposed, then the ordering cannot be fixed (quite arbitrary) in one direction (e.g, from left to right if $\mathbf{w} = [U, V]$ with $U \geq 0$) because the directions of the vector field $\mathbf{w}$ cannot be predicted.

For $K \in \mathcal{T}_h$, given p_h^- on ∂K_-, we define a bilinear form c_K by

(2.6)

$$c_K(p_h, \chi_h) = \begin{cases} (\mathbf{w} \cdot \nabla p_h, \chi_h)_K - \displaystyle\int_{\partial K_-} [p_h]_J \, \chi_h \, \mathbf{w} \cdot \mathbf{n}, & \text{if } \partial K_- \cap \Gamma = \emptyset, \\ (\mathbf{w} \cdot \nabla p_h, \chi_h)_K, & \text{if } \partial K_- \cap \Gamma \neq \emptyset \end{cases}$$

where $[p_h]_J = p_h^+ - p_h^-$ is the jump across the side of triangle. Dislike the discontinuous Galerkin method of solving the neutron transport equation, note that the triangle sides are not required to be bounded away from the characteristic direction, i.e., $|\mathbf{w}\cdot\mathbf{n}| \geq c_* > 0$.

Now, using (2.5) and (2.6) the discrete weak formulation for (2.1) can be formulated by: For each $K \in \mathcal{T}_h$, given p_h^- on ∂K_- (if it exists), find $[\mathbf{u}_h, p_h]$ in $\mathbf{V}_h \times Q_h$ such that

$$(2.7)\qquad \begin{aligned} a_K(\mathbf{u}_h, \mathbf{v}_h) - b_K(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h)_K,\ \forall \mathbf{v}_h \in \mathcal{P}_2(K),\\ c_K(p_h, \chi_h) + b_K(\mathbf{u}_h, \chi_h) &= (g, \chi_h)_K,\ \forall \chi_h \in P_1(K). \end{aligned}$$

If the first component U of the ambient flow vector field $\mathbf{w}$ is assumed to be nonnegative, problem (2.7) can be considered from triangle to triangle as the discontinuous Galerkin method for solving the neutron transport equation approximately, but note that the finite element solution $\mathbf{u}_h$ is required to be continuous on the whole domain Ω. For simplicity, we set

$$(2.8)\qquad \begin{aligned} \mathcal{B}_K([\mathbf{u}_h, p_h], [\mathbf{v}_h, \chi_h]) &= a_K(\mathbf{w}; \mathbf{u}_h, \mathbf{v}_h) - b_K(\mathbf{v}_h, p_h)\\ &\quad + c_K(\mathbf{w}; p_h, \chi_h) + b_K(\mathbf{u}_h, \chi_h),\\ \Lambda_K[\mathbf{v}_h, \chi_h] &= (\mathbf{f}, \mathbf{v}_h)_K + (g, \chi_h)_K. \end{aligned}$$

In next lemma we globally show a unique existence of the finite element solution for (2.7). We let $\mu_0 = \min\{\mu, \mu+\nu\}$ and $\gamma_0 = \frac{1}{2}|\mathrm{div}\mathbf{w}|_\infty$, and shall denote $(\cdot,\cdot)$ the inner product in L^2 space and let $\bar{C}$ be the Poincaré constant with $\bar{C}\|u\|_0 \leq \|\nabla u\|_0$ for $u \in H_0^1(\Omega)$. We assume that the constant γ_0 satisfies a condition

$$(2.9)\qquad \gamma_0 < \min\{\mu_0 \bar{C}^{-2},\ \beta_*\},$$

where β_* is the constant in (2.4).

LEMMA 2.2. Assume that the condition (2.9) holds. Then problem (2.7) has a unique solution $[\mathbf{u}_h, p_h] \in \mathbf{V}_h \times Q_h$ and the solution satisfies the inequality

$$(2.10)\qquad \begin{aligned} &\mu\|\nabla\mathbf{u}_h\|_0^2 + \|p_h\|_0 + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K_-}(p_h^+ - p_h^-)^2|\mathbf{w}\cdot\mathbf{n}|\\ &\leq C(\|\mathbf{f}\|_0^2 + \|g\|_0^2), \end{aligned}$$

where C is a constant not depending on h.

Proof. For each triangle $K \in \mathcal{T}_h$, letting $\mathbf{v}_h = \mathbf{u}_h$ and $\chi_h = p_h$ in (2.8), one get

$$\begin{aligned}
&\mathcal{B}_K([\mathbf{u}_h, p_h], [\mathbf{u}_h, p_h]) \\
&= \mu\|\nabla \mathbf{u}_h\|_{0,K}^2 + \nu\|\operatorname{div}\mathbf{u}_h\|_{0,K}^2 + (\mathbf{u}_{h,\beta}, \mathbf{u}_h)_K \\
&\quad + (p_{h,\beta}, p_h)_K - \int_{\partial K_-} [p_h]_J\, p_h^+\, \mathbf{w}\cdot\mathbf{n} \\
&\geq \mu_0\|\nabla \mathbf{u}_h\|_{0,K}^2 + \frac{1}{2}\int_{\partial K_-} |\mathbf{u}_h|^2 \mathbf{w}\cdot\mathbf{n} - \frac{1}{2}\int_K |\mathbf{u}_h|^2 \operatorname{div}\mathbf{w}\,d\mathbf{x} \\
&\quad + \int_K \mathbf{w}\cdot\nabla p_h\, p_h\, d\mathbf{x} - \int_{\partial K_-} [p_h]_J\, p_h^+\, \mathbf{w}\cdot\mathbf{n}.
\end{aligned} \tag{2.11}$$

But the bilinear form $c_K(\mathbf{w}; p_h, p_h)$ can be estimated by

$$\begin{aligned}
&\int_K \mathbf{w}\cdot\nabla p_h\, p_h\, d\mathbf{x} - \int_{\partial K_-} [p_h]_J\, p_h^+\, \mathbf{w}\cdot\mathbf{n} \\
&= \frac{1}{2}\int_{\partial K} p_h^2\, \mathbf{w}\cdot\mathbf{n} - \int_{\partial K_-} [p_h]_J\, p_h^+\, \mathbf{w}\cdot\mathbf{n} - \frac{1}{2}\int_K p_h^2 \operatorname{div}\mathbf{w}\, d\mathbf{x} \\
&= -\frac{1}{2}\int_{\partial K_-} (p_h^+)^2 |\mathbf{w}\cdot\mathbf{n}| + \frac{1}{2}\int_{\partial K_+} (p_h^-)^2 |\mathbf{w}\cdot\mathbf{n}| + \int_{\partial K_-} (p_h^+)^2 |\mathbf{w}\cdot\mathbf{n}| \\
&\quad - \int_{\partial K_-} p_h^+ p_h^- |\mathbf{w}\cdot\mathbf{n}| - \frac{1}{2}\int_K p_h^2 \operatorname{div}\mathbf{w}\, d\mathbf{x} \\
&= \frac{1}{2}\oint (p_h^-)^2 \mathbf{w}\cdot\mathbf{n} + \frac{1}{2}\int_{\partial K_-} (p_h^+ - p_h^-)^2 |\mathbf{w}\cdot\mathbf{n}| - \frac{1}{2}\int_K p_h^2 \operatorname{div}\mathbf{w}\, d\mathbf{x}.
\end{aligned} \tag{2.12}$$

Thus it follows from (2.11)-(2.12) that

$$\begin{aligned}
&\mathcal{B}_K([\mathbf{u}_h, p_h], [\mathbf{u}_h, p_h]) \\
&\geq \mu_0\|\nabla \mathbf{u}_h\|_{0,K}^2 + \frac{1}{2}\int_{\partial K} |\mathbf{u}_h|^2 \mathbf{w}\cdot\mathbf{n} - \frac{1}{2}\int_K |\mathbf{u}_h|^2 \operatorname{div}\mathbf{w}\, d\mathbf{x} \\
&\quad + \frac{1}{2}\oint_{\partial K} (p_h^-)^2 \mathbf{w}\cdot\mathbf{n} + \frac{1}{2}\int_{\partial K_-} (p_h^+ - p_h^-)^2 |\mathbf{w}\cdot\mathbf{n}| - \frac{1}{2}\int_K p_h^2 \operatorname{div}\mathbf{w}\, d\mathbf{x}.
\end{aligned}$$

Now, summing both sides of (2.13) over all triangles $K \in \mathcal{T}_h$, one has

$$\begin{aligned}
&\mathcal{B}([\mathbf{u}_h, p_h], [\mathbf{u}_h, p_h]) \\
&\geq \sum_{K\in\mathcal{T}_h} (\mu_0 \|\nabla \mathbf{u}_h\|_{0,K}^2 - \gamma_0 \|\mathbf{u}_h\|_{0,K}^2) \\
&+\frac{1}{2}\sum_{K\in\mathcal{T}_h} \int_{\partial K_-} [p_h]_J^2 |\mathbf{w}\cdot\mathbf{n}| - \gamma_0 \|p_h\|_{0,\Omega}^2 \qquad (2.13)\\
&\geq \mu_* \|\nabla \mathbf{u}_h\|_{0,\Omega}^2 + \frac{1}{2}\sum_{K\in\mathcal{T}_h} \int_{\partial K_-} [p_h]_J^2 |\mathbf{w}\cdot\mathbf{n}| - \gamma_0 \|p_h\|_{0,\Omega}^2
\end{aligned}$$

where $\mu_* = \mu_0 - \gamma_0 \bar{C}^2$. Next, computing $\Lambda_K[\mathbf{v}_h, \chi_h]$ of (2.8) and summing over all triangles and combining with (2.14) one obtains

$$\begin{aligned}
&\mu_* \|\nabla \mathbf{u}_h\|_{0,\Omega}^2 + \frac{1}{2}\sum_{K\in\mathcal{T}_h} \int_{\partial K_-} [p_h]^2 |\mathbf{w}\cdot\mathbf{n}| - \gamma_0 \|p_h\|_{0,\Omega}^2 \\
&\leq C(\|\mathbf{f}\|_{0,\Omega}^2 + \|g\|_{0,\Omega}^2). \qquad (2.14)
\end{aligned}$$

Thus using (2.15) and (2.4) and our assumption, the required inequality easily follows. A standard argument by Lax Milgram lemma shows a unique existence of solution of (2.7). □

REMARK. In showing Lemma 2.1, except that $\mathbf{w}\cdot\mathbf{n} < 0$ on ∂K_- and $\mathbf{w}\cdot\mathbf{n} \geq 0$ on ∂K_+ and $\mathbf{w}|_\Gamma = 0$ and the condition (2.9), any extra condition on $\mathbf{w}$ was not imposed.

THEOREM 2.3. Assume that the same condition in Lemma 2.1 holds. Then there is a constant K, not depending on h such that

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_1 + \|p-p_h\|_0 + \Big(\sum_{K\in\mathcal{T}_h} \int_{\partial K_-} (p_h^+ - p_h^-)^2 |\mathbf{w}\cdot\mathbf{n}|\Big)^{1/2} \qquad (2.15)\\
&\leq K \inf \Big\{ \|\nabla(\mathbf{u}-\hat{\mathbf{u}})\|_0 + \|p-\hat{p}\|_0 + \Big(\sum_{K\in\mathcal{T}_h} \int_{\partial K_-} (p-\hat{p}^-)^2\Big)^{1/2} \Big\} \\
&+Ch^{l-1}\|p\|_{l,\Omega}, \ (l \geq 1),
\end{aligned}$$

where the infimum is taken over all $\hat{\mathbf{u}} \in \mathbf{V}_h$ and $\hat{p} \in Q_h$.

Proof. Noting that $\mathbf{V}_h \subset (H_0^1(\Omega))^2$ and $M_h \subset L_0^2(\Omega)$, and subtracting (2.1) from (2.7), we have

$$\begin{aligned}(2.16)\qquad & \sum_{K\in\mathcal{T}_h} \mathcal{B}_K([\mathbf{u}_h-\hat{\mathbf{u}}, p_h-\hat{p}],[\mathbf{v}_h,\chi_h]) \\ &= \sum_{K\in\mathcal{T}_h} \mathcal{B}_K([\mathbf{u}-\hat{\mathbf{u}}, p-\hat{p}],[\mathbf{v}_h,\chi_h]),\end{aligned}$$

where $\hat{\mathbf{u}}$ and $\hat{p}$ are arbitrary elements in $\mathbf{V}_h$ and M_h respectively. Now taking $\mathbf{v}_h = \mathbf{u}_h - \hat{\mathbf{u}}$ and $\chi_h = p_h - \hat{p}$ in (2.17) and using the definition (2.8) of $\mathcal{B}_K$, one can estimate the right hand side of (2.17) as follows:

$$\begin{aligned}
& c_K(p-\hat{p}, p_h-\hat{p}) \\
&= \int_K (p-\hat{p})_\beta (p_h-\hat{p})\,d\mathbf{x} - \int_{\partial K_-} [p-\hat{p}]_J\,(p_h-\hat{p})^+ \mathbf{w}\cdot\mathbf{n} \\
&\qquad (\text{ using the integration by parts }) \\
&= -(p-\hat{p},(p_h-\hat{p})_\beta)_K - \int_K (p_h-\hat{p})(p-\hat{p})\,\mathrm{div}\,\mathbf{w}\,d\mathbf{x} \\
(2.17)\qquad &\quad + \int_{\partial K_+} (p-\hat{p})^-(p_h-\hat{p})^-\,\mathbf{w}\cdot\mathbf{n} + \int_{\partial K_-} (p-\hat{p})^-(p_h-\hat{p})^+\,\mathbf{w}\cdot\mathbf{n} \\
&= -(p-\hat{p},(p_h-\hat{p})_\beta)_K - \int_K (p_h-\hat{p})(p-\hat{p})\,\mathrm{div}\,\mathbf{w}\,d\mathbf{x} \\
&\quad + \int_{\partial K_-} (p-\hat{p})^-[p_h-\hat{p}]_J\,\mathbf{w}\cdot\mathbf{n} + \oint_{\partial K} (p-\hat{p})^-(p_h-\hat{p})^-\,\mathbf{w}\cdot\mathbf{n}
\end{aligned}$$

Now the first and second terms in (2.18) are estimated by

$$\begin{aligned}
& -(p-\hat{p},(p_h-\hat{p})_\beta)_K - \int_K (p_h-\hat{p})(p-\hat{p})\,\mathrm{div}\,\mathbf{w}\,d\mathbf{x} \\
&\le \|\mathbf{w}\|_{1,\infty}\|p_h-\hat{p}\|_{1,K}\|p-\hat{p}\|_{0,K} \\
&\le Ch^l\|p\|_{l,K}\|p_h-\hat{p}\|_{1,K} \\
(2.18)\qquad &\quad (\text{ using the inverse inequality } \|\nabla u_h\|_{0,K} \le Ch^{-1}\|u_h\|_{0,K}\ ) \\
&\le Ch^{l-1}\|p\|_{l,K}\|p_h-\hat{p}\|_{0,K} \\
&\le \epsilon\|p_h-\hat{p}\|_{0,K}^2 + C(\epsilon)h^{2(l-1)}\|p\|_{l,K}^2,\ \forall\epsilon>0.
\end{aligned}$$

Next the third term of (2.18) is estimated by

$$\tag{2.19}\begin{aligned}\int_{\partial K_-}&(p-\hat{p})^-[p_h-\hat{p}]_J\,\mathbf{w}\cdot\mathbf{n}\\ &\le \epsilon\int_{\partial K_-}[p_h-\hat{p}]_J^2|\mathbf{w}\cdot\mathbf{n}|^2 + C(\epsilon)\int_{\partial K_-}|(p-\hat{p})^-|^2,\ \forall\epsilon>0.\end{aligned}$$

Hence, combining (2.18)-(2.20), one has

$$\tag{2.20}\begin{aligned}c_K(p-\hat{p},p_h-\hat{p}) &\le \epsilon\|p_h-\hat{p}\|_{0,K}^2+\delta\int_{\partial K_-}[p_h-\hat{p}]_J^2\,|\mathbf{w}\cdot\mathbf{n}|\\ &+\ C(\epsilon)\,h^{2(l-1)}\|p\|_{l,K}^2+C(\delta)\int_{\partial K_-}|(p-\hat{p})^-|^2\\ &+\ \oint_{\partial K}(p-\hat{p})^-(p_h-\hat{p})^-\mathbf{w}\cdot\mathbf{n},\ \forall\epsilon>0,\forall\delta>0.\end{aligned}$$

Furthermore we have

$$\tag{2.21}\begin{aligned}&b_K(\mathbf{u}-\hat{\mathbf{u}},p_h-\hat{p})-b_K(\mathbf{u}_h-\hat{\mathbf{u}},p-\hat{p})\\ &\le \epsilon\|p_h-\hat{p}\|_{0,K}^2+\delta\|\nabla(\mathbf{u}_h-\hat{\mathbf{u}})\|_{0,K}^2\\ &\quad +C(\|p-\hat{p}\|_{0,K}+\|\nabla(\mathbf{u}-\hat{\mathbf{u}})\|_{0,K}),\ \forall\epsilon>0,\ \forall\delta>0,\end{aligned}$$

$$\tag{2.22}\begin{aligned}&a_K(\mathbf{u}-\hat{\mathbf{u}},\mathbf{u}_h-\hat{\mathbf{u}})\\ &\le \epsilon\|\nabla(\mathbf{u}_h-\hat{\mathbf{u}})\|_{0,K}^2+C(\epsilon)\|\nabla(\mathbf{u}-\hat{\mathbf{u}})\|_{0,K}^2,\ \forall\epsilon>0.\end{aligned}$$

Thus putting together (2.21)-(2.23), and the right hand side of (2.17) is estimated by

$$\tag{2.23}\begin{aligned}&\mathcal{B}_K([\mathbf{u}-\hat{\mathbf{u}},p-\hat{p}],[\mathbf{u}_h-\hat{\mathbf{u}},p_h-\hat{p}])\\ &\le\ \epsilon\|\nabla(\mathbf{u}_h-\hat{\mathbf{u}})\|_{0,K}^2+\delta\|p_h-\hat{p}\|_{0,K}^2+\delta_1\int_{\partial K_-}[p_h-\hat{p}]_J^2|\mathbf{w}\cdot\mathbf{n}|\\ &\quad+\oint_{\partial K}(p-\hat{p})^-(p_h-\hat{p})^-\mathbf{w}\cdot\mathbf{n}\\ &\quad+C(\epsilon,\delta,\delta_1,|\mathbf{w}|_{1,\infty})\Big(\|\nabla(\mathbf{u}-\hat{\mathbf{u}})\|_{0,K}^2+\int_{\partial K_-}|(p-\hat{p})^-|^2\Big)\\ &\quad+C(\delta)\,h^{2(l-1)}\|p\|_{l,K},\ \forall\epsilon>0,\ \forall\delta>0,\ \forall\delta_1>0.\end{aligned}$$

Furthermore the left hand side of (2.17) is bounded below in the following ways:

$$\begin{aligned} &\mathcal{B}_K([\mathbf{u}_h - \hat{\mathbf{u}}, p_h - \hat{p}], [\mathbf{u}_h - \hat{\mathbf{u}}, p_h - \hat{p}]) \\ &\geq \mu_0 \|\nabla(\mathbf{u}_h - \hat{\mathbf{u}})\|_{0,K}^2 + \frac{1}{2}\oint_{\partial K} |(\mathbf{u}_h - \hat{\mathbf{u}})^-|^2 \, \mathbf{w}\cdot\mathbf{n} \\ &-\frac{1}{2}\int_K |\mathbf{u}_h - \hat{\mathbf{u}}|^2 \operatorname{div}\mathbf{w}\,d\mathbf{x} + \frac{1}{2}\oint_{\partial K} |(p_h - \hat{p})^-|^2 \, \mathbf{w}\cdot\mathbf{n} \\ &+\frac{1}{2}\int_{\partial K_-} [p_h - \hat{p}]_J^2 |\mathbf{w}\cdot\mathbf{n}| - \frac{1}{2}\int_K (p_h - \hat{p})^2 \operatorname{div}\mathbf{w}\,d\mathbf{x}. \end{aligned} \tag{2.24}$$

Since $b(p_h - \hat{p}, \mathbf{v}_h) = a(\mathbf{u}_h - \hat{\mathbf{u}}, \mathbf{v}_h) - a(\mathbf{u} - \hat{\mathbf{u}}, \mathbf{v}_h) + b(p - \hat{p}, \mathbf{v}_h)$, and using (2.4), one has

$$\begin{aligned} &\beta_* \|p_h - \hat{p}\|_{0,\Omega} \\ &\leq C\|\nabla(\mathbf{u}_h - \hat{\mathbf{u}})\|_{0,\Omega} + C(\|\nabla(\mathbf{u} - \hat{\mathbf{u}})\|_{0,K} + \|p - \hat{p}\|_{0,\Omega}). \end{aligned} \tag{2.25}$$

Finally, noting that

$$\sum_{K\in\mathcal{T}_h} \oint_{\partial K} p^- \chi^- \, \mathbf{w}\cdot\mathbf{n} = 0$$

and combining (2.24)-(2.25) and summing its resulted inequality over all triangles, one has

$$\begin{aligned} &\mu_0 \|\nabla(\mathbf{u}_h - \hat{\mathbf{u}})\|_{0,\Omega}^2 + \frac{1}{2}\sum_{K\in\mathcal{T}_h} \int_{\partial K_-} [p_h - \hat{p}]^2 |\mathbf{w}\cdot\mathbf{n}| \\ &\leq \epsilon \|\nabla(\mathbf{u}_h - \hat{\mathbf{u}})\|_{0,\Omega}^2 + (\gamma_0 + \delta)\|p_h - \hat{p}\|_{0,\Omega}^2 \\ &+ \delta_* \sum_{K\in\mathcal{T}_h} \int_{\partial K_-} [p_h - \hat{p}]_J^2 |\mathbf{w}\cdot\mathbf{n}| \\ &+ C\Big(\|\nabla(\mathbf{u} - \hat{\mathbf{u}})\|_{0,\Omega}^2 + \|p - \hat{p}\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \int_{\partial K_-} |(p - \hat{p})^-|^2\Big) \\ &+ C\,h^{2(l-1)}\|p\|_{l,K}, \ \forall\epsilon > 0, \ \forall\delta > 0, \ \forall\delta_* > 0. \end{aligned} \tag{2.26}$$

Finally using (2.26)-(2.27) and the triangle inequality, and our assumption, the inequality (2.16) easily follows since $\hat{\mathbf{u}} \in \mathbf{V}_h$ and $\hat{p} \in M_h$ were arbitrary. □

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