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MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE ALMOST EVERYWHERE

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ABSTRACT. Under the condition of $\mathrm{Ric}_M \geq -(n-1)k$, $\mathrm{inj}_M \geq i_0$, we prove the existence of an $\epsilon > 0$ such that on the region of volume $\epsilon > 0$ the curvature condition of splitting theorem can be weakened.

1. Introduction

It is an important problem in Riemannian geometry to classify the complete Riemannian manifolds by curvature conditions. Splitting theorems are concerned about the non-compact manifolds with nonnegative curvature. Using splitting theorem, we can prove that some finite cover of a compact Riemannian manifold with nonnegative Ricci curvature can be splitted to $T^k \times N$, where N is a compact simply connected manifold and T^k is a k-torus [3]. In [6, 8], we prove sphere theorems under weaker curvature conditions than the standard Ricci curvature condition $\mathrm{Ric}_M \geq n-1$, i.e., Ricci curvature and injectivity radius bounded below and $\mathrm{Ric}_M \geq n-1$ on $M - A$ where diameter of A, $\mathrm{diam}(A)$ or volume of A, $\mathrm{vol}(A)$ are sufficiently small. Similarly to sphere theorems, we will prove a splitting theorem of compact space also holds even if the Ricci curvature conditions are not satisfied on the region of small volume.

Let $\mathcal{M}^n_{i_0,k}$ be the class of n-dimensional complete Riemannian manifolds with $\mathrm{Ric}_M \geq -(n-1)k$ and the injectivity radius $\mathrm{inj}_M \geq i_0$. We obtain the following theorem;

THEOREM 1.1. *Let $M \in \mathcal{M}^n_{i_0,k}$ and $\mathrm{diam}(M) \leq d$. Then there exists an $\epsilon > 0$ depending only on i_0, n, k, d such that if* $\mathrm{Ric}_M \geq -(n-$

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1)ϵ on $M - A$, $\mathrm{vol}(A) \leq \epsilon$, then M is diffeomorphic to $T^k \times N$ up to finite cover, where N is a simply connected space.

This theorem is a generalization of the following theorem due to Cai [2].

THEOREM 1.2. *Let $M \in \mathcal{M}(i_0, n, k)$ and $\mathrm{diam}(M) \leq d$. Then there exists an $\epsilon(i_0, n, k, d) > 0$ depending only on n, i_0, k, d such that if $\mathrm{Ric}_M \geq -(n-1)\epsilon$, then M is diffeomorphic to $T^k \times N$ up to finite cover, where N is a simply connected space.*

We use the compactness theorem due to Anderson and Cheeger [1] and the rigidity result in a fixed small ball [7, 4]. The following notation will be used; if we fix $\delta_1, \cdots, \delta_n$, then $\tau(\delta_1, \cdots, \delta_n|\epsilon) \to 0$ and $\tau(\epsilon) \to 0$ as $\epsilon \to 0$.

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2. Preliminaries

In Preliminaries, we show that the conditions of $\mathrm{Ric}_M \geq -(n-1)k$ and $\mathrm{inj}_M \geq i_0$ make exponential map almost isometric on a ball of fixed small radius which depends only on n, k, i_0.

By Brocks' estimate on the Laplacian of the distance function, we obtain the following Jacobi field estimate on $i_0/2$-ball $B(p, i_0/2)$ [4, 7, 8]. Let M be a complete Riemannian manifold with $\mathrm{Ric}_M \geq -(n-1)k$, $\mathrm{inj}_M \geq i_0$ and $\gamma(t)$ be a minimal geodesic starting from p and $Y(t)$ is a Jacobi field along γ such that $Y(0) = 0, \langle Y'(0), \gamma'(0)\rangle = 0$. Let $d(\gamma)$ be the distance from p to the cut point on γ. Then $i_0 \leq d(\gamma)$. Define $A := \nabla\nabla r = \text{Hess } r$, so $\mathrm{tr}A = \triangle r$ and $Y' = AY$. Write $A(t) = B(t) + I/t$. We know

$$\int_0^r ||B|| \leq D(i_0, n, k) r^{1/2},$$

for some constant $D(i_0, n, k)$ [4, 7]. Then we have

$$e^{-Dr^{1/2}} r||Y'(0)|| \leq ||Y||(r) \leq e^{Dr^{1/2}} r||Y'||(0)$$

for some constant $D = D(i_0, n, k)$ if $r < i_0/2$ [4, 7, 8]. In Euclidean space, we know that $D = 0$. For any $\epsilon > 0$, we can choose a uniform

$r_0 > 0$ which is depending only on n, k, i_0 such that $Dr^{1/2} < \epsilon$ for $r < r_0$. Then

$$\begin{aligned} 1-\epsilon &\le \min\left\{\frac{||d\exp(v)||}{||v||} \mid v \in T_xM,\ x \in B(p,r_0)\right\} \\ &\le \max\left\{\frac{||d\exp(v)||}{||v||} \mid v \in T_xM,\ x \in B(p,r_0)\right\} \le 1+\epsilon \end{aligned}$$

on r_0-ball by the above inequality. So we can find a uniform $r_0 > 0$ such that the exponential map is almost isometric on r_0-ball.

Furthermore, we know that

$$-(n-1)k\coth k(d(\gamma)-t) \le \Delta r(\gamma(t)) \le (n-1)k\coth kt. \tag{2.1}$$

For the proof, see [1]. Then by the Riccati equation, we get the following estimate for $r \le d(\gamma) - \delta$,

$$||A||^2 \le -\mathrm{tr}A' - \mathrm{Ric}_M.$$

Then

$$\begin{aligned} &\int_{i_0/2}^{r} ||A||^2 \\ &\le -\mathrm{tr}A(r) + \mathrm{tr}A(i_0/2) + (n-1)kd(\gamma) \\ &\le (n-1)k\coth k(d(\gamma)-r) + C(i_0,n,k) + \frac{2(n-1)}{i_0} + (n-1)kd(\gamma) \\ &\le (n-1)k\coth k\delta + C(i_0,n,k) + \frac{2(n-1)}{i_0} + (n-1)kd(\gamma) \\ &\le F(i_0,n,k,\delta,d(\gamma)), \end{aligned}$$

for some constant F depending only on $i_0, n, k, \delta, d(\gamma)$ since $\coth x$ is decreasing function. Then

$$\begin{aligned} \int_{i_0/2}^{r} ||A|| &\le (d(\gamma)\int_{i_0/2}^{r} ||A||^2))^{1/2} \\ &\le \sqrt{d(\gamma)}F(i_0,n,k,\delta,d(\gamma))^{1/2} = F_1(i_0,n,k,\delta,d(\gamma)), \end{aligned}$$

for some constant F_1.

$$\begin{aligned}\int_0^r \|B\| &= \int_0^{i_0/2} + \int_{i_0/2}^r \|B\| \\ &\leq D(i_0,n,k)\sqrt{i_0/2} + \int_{i_0/2}^r (\|A\| + \frac{1}{t}) \\ &\leq D(i_0,n,k)\sqrt{i_0/2} + F_1(i_0,n,k,\delta,d(\gamma)) + \frac{2}{i_0} \\ &\leq F_2(i_0,n,k,\delta,d(\gamma)),\end{aligned}$$

for some constant F_2. Then by the same method as above, we also get a uniform lower bound and upper bound of the $\|Y\|$ depending only on $n, k, i_0, \delta, d(\gamma)$ on $t < d(\gamma) - \delta$.

3. Estimate of the volume of the bad part

For the simplicity of argument, we only consider the sequence (M_j, g_j) in $\mathcal{M}^n_{i_0,k}$ satisfying the conditions $\mathrm{diam}(M_j) \leq d$, $\mathrm{Ric}_{M_j} \geq 0$ on $M_j - A_j$ where $\mathrm{vol}(A_j) \leq \epsilon_j$ and $\epsilon_j \to 0$. Then we know that the universal covering space of M_j, $\tilde{M}_j$ converges to a compact subset of C^α-Riemannian manifold X on compact subset. Shortly we use A_j instead of the lifting of A_j, $\tilde{A}_j$. Fix a point $p_j \in M_j$. We may assume $p_j \in \tilde{M}_j$. Let $\gamma_\theta(t) = \exp_{p_j} t\theta$ and μ is the measure on γ_θ. We use the following notations;

$$\begin{aligned} d^j(\theta) = &\text{ the distance from } p_j \text{ to the cut point in the direction } \theta, \\ &\text{where } \theta \in S^{n-1} \subset T_{p_j}M_j, \\ \Theta^j_\epsilon &= \{\theta \in S^{n-1} \subset T_{p_j}M_j \mid \mu(\gamma_\theta([0, d^j(\theta)]) \cap A_j) < \epsilon\}, \\ S^j_{\epsilon,\delta}(\theta) &= \inf\{s \mid s > \delta,\ \theta \in (\Theta^j_\epsilon)^c,\ \mu(\gamma_\theta([\delta, s]) \cap A_j) > \epsilon\}. \end{aligned}$$

Lemma 3.1. *For any fixed* $D > 0$, $\lim_{j\to\infty} \mathrm{vol}(\{\exp_{p_j} t\theta \mid \theta \in (\Theta^j_\epsilon)^c,\ S^j_{\epsilon,\delta}(\theta) \leq t < \min(d^j(\theta), D)\}) = 0$.

We can consider $\{\exp_{p_j} t\theta \mid \theta \in (\Theta^j_\epsilon)^c,\ S^j_{\epsilon,\delta}(\theta) \leq t < d^j(\theta)\}$ as a bad part for applying the Bishop-Gromov theorem. We want to show that this bad part can be ignored.

Proof. Assume that M is an element of $\mathcal{M}^n_{i_0,k}$. Let $\{u, e_1, \cdots, e_{n-1}\}$ be an orthonormal basis for the tangent space at some point $q \in M$ and let $Y_i(t)$ be a Jacobi field along $\gamma_\theta(t) = \exp t\theta$ such that $Y_i(0) = 0, Y_i'(0) = e_i$ for $i = 1, \cdots, n-1$.

Define

$$J(u,t) = \begin{cases} t^{-(n-1)}(\det g(Y_i, Y_j))^{\frac{1}{2}} & \text{if } t \leq d^j(\theta) \\ 0 & \text{if } t > d^j(\theta). \end{cases}$$

Then for a region A in the unit sphere of the tangent space T_qM,

$$\mathrm{vol}\{\exp_q t\theta \mid t_1 < t < t_2,\ \theta \in A\} = \int_A \int_{t_1}^{t_2} J(u,t) t^{n-1} dt du.$$

Let J^j and J_{-k} be the J of M_j and the space form with constant curvature $-k$, respectively and $b^j(u,t) = J^j(u,t)^{\frac{1}{n-1}} t$, $\bar{b}(u,t) = J_{-k}(u,t)^{\frac{1}{n-1}} t$. In the proof of the Bishop-Gromov inequality, we see that $\dfrac{b(u,r)}{b(u,a)} \leq \dfrac{\bar{b}(u,r)}{\bar{b}(u,a)}$, if $r > a$. Simply, we use J and b instead of J^j and b^j, respectively. We define

$$C_{t_1}^{t_2} := \max\left\{ \frac{\bar{b}(u,r)}{\bar{b}(u,s)} \mid r, s \in [t_1, t_2] \right\}.$$

If $\mathrm{vol}((\Theta^j_\epsilon)^c) \to 0$ then there are nothing to prove by the Bishop-Gromov theorem. So we assume that $\lim_{j\to\infty} \mathrm{vol}((\Theta^j_\epsilon)^c) > 0$. Let

$$\Phi^j_\epsilon := \{\theta \in (\Theta^j_\epsilon)^c \mid \int_{A_j \cap \gamma_\theta} b^{n-1}(\theta, r) > \epsilon_j^{1/2}\}.$$

Then $\mathrm{vol}(A_j) > \epsilon_j^{1/2} \mathrm{vol}(\Phi^j_\epsilon)$. So $\mathrm{vol}(\Phi^j_\epsilon) < \epsilon_j^{1/2} \to 0$ as $j \to \infty$ which is a contradiction.

Now we may assume that for every direction $\theta \in (\Theta^j_\epsilon)^c$,

$$\int_{\gamma_\theta \cap A_j} b^{n-1}(\theta, r) < \epsilon_j^{1/2}.$$

Then we know that for any $\epsilon > 0$, there exists a $c > \delta$ such that $b^{n-1}(\theta, c) < \frac{\epsilon_j^{1/2}}{\epsilon}$ and $\mu(\gamma_\theta[\delta, c] \cap A_j) < \epsilon$. From this fact, we know that $c \leq S^j_{\epsilon,\delta}$. Then by the above inequality, we get

$$b^{n-1}(\theta, r) \leq (C^D_\delta)^{n-1} b^{n-1}(\theta, c) \leq (C^D_\delta)^{n-1} \frac{\epsilon_j^{1/2}}{\epsilon} \tag{3.1}$$

for $S^j_{\epsilon,\delta} < r < D$. So $b^{n-1}(\theta, r) < \tau(\delta, \epsilon|\epsilon_j)$ for $r > S^j_{\epsilon,\delta}$.

Let

$$\tilde{A}_j(D)(\delta, \epsilon) := \{\exp t\theta \mid \theta \in (\Theta^j_\epsilon)^c,\ S^j_{\epsilon,\delta} < t < D\}.$$

Consequently, $\mathrm{vol}(\tilde{A}_j(D))(\delta, \epsilon) = \int_{(\Theta^j_\epsilon)^c} \int_{S^j_{\epsilon,\delta}}^D b^{n-1}(\theta, t) dt d\theta \to 0$ as $j \to \infty$. This completes the proof. □

REMARK 3.2. By the above proof, we know that if $\rho_j = \epsilon_j^{1/4}$, then $\rho_j \to 0$ and

$$\lim_{j \to \infty} \mathrm{vol}(\{\exp_{p_j} t\theta \mid \theta \in (\Theta^j_{\rho_j})^c,\ S^j_{\rho_j,\delta}(\theta) \leq t < \min(D, d^j(\theta))\}) = 0.$$

This value ρ_j will be used in following sections.

4. Proof of Theorem

We will prove that the limit space $X = R^k \times N$ where N contains no line. Then we can prove Theorem 1.1 by a contradiction.

Shortly, we use A_i instead of the lifting of A_i, $\tilde{A}_i$. Let γ_i be a line in $\tilde{M}_i$ and $\gamma_i(0) = p_i$. Then γ_i converge to γ in X and we may assume that $p_i \to p$. We follow the proof in [9].

We know that $\mathrm{vol}(M_i) \geq v_1(i_0, n, k)$ and $\mathrm{vol}(B(\gamma_i(t_i), 2t_i)) \leq v_2(n, k, t_i)$ for some v_1, v_2. Then the number of fundamental domains in $B(\gamma_i(t_i), 2t_i)$ is bounded by $n_i = n_1(i_0, n, k, t_i) := v_2/v_1$. Let

$$\rho_i := (n_i \epsilon_i)^{1/4}$$

as Remark 3.2 where t_i will be chosen below. Let $\gamma_{\theta_x}(0) = x$. Now we use the similar notation as previous section;

$$\Theta^i(x) = \{\theta_x \in S^{n-1} \subset T_x\tilde{M}_i \mid \mu(\gamma_{\theta_x}([0, d^i(\theta_x)]) \cap A_i) < \rho_i\},$$
$$S^i_\delta(\theta_x) = \inf\{s \mid s > \delta,\ \theta_x \in (\Theta^i(x))^c,\ \mu(\gamma_{\theta_x}([\delta, s]) \cap A_i) > \rho_i\}.$$

We use Θ^i instead of $\Theta^i(\gamma_i(t_i))$.

Let $R_i(\delta)$ be a δ-tubular neighborhood of the cut locus of $\gamma_i(t_i)$. We also know that the volume form $b_{\gamma_i(t_i)}(t)$ has a uniform lower bound $H_0(i_0, n, k, \delta, t) > 0$ on $R_i(\delta)^c$ as we see in Preliminaries.

Put D to be a bounded domain with smooth boundary in X and F_i be a diffeomorphism from D to a domain D_i in M_i. We consider D_i as D with a metric $F_i^* g_i$ where g_i is the metric on M_i. Define

$$B_i^{\pm}(x) := d(\gamma_i(\pm t_i), x) \mp t_i,$$

and

$$B^{\pm}(x) = \lim_{i \to \infty} B_i^{\pm}(x).$$

It is an essential part of the proof of Theorem 1.1 that $\triangle B^+ \equiv 0$, i.e., B^+ is a harmonic function. If we assume the almost nonnegativity of Ricci curvature, we need not check that B^+ is a harmonic function [2, 10]. But in our case, we must show that B^+ is harmonic.

Now we choose t_i such that

$$n_1(i_0, n, k, t_i)\epsilon_i \to 0,$$

$$\rho_i e^{10(n-1)kt_i} \to 0,$$

$$\frac{\rho_i e^{10(n-1)kt_i}}{H_0(i_0, n, k, \delta, t_i)} \to 0$$

and $t_i \to \infty$ as $i \to \infty$. It is possible to find such t_i by passing to a subsequence if necessary since we know $\epsilon_i \to 0$. We will show that the limit of B_i^+ has properties as Busemann function. Let $d^i(\theta)$ be the distance from $\gamma_i(t_i)$ to the cut point in the direction $\theta \in S^{n-1} \subset T_{\gamma_i(t_i)}M_i$.

Then we also know that $D_i \subset B(\gamma_i(t_i), 2t_i)$ as $i \to \infty$ and the volume of bad part

$$W_i = \{\exp_{\gamma_i(t_i)} t\theta \mid \theta \in (\Theta^i)^c,\ S^i_\delta(\theta) \le t < \min(d^i(\theta), 2t_i)\}$$

converges to 0 as $i \to \infty$ by the choice of t_i, i.e., γ_i meets at most $n_1(i_0, n, k, t_i)$ fundamental domains so we may consider the volume of lifting of A_i as $n_1(i_0, n, k, t_i)\epsilon_i \to 0$. Precisely we get that for some constant $H_1(n, k, \delta)$

$$(4.1) \qquad b^{n-1}_{\gamma_i(t_i)}(r) \le (C^{2t_i}_\delta)^{n-1}\frac{(n_i\epsilon_i)^{1/2}}{\rho_i} \le H_1(n, k, \delta)\rho_i e^{2(n-1)kt_i},$$

as (3.1) since the exponential growth rate for C^r_δ is less than $(n-1)k$. Then

$$\mathrm{vol}(W_i) = \int_{(\Theta^i)^c}\int_0^{2t_i} b^{n-1}_{\gamma_i(t_i)} \le \omega_{n-1}H_1(n, k, \delta)\rho_i t_i e^{2(n-1)kt_i} \to 0,$$

by the choice of t_i, where ω_{n-1} is the volume of the standard $(n-1)$-sphere. If i is sufficiently large, $\triangle B^+_i(x)$ has an upper bound $(n-1)\coth kd(\gamma_i(t_i), x) \le H_2(n, k)$ for some constant $H_2(n, k)$. Since the volume of W_i converges to 0, we may consider the integration only on $\{\exp_{\gamma_i(t_i)}(t\Theta^i)\}$ for computing the upper bound of $\lim_{i\to\infty}\int_{D_i}\triangle B^+_i$.

Let $\gamma_\theta \subset M_i$ be a geodesic from q in direction θ and $\Phi : [0, \infty) \times S^{n-1} \to M_i$ such that $\Phi(r, \theta) = \exp_q(r\theta)$. Set $\Phi^* v_{g_i} = a(\theta, r)drd\theta$ and $b = a^{\frac{1}{n-1}}$, where v_{g_i} is the volume form of M_i. Then we know that

$$b'' + \frac{\mathrm{Ric}(\gamma', \gamma')b}{n-1} \le 0.$$

For the proof, see [5]. In the case of the space of constant curvatures, the equality holds. So for R^n,

$$\bar{b}'' = 0.$$

In this section, we only consider the direction $\theta \in \Theta^i$. If $\gamma_\theta(r) \in A^c_i$, then

$$(b''\bar{b} - \bar{b}''b)(r) = (b''\bar{b})(r) \le 0,$$

and if $\gamma_\theta(r) \in A_i$ and $r \le d_i(\theta)$ then

$$(4.2)\qquad \begin{aligned}(b''\bar{b} - \bar{b}''b)(r) &= ((b''+b)\bar{b})(r) \le (k+1)b(r)\bar{b}(r)\\ &\le C_1(i_0,n,k)e^{2(n-1)kr},\end{aligned}$$

for some constant C_1 depending only on i_0, n, k. From this, we get

$$(b'\bar{b} - \bar{b}'b)(r) = \int_0^r b''\bar{b} - \bar{b}''b \le C_1(i_0,n,k)\rho_i e^{2(n-1)kr}.$$

Then we have

(4.3)

$$\left(\frac{b'}{b} - \frac{\bar{b}'}{\bar{b}}\right)(r) = \frac{\int_0^r b''\bar{b} - \bar{b}''b}{b(r)\bar{b}(r)} \le \frac{\rho_i C_1(i_0,n,k)e^{2(n-1)kr}}{b(r)\bar{b}(r)},$$

$$\frac{b'}{b} \le \frac{\bar{b}'}{\bar{b}} + \frac{\rho_i C_1(i_0,n,k)e^{2(n-1)kr}}{b(r)\bar{b}(r)} \le 2\frac{n-1}{t_i} + 2\frac{C_1}{t_i}\frac{e^{4(n-1)kt_i}\rho_i}{H_0} \to 0$$

on $R_i(\delta)^c$ as $i \to \infty$ by the choice of t_i and we may assume $t_i/2 < r < 2t_i$.

Combining (4.2), (4.3) with the fact $(n-1)\dfrac{b'_{\gamma_i(t_i)}}{b_{\gamma_i(t_i)}} = \triangle B_i^+$, we get

$$(4.4)\qquad \begin{aligned}\int_{U_i} \triangle B_i^+ dV &\le \left(\int_{U_i - R_i(\delta)} + \int_{U_i \cap R_i(\delta)}\right) \triangle B_i^+ dV\\ &\le \int_{U_i - R_i(\delta)} \triangle B_i^+ dV + H_2(n,k)\int_{U_i \cap R_i(\delta)} dV\\ &< \tau(\delta|\epsilon_i) + \tau(\delta),\end{aligned}$$

for any fixed $U \subset D$. We can choose $\delta > 0$ arbitrarily small.

LEMMA 4.1. *$B^+(x) \ge \dfrac{1}{\omega_{n-1}R^n}\int_{B(x,R)} B^+$ for fixed small $i_0/2 > R > 0$ and $x \in D$.*

Proof. Let (r,θ) be the normal coordinate system centered at x. The metric g^i of D_i can be expressed as $g^i = dr^2 + r^2 g^i_{lk} d\theta_l d\theta_k$, $1 \le l, k \le n-1$. Let $G^i = \det(g^i_{lk})$. It is known that

$$\triangle r = \frac{n-1}{r} + \frac{\partial \log\sqrt{G}}{\partial r}.$$

We know that $H_4 \leq \sqrt{G}(r) \leq H_5$ for $r \leq i_0/2$ from Preliminaries. Using this fact with (4.3) we obtain

$$\lim_{i\to\infty} \frac{1}{\sqrt{G^i}} \frac{\partial\sqrt{G^i}}{\partial r} = \lim_{i\to\infty} \triangle_i r - \frac{n-1}{r} \leq \frac{\rho_i C_1(i_0, n, k) e^{2(n-1)kr}}{b(r)\bar{b}(r)} \to 0,$$

on $V_i := \{\exp t\Theta^i(x) \mid t \leq d^i(\theta_x)\}$. Furthermore on V_i,

(4.5)
$$\lim_{i\to\infty} B_i^+ \frac{\partial\sqrt{G^i}}{\partial r} \leq \lim_{i\to\infty} 2t_i \frac{\partial\sqrt{G^i}}{\partial r} \leq \frac{2t_i\rho_i C_2(i_0, n, k) e^{3(n-1)kr}}{b(r)\bar{b}(r)} \to 0.$$

by the choice of t_i and $\sqrt{G^i} \leq C_3(i_0, n, k)e^{(n-1)kr}$ for some constants C_2, C_3.

Take $\phi_i^k \in C_0^\infty(R_i(\delta))$ such that $||\phi_i^k|| \leq 2$ and $\lim_{k\to\infty} \phi_i^k = 1$ in the distribution sense and $\phi_i^k = 0$ on the cut locus of $\gamma_i(t_i)$, where $C_0^\infty(D)$ is the class of C^∞-functions with compact support in D. Then $B_i^+\phi_i^k$ is a smooth function. Since $B_i^+\phi_i^k \to B_i^+$ in distribution sense, we also have $\triangle(B_i^+\phi_i^l) \to \triangle B_i^+$ in distribution sense. By (4.4) and divergence theorem,

$$\begin{aligned} 0 &\geq \lim_{i\to\infty} \int_{B(x,t)} \triangle_i B_i^+ = \lim_{i\to\infty}\lim_{k\to\infty} \int_{B(x,t)} \triangle_i (B_i^+\phi_i^k) \\ &= \lim_{i\to\infty}\lim_{k\to\infty} \int_{\partial B(x,t)} \frac{\partial(B_i^+\phi_i^k)}{\partial r} t^{n-1}\sqrt{G^i}d\theta, \end{aligned} \tag{4.6}$$

where $0 < t < R$. We know that

$$(B_i^+\phi_i^k)\frac{1}{\sqrt{G^i}}\frac{\partial\sqrt{G^i}}{\partial r} \leq H_6(i_0, n, k, t)t_i$$

for some constant H_6 since

$$||\triangle r - \frac{n-1}{r}|| \leq C(i_0, n, k)$$

by [4]. Also we have

$$t_i \mathrm{vol}(V_i^c \cap B(x,t)) \leq H_1 t_i \rho_i e^{2(n-1)kt_i} \to 0$$

from (4.1). So we obtain

$$
\begin{aligned}
&\lim_{i\to\infty}\int_0^r\int_{\partial B(x,t)\cap V_i^c} B_i^+\phi_i^k\frac{\partial\sqrt{G^i}}{\partial r}d\theta dt \\
(4.7)\qquad &= \lim_{i\to\infty}\int_0^r\int_{\partial B(x,t)\cap V_i^c} B_i^+\phi_i^k\frac{1}{\sqrt{G^i}}\frac{\partial\sqrt{G^i}}{\partial r}\sqrt{G^i}d\theta dt = 0.
\end{aligned}
$$

Using (4.6),

$$
\begin{aligned}
&\lim_{i\to\infty}\lim_{k\to\infty}\int_{\partial B(x,t)} B_i^+\phi_i^k\frac{\partial\sqrt{G^i}}{\partial r}d\theta \\
&\geq \lim_{i\to\infty}\lim_{k\to\infty}\frac{1}{t^{n-1}}\int_{B(x,t)}\triangle_i(B_i^+\phi_i^k) \\
&\qquad + \lim_{i\to\infty}\lim_{k\to\infty}\int_{\partial B(x,t)} B_i^+\phi_i^k\frac{\partial\sqrt{G^i}}{\partial r}d\theta \\
(4.8)\qquad &\geq \lim_{i\to\infty}\lim_{k\to\infty}\left(\int_{\partial B(x,t)-V_i}+\int_{\partial B(x,t)\cap V_i}\right) \\
&\qquad \left(\frac{\partial(B_i^+\phi_i^k)}{\partial r}\sqrt{G^i}+(B_i^+\phi_i^k)\frac{\partial\sqrt{G^i}}{\partial r}\right)d\theta \\
&= \lim_{i\to\infty}\lim_{k\to\infty}\frac{d}{dt}\int_{\partial B(x,t)} B_i^+\phi_i^k\sqrt{G^i}d\theta.
\end{aligned}
$$

Then we get

$$
\omega_{n-1}B^+ = \lim_{i\to\infty}\omega_{n-1}B_i^+ = \lim_{i\to\infty}\lim_{k\to\infty}\omega_{n-1}B_i^+\phi_i^k
$$

in distribution sense and from (4.7) and (4.8),

$$
\begin{aligned}
&\lim_{i\to\infty}\lim_{k\to\infty}\omega_{n-1}B_i^+\phi_i^k(x) \\
&\geq \lim_{i\to\infty}\lim_{k\to\infty}\left(\frac{1}{t^{n-1}}\int_{\partial B(x,r)} B_i^+\phi_i^k - \int_0^r\int_{\partial B(x,t)} B_i^+\phi_i^k\frac{\partial\sqrt{G^i}}{\partial r}\right)d\theta dt \\
&\geq \lim_{i\to\infty}\lim_{k\to\infty}\frac{1}{t^{n-1}}\int_{\partial B(x,r)} B_i^+\phi_i^k.
\end{aligned}
$$

Integrating this over $(0, R)$, we obtain that

$$B^+(x) \geq \lim_{i\to\infty}\lim_{k\to\infty}\frac{1}{t^{n-1}}\int_{B(x,r)} B_i^+\phi_i^k = \frac{1}{t^{n-1}}\int_{B(x,r)} B^+$$

in distribution sense. Since B^+ is continuous, we get Lemma 4.1. □

We easily get that $\triangle B^+ \leq 0$, i.e., for any nonnegative C_0^∞-function h and any $\delta > 0$,

$$\begin{aligned}\int_D h\triangle B^+ &= \int_D \triangle h B^+ = \lim_{i\to\infty}\int_{D_i}\triangle_i h B_i^+ = \lim_{i\to\infty}\int_{D_i} h\triangle_i B_i^+ \\ &= \lim_{i\to\infty}\left(\int_{D_i-R_i(\delta)} + \int_{D_i\cap R_i(\delta)}\right) h\triangle_i B_i^+ \\ &\leq \tau(\delta) + \lim_{i\to\infty}\tau(\delta|\epsilon_i) = \tau(\delta),\end{aligned}$$

since $g^i \longrightarrow g$ in $L^{1,p}$ or C^α-norm so the coefficients of Laplacian operator for harmonic coordinates converges in C^α-norm [1]. Then by the same argument as (4.4), we get the above inequality. Also we know that $\triangle_i B_i^+ \to \triangle B^+$ in distribution sense by the above argument.

By the same argument as [3, 9], we get $\triangle(B^+ + B^-) \equiv 0$. From $\triangle B^+$, $\triangle B^- \leq 0$, we obtain $\triangle B^+ \equiv 0$ so B^+ is harmonic.

In [3], from $\triangle B^+ \equiv 0$ we get that $\mathrm{Hess}(B^+) = 0$ and ∇B^+ is a parallel vector field. It is also an important step to show that $\mathrm{Hess}(B^+) = 0$ in our case. We will follow Cai's proof. Define b_i by the following Dirichlet problem;

$$\triangle_i b_i = 0$$
$$b_i|_{\partial D} = B^+.$$

Then we can prove Lemma 3.1 in [2].

LEMMA 4.2. *$|\nabla_i b_i|^2$ converges, in the strong $L^{1,q}$-topology (for $1 < q < p$), to $|\nabla B^+|^2$.*

Proof. The only difference of proof occurs when we apply the Bochner-Weitzenböck formula. But we only need the integration of $\triangle_i|\nabla_i b_i|^2$ so there are no obstruction to prove this lemma. □

From this lemma, we get $\mathrm{Hess}(B^+) = 0$ in L^q and can prove the remainder of the proof by the same argument as [2].

Now we obtain that there exists an $\epsilon > 0$ depending only on n, k, i_0, d such that if $\mathrm{Ric}_M \geq 0$ on $M - A$ where $\mathrm{vol}(A) \leq \epsilon$ then M is diffeomorphic to $T^k \times N$ up to finite cover, where N is a compact simply connected space. But the complete proof of Theorem 1.1 is the same as the above argument.

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