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REDUCED CROSSED PRODUCTS BY SEMIGROUPS OF AUTOMORPHISMS

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ABSTRACT. Given a C^*-dynamical system $(\mathcal{A}, G, \alpha)$ with a locally compact group G, two kinds of C^*-algebras are made from it, called the full C^*-crossed product and the reduced C^*-crossed product. In this paper, we extend the theory of the classical C^*-crossed product to the C^*-dynamical system $(\mathcal{A}, M, \alpha)$ with a left-cancellative semigroup M with unit. We construct a new C^*-algebra $\mathcal{A} \rtimes_{\alpha r} M$, the reduced crossed product of A by the semigroup M under the action α and investigate some properties of $\mathcal{A} \rtimes_{\alpha r} M$.

1. Introduction

The theory of crossed products of C^*-algebras by groups of automorphisms is an interesting and important area of the theory of operator algebras and has been much developed for the recent decades. From the importance and the success of that theory, it is natural to attempt to extend it to a more general situation by developing the theory of crossed products by semigroups of automorphisms (cf. [3,8]) or even of endomorphisms (cf. [5,6,10]). We say crossed products of C^*-algebras by groups of automorphisms as the classical crossed products. In recent years a number of papers have appeared that are concerned with such non-classical theories of crossed products of C^*-algebras (cf. [5,6,8,10, etc.]). The theory of crossed products by semigroups of automorphisms is one way to do this, which was introduced and developed in the paper [8]. In [8], G. J. Murphy defined and analyzed the concept of the full crossed product of C^*-algebras by semigroups of automorphisms.

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In this paper we define the reduced C^*-crossed product of a C^*-algebra by the semigroup of automorphisms by using the universal property of the crossed product defined by G. J. Murphy and study its properties. This C^*-algebra is the counterpart of the reduced crossed product of a C^*-algebra by the group of automorphisms, which is the rich source of examples of operator algebras.

Though the reduced crossed product of a C^*-algebra by the group of automorphisms has the deep relation with the left regular unitary representation of that group, the regular isometric representation is the source of the construction of the reduced crossed product of a C^*-algebra by the semigroup of automorphisms. Some results in the classical crossed products theory hold in crossed products by semigroups of automorphisms (cf. Theorem 3.2 and Theorem 3.4). However there are significant differences (cf. Proposition 4.2).

Given a group G, the classical reduced crossed product construction applied to the trivial C^*-dynamical system $(\mathbb{C}, G, \alpha)$ gives rise to the reduced group C^*-algebra $C^*_{red}(G)$, which is generated by the regular unitary representation of G. Analogously, we have its corresponding C^*-algebra $C^*_{red}(M)$, the reduced semigroup C^*-algebra for a left-cancellative semigroup M, which is generated by the regular isometric representation of M. For an abelian group G, $C^*_{red}(G)$ is abelian, but $C^*_{red}(M)$ is not abelian even though M is abelian. Considering that the reduced group C^*-algebras give lots of good examples of operator algebras, we think that reduced semigroup C^*-algebras will also give good examples of C^*-algebras (cf. [1,2]) and (cf. Example 1 and 2).

When the semigroup is the particular one, the reduced semigroup C^*-algebra $C^*_{red}(M)$ has been studied in the term of the another C^*-algebras (cf. [1,2], etc.). For example, if $M = \mathbb{N}$, the additive semigroup of natural numbers and 0, then $C^*_{red}(\mathbb{N})$ is identifiable with the Toeplitz algebra $\mathcal{T}$ generated by all Toeplitz operators having continuous symbols on the unit circle.

2. Construction of the crossed product

Let M denote a semigroup with unit e, and let $\mathcal{B}$ be a unital C^*-algebra with unit $I_{\mathcal{B}}$. We call a map $W : M \to \mathcal{B}, x \mapsto W_x$ an isometric homomorphism if $W_e = I_{\mathcal{B}}$, W_x is an isometry and $W_{xy} = W_x W_y$ for all $x, y \in M$. If $\mathcal{B} = B(H)$, the algebra of all bounded linear operators on a Hilbert space H, then we call (H, W) an isometric representation of M on H.

If M is left-cancellative, then an isometric representation of M can be constructed as follows, which is important for the sequel. To be specific, let H be a non-zero Hilbert space and $l^2(M, H)$ denote the Hilbert space of all norm square-summable maps f from M to H (i.e., $\sum \|f(x)\|^2 < +\infty$) with the norm and scalar product given by $\|f\| = (\sum \|f(x)\|^2)^{(1/2)}$ and $\langle f, g\rangle = \sum \langle f(x), g(x)\rangle$ for $x \in M$. For each $x \in M$, a map $\mathcal{L}_x$ on $l^2(M, H)$ is defined by the equation

$$(\mathcal{L}_x f)(z) = \begin{cases} f(y), & \text{if } z = xy \text{ for some } y \in M, \\ 0, & \text{if } z \notin xM. \end{cases}$$

Then $\mathcal{L}_x$ is an isometry for each $x \in M$ and $\mathcal{L}_{xy} = \mathcal{L}_x \mathcal{L}_y$ for all $x, y \in M$. So the map $\mathcal{L} : M \to B(l^2(M, H))$, $x \mapsto \mathcal{L}_x$ is an isometric homomorphism. We call $(l^2(M, H), \mathcal{L})$ the regular isometric representation of M on $l^2(M, H)$.

A C^*-dynamical system will refer in this paper to a triple $(\mathcal{A}, M, \alpha)$ where $\mathcal{A}$ is a C^*-algebra, M is a left-cancellative semigroup with unit, and α is a homomorphism from M to the group $Aut(\mathcal{A})$ of automorphisms on $\mathcal{A}$. For a C^*-algebra $\mathcal{A}$, $\mathcal{M}(\mathcal{A})$ denotes the multiplier algebra of $\mathcal{A}$. Let $\mathcal{B}$ be a C^*-algebra with the multiplier algebra $\mathcal{M}(\mathcal{B})$. A covariant homomorphism from $(\mathcal{A}, M, \alpha)$ to $\mathcal{B}$ is a pair (ϕ, W) where $\phi : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism, $W : M \to \mathcal{M}(\mathcal{B})$ is an isometric homomorphism, and ϕ and W interact via the equation

$$\phi(\alpha_x(a))W_x = W_x \phi(a)$$

for $x \in M$ and $a \in \mathcal{A}$. If $\mathcal{B}$ is the algebra $B(H)$ of the bounded linear operators for a Hilbert space H, we call (H, ϕ, W) a covariant representation of $(\mathcal{A}, M, \alpha)$.

Let (H, ϕ) be a representation of $\mathcal{A}$ with $H \neq 0$. For $a \in \mathcal{A}$ define $\bar{\phi}(a) \in B(l^2(M, H))$ by the formula

$$(\bar{\phi}(a)f)(x) = \phi(\alpha_x^{-1}(a))f(x)$$

for all $f \in l^2(M,H)$ and $x \in M$. The map $\bar{\phi} : \mathcal{A} \to B(l^2(M,H))$ is a *-homomorphism, and it is easily verified that $(l^2(M,H), \bar{\phi}, \mathcal{L})$ is a covariant representation of $(\mathcal{A}, M, \alpha)$, said to be induced by (H, ϕ), where $\mathcal{L}$ is the regular isometric representation of M on $l^2(M,H)$. Note that if (H,ϕ) is a faithful (respectively non-degenerate) representation of $\mathcal{A}$, then $(l^2(M,H), \bar{\phi})$ is also a faithful (respectively non-degenerate) representation of $\mathcal{A}$. In [8], G. J. Murphy defined the crossed product $A \rtimes_\alpha M$ of $\mathcal{A}$ by the semigroup M under the action α. Then there is a canonical *-homomorphism $\rho : \mathcal{A} \to \mathcal{A} \rtimes_\alpha M$ and an isometric homomorphism $V : M \to \mathcal{M}(\mathcal{A} \rtimes_\alpha M)$ such that (ρ, V) is a covariant homomorphism from $(\mathcal{A}, M, \alpha)$ to $\mathcal{A} \rtimes_\alpha M$. In fact the C^*-algebra $\mathcal{A} \rtimes_\alpha M$ is generated by $\{\rho(a)V_x \mid a \in \mathcal{A}, x \in M\}$. The crossed product $\mathcal{A} \rtimes_\alpha M$ has the important universal property as follows ([8], Proposition 1.1): if (ϕ, W) is any covariant homomorphism from $(\mathcal{A}, M, \alpha)$ to a unital C^*-algebra $\mathcal{B}$, there exists a unique *-homomorphism $\phi \times W : \mathcal{A} \rtimes_\alpha M \to \mathcal{B}$ such that

$$(\phi \times W)(\rho(a)V_x) = \phi(a)W_x$$

for $a \in A, x \in M$.

If $\mathcal{A} = \mathbb{C}$, the complex field, in which the action α of M on $\mathcal{A}$ is necessarily trivial, then $\mathbb{C} \rtimes_\alpha M$ is the universal C^*-algebra generated by the semigroup of isometries on M. $\mathbb{C} \rtimes_\alpha M$ is denoted by $C^*(M)$ and called the semigroup C^*-algebra of M.

Let (π_u, H_u) be the universal representation of $\mathcal{A}$ and $(l^2(M,H_u), \mathcal{L})$ be the regular isometric representation of M on $l^2(M,H_u)$. Then $(\bar{\pi}_u, \mathcal{L})$ is a covariant representation of $(\mathcal{A}, M, \alpha)$. By the universal property of the C^*-algebra $\mathcal{A} \rtimes_\alpha M$, there exists a unique *-homomorphism $\bar{\pi}_u \times \mathcal{L} : \mathcal{A} \rtimes_\alpha M \to B(l^2(M,H_u))$ such that

$$(\bar{\pi}_u \times \mathcal{L})(\rho(a)V_x) = \bar{\pi}_u(\rho(a))\mathcal{L}_x$$

for $a \in A, x \in M$.

Definition. Let M be a left-cancellative semigroup with unit e and $(\mathcal{A}, M, \alpha)$ be a C^*-dynamical system. Let (π_u, H_u) be the universal representation of $\mathcal{A}$ and $\mathcal{L}$ be the regular isometric representation of M on $l^2(M,H_u)$. We call $(\bar{\pi}_u \times \mathcal{L})(\mathcal{A} \rtimes_\alpha M)$ the reduced C^*-crossed product of $\mathcal{A}$ by the semigroup M under the action α.

We denote the reduced crossed product of $\mathcal{A}$ by the semigroup M under the action α by $\mathcal{A} \rtimes_{\alpha r} M$. Since $\bar{\pi}_u$ and ρ are injective, $\bar{\pi}_u(\rho(a))$ can be identified with a for each $a \in \mathcal{A}$. Hence $\mathcal{A} \rtimes_{\alpha r} M$ is generated by $\{a\mathcal{L}_x | a \in \mathcal{A}, x \in M\}$. If $\mathcal{A} = \mathbb{C}$, then $\mathbb{C} \rtimes_{\alpha r} M$ is the C^*-algebra generated by the regular isometric representation $\mathcal{L}$ of M on $l^2(M)$. We denote $\mathbb{C} \rtimes_{\alpha r} M$ by $C^*_{red}(M)$, and call it the reduced semigroup C^*-algebra of M.

3. Some properties of $\mathcal{A} \rtimes_{\alpha r} M$

We conjecture many aspects of extended theories developed in crossed products by semigroups of automorphisms are analogous to results of the original classical theory. Nevertheless there are significant differences.

Throughout the rest of this paper a semigroup M denotes a left-cancellative semigroup with unit, $\mathcal{A}$ denotes a C^*-algebra, H_u denotes the universal representation of $\mathcal{A}$, and $\mathcal{L}$ denotes the regular isometric representation on $l^2(M, H_u)$.

By the covariance relation $\mathcal{A} \rtimes_{\alpha r} M$ is the closed linear span of $a\mathcal{L}_{x_1}\mathcal{L}^*_{y_1} \dots \mathcal{L}_{x_n}\mathcal{L}^*_{y_n}$ for $a \in \mathcal{A}$ and x_i, $y_i \in M$ for $1 \le i \le n$.

LEMMA 3.1. *If the net $\{e_\lambda | \lambda \in I\}$ is an approximate unit of $\mathcal{A}$, then $\{e_\lambda | \lambda \in I\}$ is also an approximate unit of $\mathcal{A} \rtimes_{\alpha r} M$.*

Proof. For $\sum_{i=1}^k a_i \mathcal{L}_{x_{i_1}} \mathcal{L}^*_{y_{i_1}} \dots \mathcal{L}_{x_{i_{ni}}} \mathcal{L}^*_{y_{i_{ni}}}$ where $x_{i_1}, y_{i_1}, \dots, x_{i_{ni}}, y_{i_{ni}} \in M$ and $a_i \in \mathcal{A}$, we have

$$\begin{aligned}
&\lim \| \sum a_i \mathcal{L}_{x_{i_1}} \mathcal{L}^*_{y_{i_1}} \dots \mathcal{L}_{x_{i_{ni}}} \mathcal{L}^*_{y_{i_{ni}}} e_\lambda - \sum a_i \mathcal{L}_{x_{i_1}} \mathcal{L}^*_{y_{i_1}} \dots \mathcal{L}_{x_{i_{ni}}} \mathcal{L}^*_{y_{i_{ni}}} \| \\
&= \lim \| (\alpha_{x_{i_1}} \alpha^{-1}_{y_{i_1}} \dots \alpha_{x_{i_{ni}}} \alpha^{-1}_{y_{i_{ni}}} (e_\lambda) - 1)(\sum a_i \mathcal{L}_{x_{i_1}} \mathcal{L}^*_{y_{i_1}} \dots \mathcal{L}_{x_{i_{ni}}} \mathcal{L}^*_{y_{i_{ni}}}) \| \\
&= 0.
\end{aligned}$$

Since $\{a\mathcal{L}_{x_1}\mathcal{L}^*_{y_1} \dots \mathcal{L}_{x_n}\mathcal{L}^*_{y_n} \mid a \in \mathcal{A} \text{ and } x_i, y_i \in M\}$ is total in $\mathcal{A} \rtimes_{\alpha r} M$, $\{e_\lambda | \lambda \in I\}$ is also the approximate unit of $\mathcal{A} \rtimes_{\alpha r} M$. □

The symbol $\mathcal{A} \otimes_{min} \mathcal{B}$ denotes the minimal C^*-tensor product of two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. The following result is one of the very exact analogue to that of the theory of the classical crossed products.

THEOREM 3.2. *If the action α of $(\mathcal{A}, M, \alpha)$ is trivial, then there is a *-isomorphism ϕ from $\mathcal{A} \rtimes_{\alpha r} M$ onto $\mathcal{A} \otimes_{min} C^*_{red}(M)$ such that*

$$\phi(a\mathcal{L}_x) = a \otimes V^r_x$$

for $a \in \mathcal{A}, x \in M$ where $\mathcal{L}$ and V^r are the regular isometric representations on $l^2(M, H_u)$ and $l^2(M)$, respectively.

Proof. Let H_u be the universal representation of $\mathcal{A}$. For each $\xi \in H_u$ and s, $t \in M$ we shall write a function $\tilde{\xi}^{(t)} : M \to H_u$,

$$\tilde{\xi}^{(t)}(s) = \begin{cases} \xi, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

Then $\{\widetilde{\xi}^{(t)} | t \in M\}$ is the orthonormal basis of $l^2(M, H_u)$. If we define a map η_t by the equation

$$\eta_t(s) = \begin{cases} 1, & \text{if } s = t, \\ 0, & \text{if } s \neq t, \end{cases}$$

then $\{\eta_t | t \in M\}$ is the canonical orthonormal basis for $l^2(M)$. There is a natural unitary operator $U : l^2(M, H_u) \to H_u \otimes l^2(M)$ defined by $\tilde{\xi}^{(t)} \to \xi \otimes \eta_t$ for all $\xi \in H$ and $t \in G$. Since α is the trivial action, we have

$$\begin{aligned} (U\bar{\pi}_u(a)\mathcal{L}_x U^*)(\xi \otimes \eta_t) =& (U\bar{\pi}_u(a))\tilde{\xi}^{(xt)} \\ =& U(\widetilde{\pi_u(a)\xi})^{(xt)} \\ =& (\pi_u(a) \otimes V^r_x)(\xi \otimes \eta_t) \end{aligned}$$

for each $\xi \otimes \eta_t \in H_u \otimes l^2(M)$. Since $\mathcal{A} \rtimes_{\alpha r} M$ is generated by $\{a\mathcal{L}_x | \ a \in A, \ x \in M\}$ and $C^*_{red}(M)$ is generated by $\{V^r_x | x \in M\}$, we can define a map $\phi : \mathcal{A} \rtimes_{\alpha r} M \to B(H_u) \otimes B(l^2(M))$ by $\phi(x) = UxU^*$ for each $x \in \mathcal{A} \rtimes_{\alpha r} M$ and have $\phi(\mathcal{A} \rtimes_{\alpha r} M) = \mathcal{A} \otimes_{min} C^*_{red}(M)$ since the norm of the tensor products of represented C^*-algebras acting on the Hilbert spaces $H_1, H_2, \ldots, H_n$, respectively is equal to the spatial C^*-norm on $\otimes^n_{i=1} \mathcal{A}_i$ ([4], Chapter 11). □

Some of important results of the theory of C^*-algebras are concerned with giving conditions on a C^*-dynamical system which ensure the crossed product is simple or prime. Crossed products by semigroups are rarely simple though crossed products by groups give important examples of the simple C^*-algebras. But in many cases, crossed products of semigroups are prime. In some sense, the prime C^*-algebras are the more appropriate analogue of factors in the theory of von Neumann algebras rather than the simple C^*-algebras.

Let $(\mathcal{A}, M, \alpha)$ be a C^*-dynamical system and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. $\mathcal{B}$ is doubly M-invariant if $\alpha_x(\mathcal{B}) = \mathcal{B}$ for all $x \in M$. Then $(\mathcal{B}, M, \alpha|_{\mathcal{B}})$ becomes a C^*-dynamical system where $\alpha|_{\mathcal{B}} : M \to Aut(\mathcal{B})$ is a $*$-homomorphism. Set $\widetilde{\mathcal{B}} =$ closed linear span of $\{b\mathcal{L}_{x_1}\mathcal{L}^*_{y_1} \dots \mathcal{L}_{x_n}\mathcal{L}^*_{y_n} | b \in \mathcal{B}, x_1, y_1, \dots, x_n, y_n \in M\}$. Then $\widetilde{\mathcal{B}}$ is the C^*-subalgebra of $\mathcal{A} \rtimes_{\alpha r} M$.

PROPOSITION 3.3. *Let $(\mathcal{A}, M, \alpha)$ be a C^*-dynamical system and $\mathcal{B}$ be a doubly M-invariant hereditary C^*-subalgebra of $\mathcal{A}$. Then $\widetilde{\mathcal{B}}$ is a hereditary C^*- subalgebra of $\mathcal{A} \rtimes_{\alpha r} M$.*

Proof. Let $L_{\mathcal{B}}$ be a closed left ideal of $\mathcal{A}$ such that $L_{\mathcal{B}} \cap L^*_{\mathcal{B}} = \mathcal{B}$ ([9], Theorem 1.5.2). Since $\mathcal{B}$ is doubly M-invariant, $L_{\mathcal{B}}$ is also doubly M-invariant. If we consider elements $a' = a\mathcal{L}_{x_1}\mathcal{L}^*_{y_1} \dots \mathcal{L}_{x_n}\mathcal{L}^*{}_{y_n}$ and $b' = b\mathcal{L}_{s_1}\mathcal{L}^*{}_{t_1} \dots \mathcal{L}_{s_k}\mathcal{L}^*{}_{t_k}$, then

$$a'b' = a\alpha_{x_1}\alpha^{-1}_{y_1} \dots \alpha_{x_n}\alpha^{-1}_{y_n}(b)\mathcal{L}_{x_1}\mathcal{L}^*_{y_1} \dots \mathcal{L}_{x_n}\mathcal{L}^*_{y_n}\mathcal{L}_{s_1}\mathcal{L}^*{}_{t_1} \dots \mathcal{L}_{s_k}\mathcal{L}^*{}_{t_k}.$$

Put $L_{\widetilde{\mathcal{B}}}=$ closed linear span of $\{b\mathcal{L}_{x_1}\mathcal{L}^*{}_{y_1} \dots \mathcal{L}_{x_n}\mathcal{L}^*_{y_n} | b \in L_{\mathcal{B}},\ x_i, y_i, \in M\}$. Since $L_{\mathcal{B}}$ is doubly M-invariant, $a'b' \in L_{\widetilde{\mathcal{B}}}$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Hence $L_{\widetilde{\mathcal{B}}}$ is a closed left ideal of $\mathcal{A} \rtimes_{\alpha r} M$ and $L_{\widetilde{\mathcal{B}}} \cap (L_{\widetilde{\mathcal{B}}})^* = \widetilde{\mathcal{B}}$. □

Let (A, M, α) be a C^*-dynamical system. A is M-prime if any two non-zero doubly M-invariant closed ideals have a non-zero intersection.

THEOREM 3.4. *Let M be a left-cancellative semigroup. If $\mathcal{A} \rtimes_{\alpha r} M$ is prime, then $\mathcal{A}$ is M-prime.*

Proof. Let $\mathcal{I}$ be a doubly M-invariant closed ideal of $\mathcal{A}$. Let $\widetilde{\mathcal{I}}$ be the closed linear span of $\{a\mathcal{L}_{x_1}\mathcal{L}^*{}_{y_1} \dots \mathcal{L}_{x_n}\mathcal{L}^*_{y_n} | a \in \mathcal{I}$ and $x_i, y_i \in M\}$. Then we can see that $\widetilde{\mathcal{I}}$ is a closed ideal of $\mathcal{A} \rtimes_{\alpha r} M$ from the similar computation of Proposition 3.3. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are two non-zero doubly M-invariant closed ideals of $\mathcal{A}$. Let $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{J}}$ be the corresponding closed ideals of $\mathcal{A} \rtimes_{\alpha r} M$ induced by $\mathcal{I}$ and $\mathcal{J}$, respectively. Since $\mathcal{A} \rtimes_{\alpha r} M$ is prime, $\widetilde{\mathcal{I}}\widetilde{\mathcal{J}} \neq \{0\}$. Let $\{u_i\}_i$ and $\{v_j\}_j$ be the approximate unit of $\mathcal{I}$ and $\mathcal{J}$, respectively. Then we can see that both $\{u_i\}_i$ and $\{v_j\}_j$ are the approximate units of $\widetilde{\mathcal{I}}\widetilde{\mathcal{J}}$ from the Lemma 3.1. So $u_i v_j \neq 0$ for some indices i and j. Hence $\mathcal{I} \cap \mathcal{J} \neq \{0\}$.□

4. Reduced semigroup C^*-algebras

Let G be a partially ordered abelian group, G^+ be its positive cone, and $\mathbb{T}$ be the circle group. Let $\widehat{G}$ be the dual group of G and $\epsilon_x : \widehat{G} \to \mathbb{T}$ be the evaluation homomorphism defined by $\epsilon_x(\gamma) = \gamma(x)$ for $x \in G$ and $\gamma \in \widehat{G}$. Then $\{\epsilon_x | x \in G\}$ forms an orthonormal basis for the Hilbert space $L^2(\widehat{G})$. If P_G denotes the linear span of $\{\epsilon_x |\ x \in G\}$, it follows from the Stone-Weierstrass theorem that P_G is dense in the space $C(\widehat{G})$ of continuous functions on $\widehat{G}$ under the sup-norm topology. Denote by $H^2(\widehat{G})$ the closed subspace of $L^2(\widehat{G})$ having an orthonormal basis $\{\epsilon_x | x \in G^+\}$, and let $P \in B(L^2(\widehat{G}))$ be the projection onto $H^2(\widehat{G})$. If $\phi \in L^\infty(\widehat{G})$, we define $T_\phi \in B(H^2(\widehat{G}))$, by setting $T_\phi(f) = P(\phi f)$. Denote the C^*-subalgebra of $B(H^2(\widehat{G}))$ generated by T_ϕ for all $\phi \in C(\widehat{G})$ by $\mathcal{T}^r(G)$ and it is called the reduced Toeplitz algebra of G; see [7].

PROPOSITION 4.1. *Let G be a partially ordered abelian group with its positive cone G^+. Then $C^*_{red}(G^+)$ is *-isomorphic to the reduced Toeplitz algebra $\mathcal{T}^r(G)$.*

Proof. Let $H^2(\widehat{G})$ be the Hilbert subspace of $L^2(\widehat{G})$ having the orthonormal basis $\{\epsilon_x | x \in G^+\}$. Let U be an isometry from $H^2(\widehat{G})$ onto $l^2(G^+)$ defined by

$$U\left(\sum \hat{f}(x)\epsilon_x\right) = \xi_f$$

where $\xi_f(x) = \hat{f}(x), f \in H^2(\widehat{G})$ and $x \in G^+$.

$$\begin{aligned}(UT_{\epsilon_x}U^*)(\xi)(z) &= (UT_{\epsilon_x})(f_\xi)(z)\\ &= UP\left(\sum \xi(y)\epsilon_y\epsilon_x\right)(z)\\ &= \begin{cases} \xi(y), & z = xy,\\ 0, & z \notin xG^+,\end{cases}\end{aligned}$$

where $f_\xi = \sum \xi(x)\epsilon_x$. Let $\mathcal{L}$ be the regular isometric representation of G^+ on $l^2(G^+)$. It follows from the above calculation that $UT_{\epsilon_x}U^* = \mathcal{L}_x$ for each $x \in G^+$. Since $\mathcal{T}^r(G)$ is generated by $\{T_{\epsilon_x} | x \in G^+\}$ by the Stone-Weierstrass theorem, we can have that $C^*_{red}(G^+) = U\mathcal{T}^r(G)U^*$.□

Reduced group C^*-algebras have the important role in the theory of operators, because they are the important sources of simple C^*-algebras. Though reduced semigroup C^*-algebras are rarely simple, they are prime in many cases.

PROPOSITION 4.2. *Let G be an abelian group and M be a subsemigroup of G. If M has any non-invertible element, then $C^*_{red}(M)$ is not simple.*

Proof. Suppose that $C^*_{red}(M)$ is simple. Let $\lambda : G \to B(l^2(G))$ be the left regular representation of G and $\mathcal{L} : M \to B(l^2(M))$ be the regular isometric representation. We define a map $\epsilon : C^*_{red}(M) \to C^*_{red}(G)$ by $\epsilon(\sum \mu_i \mathcal{L}_{x_{i_1}} \mathcal{L}^*_{y_{i_1}} \ldots \mathcal{L}_{x_{i_{ni}}} \mathcal{L}^*_{y_{i_{ni}}}) = \sum \mu_i \lambda_{x_{i_1} y_{i_1}^{-1} \ldots x_{i_{ni}} y_{i_{ni}}^{-1}}$, where $\mu_i \in \mathbb{C}$. Then ϵ is well defined *-homomorphism (cf. [3]). Since ϵ is injective, $a_1\mathcal{L}_x = a_2\mathcal{L}_x$ implies that $a_1 = a_2$ for all $a_1, a_2 \in C^*_{red}(M)$. So if we consider $(a\mathcal{L}_x\mathcal{L}_x^*)\mathcal{L}_x = a\mathcal{L}_x$ for $a \in C^*_{red}(M)$, then $a\mathcal{L}_x\mathcal{L}_x^* = a$. This implies $\mathcal{L}_x$ is unitary for all $x \in M$, which contradicts to the fact that M has at least a non-invertible element. □

We have some interesting examples of the reduced semigroup C^*-algebra.

EXAMPLE 1. Let $\mathcal{F}_n (n \geq 2)$ be a free group with n generators $z_1, z_2, \ldots, z_n$ and $\mathcal{F}_n^+ = \{z_{i1}^{\epsilon_{i1}} z_{i2}^{\epsilon_{i2}} \ldots z_{ik}^{\epsilon_{ik}} | (i_1, i_2, \cdots, i_k) \in \{1, 2, \cdots, n\}^k, \epsilon_{ij} \geq 0\}$ be the subsemigroup of $\mathcal{F}_n$. For a sequence $\mu = (i_1, i_2, \ldots, i_k)$ in $\{1, 2, \ldots, n\}^k$, set $z_\mu = z_{i_1} z_{i_2} \ldots z_{i_k}$. For $\mu = (i_1, i_2, \ldots, i_k)$ in $\{1, 2, \ldots, n\}^k$ and $\nu = (j_1, j_2, \ldots, j_l)$ in $\{1, 2, \ldots, n\}^l$, we define

$$\delta_{z_\mu}(z_\nu) = \begin{cases} 1, & \text{if } \mu = \nu, \\ 0, & \text{if } \mu \neq \nu. \end{cases}$$

Then $\{\delta_{z_\mu} | z_\mu \in \mathcal{F}_n^+\}$ is an orthonormal basis of $l^2(\mathcal{F}_n^+)$. Let $\mathcal{L}$ be the regular isometric representation of $\mathcal{F}_n^+$ on $l^2(\mathcal{F}_n^+)$. Then we have

$$\mathcal{L}^*_{z_i}(\delta_{z_\mu}) = \begin{cases} \delta_{z_\nu}, & \text{if } \mu = z_i\nu \text{ for some } \nu \in \mathcal{F}_n^+, \\ 0, & \text{otherwise.} \end{cases}$$

So $\mathcal{L}_{z_i}^*\mathcal{L}_{z_j} = 0$ if $i \neq j$. Hence $\mathcal{L}_{z_i}$'s are isometries with orthogonal ranges. If we put $P = I - \sum_{i=1}^n \mathcal{L}_{z_i}\mathcal{L}_{z_i}^*$, then P is the rank one operator. Let J be the closed ideal generated by P. Since

$$(\mathcal{L}_{z_\mu} P \mathcal{L}_{z_\nu}^*)(\mathcal{L}_{z_\alpha} P \mathcal{L}_{z_\beta}^*) = \begin{cases} \mathcal{L}_{z_\mu} P \mathcal{L}_{z_\beta}^*, & \text{if } \nu = \alpha, \\ 0, & \text{if } \nu \neq \alpha, \end{cases}$$

$\{(\mathcal{L}_{z_\mu} P \mathcal{L}_{z_\nu}^*)(\mathcal{L}_{z_\alpha} P \mathcal{L}_{z_\beta}^*) | \mu, \nu, \alpha, \beta \in \mathcal{F}_n^+\}$ forms a set of matrix units for an algebra isomorphic to the compact operator algebra on a separable Hilbert space. That is, J is isomorphic to $K(l^2(\mathcal{F}_n^+))$. Then for the equivalence class $[\mathcal{L}_{z_i}]$ in $C_{red}^*(\mathcal{F}_n^+)/K(l^2(\mathcal{F}_n^+))$ we have $\sum [\mathcal{L}_{z_i}][\mathcal{L}_{zi}]^*$ is the identity in $C_{red}^*(\mathcal{F}_n^+)/K(l^2(\mathcal{F}_n^+))$. Hence $C_{red}^*(\mathcal{F}_n^+)$ is isomorphic to the Cuntz-Toeplitz extension of Cuntz algebra $\mathcal{O}_n$ by the compact operator algebra $K(l^2(\mathcal{F}_n^+))$. So we have the short exact sequence

$$0 \to K(l^2(\mathcal{F}_n^+)) \to C_{red}^*(\mathcal{F}_n^+) \to \mathcal{O}_n \to 0.$$

The reduced semigroup C^*-algebra $C_{red}^*(M)$ is a quotient algebra of the semigroup C^*-algebra $C^*(M)$. We have a natural question when $C_{red}^*(M)$ is isomorphic to $C^*(M)$. By Coburn's result [1] we can see that both $C_{red}^*(\mathbb{N})$ and $C^*(\mathbb{N})$ are isomorphic to the classical Toeplitz algebra. It was shown that $C_{red}^*(M)$ is isomorphic to $C^*(M)$ where M is the positive cone of subgroup of $\mathbb{R}$, the real number group in the another term ([2], Theorem 2). In [7] the above open question was solved partially in the another form, but it is still open. We suspect that there are many examples of reduced crossed products by semigroups of automorphisms that are not isomorphic to crossed products by semigroups of automorphisms.

However it is still unknown the necessary and sufficient condition when $C_{red}^*(M)$ is prime, we suspect that there are many prime reduced semigroup C^*-algebras. The following example is one of them.

EXAMPLE 2. Let $\mathcal{L}$ be the regular isometric representation of $\mathbb{N}^k (k > 1)$, the direct sum of k copies of the additive semigroup of non-negative integers. Let e_i be the element with i-th component 1 and other component 0 for $i = 1, \ldots, k$, and we denote $\mathcal{L}_{e_i}$ by $\mathcal{L}_i$. $C_{red}^*(\mathbb{N}^k)$ is the C^*-algebra generated by k commuting unilateral shifts $\mathcal{L}_i$. Put $p_i = 1 - \mathcal{L}_i\mathcal{L}_i^*$, then $\prod p_i$ is a rank one operator. Since $C_{red}^*(\mathbb{N}^k)$ is irreducible on $B(l^2(\mathbb{N}^k))$ and the compact operator algebra $K(l^2(\mathbb{N}^k))$ of $l^2(\mathbb{N}^k)$ has non-empty intersection with $C^*_{red}(\mathbb{N}^k)$, $K(l^2(\mathbb{N}^k))$ is contained in $C^*_{red}(\mathbb{N}^k)$. Let J be an ideal of $C^*_{red}(\mathbb{N}^k)$. If x is a non-zero element in J, xk is a compact operator for each $k \in K(l^2(\mathbb{N}^k))$. Since J is also irreducible, $K(l^2(\mathbb{N}^k))$ is contained in J ([9], Lemma 6.1.4). So $K(l^2(\mathbb{N}^k))$ is the minimal ideal of $C^*_{red}(\mathbb{N}^k)$. Hence $C^*_{red}(\mathbb{N}^k)$ is prime.

Let $\mathcal{C}(C^*_{red}(\mathbb{N}^k))$ be the commutator ideal of $C^*_{red}(\mathbb{N}^k)$. Then $K(l^2(\mathbb{N}^k))$ is contained in $\mathcal{C}(C^*_{red}(\mathbb{N}^k))$ because $\mathcal{C}(C^*_{red}(\mathbb{N}^k))$ irreducibly acts on $B(l^2(\mathbb{N}^k))$ and $\prod p_i$ is contained in $\mathcal{C}(C^*_{red}(\mathbb{N}^k)) \cap K(l^2(\mathbb{N}^k))$. And we have $C^*_{red}(\mathbb{N}^k)/\mathcal{C}(C^*_{red}(\mathbb{N}^k))$ is isomorphic to $C(\mathbb{T}^k)$, the space of continuous functions on $\mathbb{T}^k$, the product of k copies of the circle group $\mathbb{T}$. The structure of $C^*(\mathbb{N}^k)$ is much more complicate and $C^*_{red}(\mathbb{N}^k)$ is not isomorphic to $C^*(\mathbb{N}^k)$.

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