ROUGH ISOMETRY, HARMONIC FUNCTIONS AND HARMONIC MAPS ON A COMPLETE RIEMANNIAN MANIFOLD

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ABSTRACT. We prove that if a given complete Riemannian manifold is roughly isometric to a complete Riemannian manifold satisfying the volume doubling condition, the Poincaré inequality and the finite covering condition at infinity on each end, then every positive harmonic function on the manifold is asymptotically constant at infinity on each end. This result is a direct generalization of those of Yau and of Li and Tam.

1. Introduction

The Liouville theorem has long been an interesting topic of study to analysts and geometers. The classical Liouville theorem states that every bounded entire function on $\mathbb{R}^2$ must be constant. In 1975, Yau [19] proved a remarkable result that every positive harmonic function on a complete Riemannian manifold with nonnegative Ricci curvature must be also constant. Later, Li and Tam [12] pointed out that every positive harmonic function on a complete Riemannian manifold with nonnegative sectional curvature outside a compact subset is asymptotically constant at infinity on each end.

In the above works, the curvature plays a crucial role in solving the each problem. A typical method to generalize these results is to relax the curvature assumption. We know that if a complete Riemannian manifold satisfies the volume doubling condition, the Sobolev inequality and the Poincaré inequality, then the Harnack inequality for the

positive harmonic function holds on the manifold. This implies that every positive harmonic function on the manifold must be constant. Thus the solvability of the problem depends on much cruder quantities rather than the local concepts like the curvature.

On the other hand, Kanai [8], [9] and [10] introduced the rough isometry, which is a map preserving some analytic quantities like the volume doubling condition and the Sobolev constant and so on. In the viewpoint of the rough isometry, Choi and the second author [5] proved that if a complete Riemannian manifold is roughly isometric to a manifold satisfying the volume doubling condition, the Sobolev inequality and the Poincaré inequality, then the Harnack inequality for the positive harmonic function holds on the manifold. (Also see [6].) It is well known that any complete Riemannian manifolds with nonnegative Ricci curvature satisfy these conditions. Therefore, this result is a generalization of Yau's result.

Now we consider the result of Li and Tam [15]. First, they classify the ends of a complete Riemannian manifold, which are defined in §2, by the volume growth rate as follows: An end D of a complete Riemannian manifold is called a large end, if

$$\int_{r_0}^{\infty} \frac{t}{\text{vol}(B_t(o) \cap D)} dt < \infty,$$

for some $r_0 > 0$. Otherwise, D is called a small end. With this classification, they proved the following theorem:

THEOREM (LI AND TAM). *Let M be a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and the finite first Betti number. Suppose that M has l-large ends and s-small ends, respectively. Then the dimension of the space spanned by harmonic functions which are bounded on one side at each end of M equals $l + s$. And the dimension of the space of bounded harmonic functions M equals l. Furthermore, every bounded harmonic function on M has the finite Dirichlet integral. If M has at least one large end, then the dimension of the space spanned by positive harmonic functions on M equals $l + s$. In particular, if M has only small ends, then every positive harmonic function on M is constant.*

In this paper, we have generalized the result of Li and Tam in the viewpoint of the rough isometry. Our first question is whether the

Harnack inequality of the sphere version at infinity holds on each end being roughly isometric to an end satisfying it. We have pointed out that the volume doubling condition (VD), the Poincaré inequality (P) and the finite covering condition (FC) at infinity of each end, which are defined in §3, enable the Harnack inequality of the sphere version at infinity to hold on each end.

On the other hand, Liu [16] proved that if a complete Riemannian manifold has the nonnegative Ricci curvature outside a compact set, then each sphere centered at origin is covered by finitely many balls. Thus it follows that if the Ricci curvature of a complete Riemannian manifold is nonnegative outside a compact set and the manifold has the finite first Betti number, then each end of the manifold satisfies the finite covering condition (See [15]), hence these three conditions. Now our concerning is whether each of these conditions is preserved through a rough isometry. In §3, the Harnack inequality of the sphere version at infinity holds on the end being roughly isometric to an end satisfying the volume doubling condition, the Poincaré inequality and the finite covering condition at infinity. On the other hand, we can also prove that the parabolicity of each end, which is defined in §2, is invariant under a rough isometry. (See §2.) As a consequence of these results, we have the following main theorem in §3:

MAIN THEOREM. *Let N be a complete Riemannian manifold roughly isometric to a complete Riemannian manifold M which satisfies the volume doubling condition (VD), the Poincaré inequality (P) and the finite covering condition (FC) at infinity of each end. Suppose that M has l-nonparabolic ends and s-parabolic ends, respectively. Then the dimension of the space spanned by harmonic functions which are bounded on one side at infinity of each end of N equals $l + s$. And the dimension of the space of bounded harmonic functions on N equals l. Furthermore, every bounded harmonic function on M has the finite Dirichlet integral. If M has at least one nonparabolic end, then the dimension of the space spanned by positive harmonic functions on N equals $l + s$. In particular, if M has only parabolic ends, then every positive harmonic function on N is constant.*

Note that there are no curvature assumptions imposed on our main theorem.

In [18], Varopoulos proved that each small end is parabolic. And Li and Tam [15] proved that each large end of M is nonparabolic if M is a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and the finite first Betti number. Since we know that the volume growth rate is invariant under rough isometries, our main result can be rephrased in the viewpoint of large ends and small ends as follows:

COROLLARY. *Let M be a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and the finite first Betti number. Suppose that M has l-large ends and s-small ends, respectively. Let N be a complete Riemannian manifold being roughly isometric to M. Then the dimension of the space spanned by harmonic functions which are bounded on one side at infinity of each end of N equals $l + s$. And the dimension of the space of bounded harmonic functions on N equals l. Furthermore, every bounded harmonic function on N has the finite Dirichlet integral. If M has at least one nonparabolic end, then the dimension of the space spanned by positive harmonic functions on N equals $l + s$. In particular, if M has only small ends, then every positive harmonic function on N is constant.*

This corollary is a generalization of the result of Li and Tam [15].

Let us turn our concern to harmonic maps. Cheng [3] proved the Liouville type theorem for harmonic maps as in [19]. Cheng proved that every harmonic map with bounded image from a complete Riemannian manifold with nonnegative Ricci curvature into a Cartan-Hadamard manifold is constant. In [4], Choi generalized Cheng's result. To be precise, Choi proved that every harmonic map from a complete Riemannian manifold with nonnegative Ricci curvature into a complete Riemannian manifold with sectional curvature bounded above by a constant is constant if the image is contained in a geodesic ball which lies inside the cut locus of the center of the ball. Later, Kendall [11] obtained the same result in the case when the domain manifold supports no nonconstant bounded harmonic functions.

On the other hand, Avilés, Choi and Micallef [1] estimated the difference between a harmonic map and the harmonic extension of its boundary data. Recently, using the estimate in [1], Sung, Tam and Wang [17] proved that if every bounded harmonic function is asymptotically

constant at infinity on each nonparabolic end, then every harmonic map with bounded image is also asymptotically constant at infinity on each nonparabolic end, in the case when the image is contained in a geodesic ball, which lies inside the cut locus of the center of the ball. From this result, in §4, we have immediately that every harmonic map with bounded image is asymptotically constant at infinity on each nonparabolic end, whenever each nonparabolic end is roughly isometric to an end satisfying the volume doubling condition, the Poincaré inequality and the finite covering condition at infinity, and the image is contained in a geodesic ball which lies inside the cut locus of the center of the ball. In §4, we also remark the existence and the uniqueness theorem about the bounded harmonic map.

2. Rough isometry and ends

In this section, we collect some definitions and results on rough isometries and ends. The notion of the rough isometry is introduced by Kanai in [8], [9] and [10].

A rough isometry $\varphi : X \to Y$ between two metric spaces X and Y is a (not necessarily continuous) map satisfying the following conditions:

($R1$)   there exists a constant $\tau > 0$ such that

$$Y = \bigcup_{x \in X} B_\tau(\varphi(x)),$$

where $B_\tau(\varphi(x))$ means the τ-neighborhood of $\varphi(x)$;

($R2$)   there exist constants $a \geq 1$ and $b \geq 0$ such that

$$\frac{1}{a} d(x_1, x_2) - b \leq d(\varphi(x_1), \varphi(x_2)) \leq a d(x_1, x_2) + b$$

for all $x_1, x_2 \in X$, where d denotes the distances of X and Y induced from their metrics, respectively.

It is easy to prove that being roughly isometric is an equivalence relation. (See [9].) In [8], [9] and [10], Kanai assumed that any complete Riemannian manifold M satisfies the following conditions:

$(K1)$ the Ricci curvature of the manifold M is bounded below by a constant;

$(K2)$ the injectivity radius is positive, i.e., $\mathrm{inj}(M) > 0$.

Recently, these conditions were replaced by Coulhon and Saloff-Coste [6] with the local volume comparison condition and the local volume doubling condition as follows: For a given rough isometry $\varphi : M \to N$ satisfying the conditions $(R1)$ and $(R2)$,

$(VC)_{loc}$ there exists a constant $C > 0$ such that

$$\frac{1}{C} \operatorname{vol} B_1(x) \leq \operatorname{vol} B_1(\varphi(x)) \leq C \operatorname{vol} B_1(x)$$

for all x in M;

$(VD)_{loc}$ there exists a constant $C_r < \infty$ depending only on r such that

$$\operatorname{vol} B_{2r}(x) \leq C_r \operatorname{vol} B_r(x)$$

for all x in M (in N, respectively).

These conditions are the technical improvements of the conditions $(K1)$ and $(K2)$.

Because rough isometries do not preserve the local properties of manifolds, it is needed to add some local conditions on the Riemannian manifolds M and N, respectively. We assume, with the condition $(VD)_{loc}$, the local Poincaré inequality on both M and N as follows:

$(P)_{loc}$ there exists a constant $C_r < \infty$ depending only on $r > 0$ such that

$$\int_{B_r(x)} |\nabla f| \geq C_r \int_{B_r(x)} |f - \bar{f}|$$

for each x in M (in N, respectively) and for all $f \in C^\infty(B_r(x))$, where $\bar{f} = (\operatorname{vol} B_r(x))^{-1} \int_{B_r(x)} f$.

To add this assumption is reasonable, because the condition $(P)_{loc}$ holds on a complete Riemannian manifold if its Ricci curvature is bounded below. (See [2].)

From now on, when we say that a map $\varphi : M \to N$ is a rough isometry between complete Riemannian manifolds M and N, it means that the map φ satisfies the conditions $(R1)$, $(R2)$ and $(VC)_{loc}$, and the complete Riemannian manifolds M and N satisfy the conditions $(VD)_{loc}$ and $(P)_{loc}$, unless otherwise specified.

Now we define ends of a complete Riemannian manifold, which enable us to forecast infinite behaviors of harmonic functions. Let o be a fixed point of M and $\sharp(r)$ denote the number of unbounded components of $M \setminus B_r(o)$. It is easy to prove that $\sharp(r)$ is nondecreasing in $r > 0$. Let $\lim_{r \to \infty} \sharp(r) = k$, where k may be infinity, then we say that the number of ends of M is k. If k is finite, then we can choose $r_0 > 0$ such that $\sharp(r) = k$ for all $r \geq r_0$. In this case, there exist mutually disjoint unbounded components $D_1, D_2, \cdots, D_k$ of $M \setminus B_{r_0}(o)$ and we call each D_i an end of the complete Riemannian manifold M for $i = 1, 2, \cdots, k$.

Let $\varphi : M \to N$ be a rough isometry between complete Riemannian manifolds M and N, and let o and o' be fixed points in M and N, respectively, such that $o' = \varphi(o)$. Then we can prove that the number of ends of N is equal to that of M as follows:

LEMMA 2.1. *Let M and N be the complete Riemannian manifolds (not necessarily satisfying the conditions $(VD)_{loc}$ and $(P)_{loc}$) and $\varphi : M \to N$ be a rough isometry (not necessarily satisfying the condition $(VC)_{loc}$). Suppose that the number of ends of M is finite. Then the number of ends of N is equal to that of M.*

Proof. Let k be the number of ends of M. Then there exist mutually disjoint unbounded components $D_1, D_2 \cdots, D_k$ of $M \setminus B_{r_0}(o)$ for some $r_0 > 0$ such that

$$D_1 \cup D_2 \cup \cdots \cup D_k \subset M \setminus B_{r_0}(o).$$

And we can choose a constant $c > 0$ such that all bounded components of $M \setminus B_{r_0}(o)$ are contained in $B_{r_0+c}(o)$.

First, we will show that the number of ends of N is less than or equal to k. Otherwise, we can choose a constant $t_0 > a(r_0 + c) + b$ such that the number of unbounded components of $N \setminus B_{t_0}(o')$ is

greater than k. And there exist mutually disjoint unbounded components $E_1, E_2, \cdots, E_{k+1}$ in $N \setminus B_{t_0}(o')$ for some $j = 1, 2, \cdots, k + 1$.

Note that $B_r(\varphi(D_1 \cup D_2 \cup \cdots \cup D_k)) \supset E_1 \cup E_2 \cup \cdots \cup E_{k+1}$. Then for some $i = 1, 2, \cdots, k$, we can choose a sequence $\{x_n\} \subset D_i$ and some $j = 1, 2, \cdots, k + 1$ satisfying the followings:

(i)
$$d(x_n, x_{n+1}) = 1$$

for each $n \in \mathbb{N}$;

(ii)
$$\lim_{n \to \infty} d(\varphi(y_n), B_{t_0}(o')) = \infty,$$

where $\{y_n\}$ is a subsequence of $\{x_n\}$ and $\varphi(y_n) \in E_j$ for all $n \in \mathbb{N}$;

(iii)
$$\lim_{n \to \infty} d(\varphi(z_n), B_{t_0}(o')) = \infty,$$

where $\{z_n\}$ is the complement of $\{y_n\}$ in $\{x_n\}$ and $\varphi(z_n) \in (N \setminus B_{t_0}(o')) \setminus E_j$ for all $n \in \mathbb{N}$.

Choose a sufficiently large $n_0 \in \mathbb{N}$ such that

$$\varphi(x_{n_0}) \in E_j \quad \text{and} \quad \varphi(x_{n_0+1}) \in (N \setminus B_{t_0}(o')) \setminus E_j,$$

where $x_{n_0} \in \{y_n\}$ and $x_{n_0+1} \in \{z_n\}$. For such an $n_0 \in \mathbb{N}$, we have

$$d(\varphi(x_{n_0}), B_{t_0}(o')) > a + b \quad \text{and} \quad d(\varphi(x_{n_0+1}), B_{t_0}(o')) > a + b.$$

Since $d(\varphi(x_{n_0}), \varphi(x_{n_0+1})) \leq a + b$, this is a contradiction. Therefore, the number of ends of N is less than or equal to k. And since there exists a rough isometry $\psi : N \to M$, the number of ends of N must be k.   □

REMARK. In Lemma 2.1, we have proved that the number of ends of a manifold is preserved under a rough isometry. We now show that each end of M corresponds to each end of N through the rough isometry $\varphi : M \to N$.

Let the number of ends of M be $k \in \mathbb{N}$ and let $E_1, E_2 \cdots, E_k$ be ends of N. Then $E_1, E_2, \cdots, E_k$ are mutually disjoint in $N \setminus B_{t_0}(o')$ for some

$t_0 > 0$. For each $i = 1, 2, \cdots, k$, there exist some $j_i = 1, 2, \cdots, k$ and a constant $r_i > 0$ satisfying $\varphi(D_i \setminus B_{r_i}(o)) \subset E_{j_i}$. Hence we can choose a constant $\tilde{r} > 0$ such that $\varphi(D_i \setminus B_{\tilde{r}}(o)) \subset E_{j_i}$ for all $i = 1, 2, \cdots, k$. We will show that if $\varphi(D_{i_1} \setminus B_{\tilde{r}}(o)) \subset E_{j_{i_1}}$ and $\varphi(D_{i_2} \setminus B_{\tilde{r}}(o)) \subset E_{j_{i_2}}$ for $i_1 \neq i_2$, then $j_{i_1} \neq j_{i_2}$. Otherwise, $\varphi((D_{i_1} \cup D_{i_2}) \setminus B_{\tilde{r}}(o)) \subset E_{j_{i_1}}$. Then there exists a sequence $\{x_n\} \subset E_{j_{i_1}}$ satisfying the followings:

(i)
$$d(x_n, x_{n+1}) = 1$$

for all $n \in \mathbb{N}$;

(ii)
$$\lim_{n \to \infty} d(y_n, B_{t_0}(o')) = \infty,$$

where $\{y_n\}$ is a subsequence of $\{x_n\}$ and $y_n \in B_\tau(\varphi(D_{i_1} \setminus B_{\tilde{r}}(o)))$ for all $n \in \mathbb{N}$;

(iii)
$$\lim_{n \to \infty} d(z_n, B_{t_0}(o')) = \infty,$$

where $\{z_n\}$ is the complement of $\{y_n\}$ in $\{x_n\}$ and $z_n \in B_\tau(\varphi((M \setminus B_{\tilde{r}}(o)) \setminus D_{i_1}))$ for all $n \in \mathbb{N}$.

We can choose a sufficiently large $n_0 \in \mathbb{N}$ such that $x_{n_0} \in \{y_n\}$ and $x_{n_0+1} \in \{z_{n+1}\}$. This is a contradiction to $d(x_{n_0}, x_{n_0+1}) = 1$. Let us rearrange j_i such that $j_i = i$. Then the restriction map φ on each end D_i becomes a rough isometry $\varphi|_{D_i} : D_i \to E_i$.

From this result, we have the following lemma:

LEMMA 2.2. *Let M and N be the complete Riemannian manifolds (not necessarily satisfying the conditions $(VD)_{loc}$ and $(P)_{loc}$) and $\varphi : M \to N$ be a rough isometry (not necessarily satisfying the condition $(VC)_{loc}$). Let D_i and E_i be ends of M and N, respectively, for $i = 1, 2, \cdots, k$. Then the rough isometry φ induces a rough isometry $\varphi|_{D_i} : D_i \to E_i$ for each $i = 1, 2, \cdots, k$ satisfying $\varphi(D_i) \subset E_i$ and $B_\tau(\varphi(D_i)) \supset E_i \setminus B_{t_1}(o')$ for some $t_i > t_0$.*

We now discuss the parabolicity of ends of a complete Riemannian manifold. The parabolicity of an end, together with the Harnack inequality of the sphere version at infinity on the end, determines whether each positive harmonic function converges uniformly at infinity on the

end or not. All ends are divided into two classes by the parabolicity. We say that an end D of a complete Riemannian manifold M is a non-parabolic end, if for some $r_1 > 0$, there exist a continuous function u defined on $D \setminus B_{r_1}(o)$ and a sequence $\{x_n\}$ on $D \setminus \overline{B}_{r_1}(o)$ such that

$$\begin{cases} \Delta u = 0 & \text{on } D \setminus \overline{B}_{r_1}(o) \\ u \equiv 1 & \text{on } \partial B_{r_1}(o) \cap D \\ u(x_n) \to 0 & \text{as } x_n \to \infty, \end{cases}$$

where Δ is the Laplacian on the manifold M. Otherwise, the end D is called a parabolic end. Now define the capacity of each end D of a complete Riemannian manifold M by

$$\mathrm{Cap}(D \setminus B_{r_1}(o))$$
$$= \inf \left\{ \int_{D \setminus B_{r_1}(o)} |\nabla u|^2 : u \in C_0^\infty(D \setminus B_{r_1}(o)) \text{ and } u|_{\partial B_{r_1}(o) \cap D} = 1 \right\},$$

where $C_0^\infty(D \setminus B_{r_1}(o))$ means the set of all smooth functions vanishing at infinity and continuous up to the interior boundary $\partial B_{r_1}(o) \cap D$. It is easy to prove that the positivity of $\mathrm{Cap}(D \setminus B_{r_1}(o))$ is equivalent to the nonparabolicity of the end D.

In [10], Kanai proved that the positivity of the capacity of the whole manifold is invariant under rough isometries. Using Kanai's program, it is easy to prove that the positivity of the capacity of each end is invariant under rough isometries as follows:

LEMMA 2.3. *Let D and E be ends of complete Riemannian manifolds M and N, respectively, satisfying the conditions $(VD)_{loc}$ and $(P)_{loc}$. Let $\varphi : D \to E$ be a rough isometry satisfying the condition $(VC)_{loc}$. Suppose that $\mathrm{Cap}(D \setminus B_{r_1}(o)) > 0$ for some $r_1 > 0$. Then we have $\mathrm{Cap}(E \setminus B_{R_0}(o')) > 0$, where $R_0 \geq 2(ar_1 + b)$.*

From Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have the consequence that the rough isometry $\varphi : M \to N$ induces a rough isometry $\varphi|_D : D \to E$ preserving the parabolicity.

3. Proof of the main theorem

First, we study the Harnack inequality of the sphere version at infinity on each end of a complete Riemannian manifold, which plays a

crucial role in proving that every positive harmonic function converges uniformly at infinity on each end. To prove the Harnack inequality of the sphere version at infinity on each end, we need to add some analytic conditions on each end of the manifold as follows: In (VD) and (FC) below, D is an end of a complete Riemannian manifold M and each ball $B_r(x)$ denotes the intersection $B_r(x) \cap D$ with the end D.

(VD) for given $0 < \alpha < 1/2$, there is a constant $C < \infty$ depending only on α such that for any point $x \in \partial B_R(o) \cap D$ and any $0 < r < R/2$,

$$\operatorname{vol} B_r(x) \leq C \operatorname{vol} B_{\alpha r}(x),$$

     where R is sufficiently large;

(P) there exist a constant $C < \infty$ and an integer $n \in \mathbb{N}$ such that for any point $x \in \partial B_R(o) \cap D$, any $0 < r < R/2$ and all $f \in C^\infty(B_r(x))$,

$$\int_{B_{r/n}(x)} |f - \overline{f}|^2 \leq Cr^2 \int_{B_r(x)} |\nabla f|^2,$$

     where R is sufficiently large and $\overline{f} = (\operatorname{vol} B_{r/n}(x))^{-1} \int_{B_{r/n}(x)} f$;

(FC) for given $0 < \alpha < 1/4$, there exist an integer $m = m(\alpha)$ and points $x_1, x_2, \cdots, x_m \in \partial C_R(o) \cap D$ such that for sufficiently large $R > 0$

$$\partial C_R(o) \cap D \subset \bigcup_{i=1}^{m} B_{\alpha R}(x_i)$$

     and $\bigcup_{i=1}^{m} B_{\alpha R}(x_i)$ is connected, where $C_R(o)$ denotes the unbounded component of $D \setminus B_R(o)$.

By the results in [16] and [15], the condition (FC) holds on a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and the finite first Betti number.

By using (VD), (P) and the result in [7], we have the Harnack inequality on each ball $B_{\alpha R}(x_i)$ in (FC). Hence, the sphere $\partial C_R(o) \cap D$

is covered by m-balls on which the Harnack inequality holds. Consequently, we can achieve the Harnack inequality of the sphere version at infinity on the end D satisfying the conditions (VD), (P) and (FC). Now we will prove that the Harnack inequality of the sphere version at infinity holds also on the end being roughly isometric to the end D.

THEOREM 3.1 (THE HARNACK INEQUALITY). *Let D and E be ends of complete Riemannian manifolds M and N, respectively, and D satisfy the conditions (VD), (P) and (FC). Let $\varphi : D \to E$ be a rough isometry. Then there exist a constant $C < \infty$ and a sequence of hypersurfaces $\{H_R\}$ in E such that for any nonnegative harmonic function f defined on the end E,*

$$\sup_{H_R \cap E} f \leq C \inf_{H_R \cap E} f.$$

In particular, $d(o', H_R) \to \infty$ as $R \to \infty$ and H_R divides E into two parts A_R and U_R, where A_R is a bounded subset of E and U_R is the unbounded component of E.

First, we prove that the end E satisfies the conditions (VD) and (P).

LEMMA 3.2. *Let D and E be ends of complete Riemannian manifolds M and N, respectively, and D satisfy the condition (VD). Let $\varphi : D \to E$ be a rough isometry. Then for each $0 < \alpha < 1/2$, there exists a constant $C < \infty$ depending only on α such that for any point $y \in \partial B_R(o') \cap E$ and any $0 < r < R/8a^2$,*

$$\operatorname{vol} B_r(y) \leq C \operatorname{vol} B_{2a^2 \alpha r}(y).$$

Proof. Since we assume the condition $(VD)_{loc}$, we have only to prove the statement for sufficiently large $r > 0$. For each $y \in \partial B_R(o') \cap E$, there exists a point $x \in D$ such that $d(y, \varphi(x)) < \tau$. From the conditions $(VD)_{loc}$, $(VC)_{loc}$ and the definition of the rough isometry, we have

$$\operatorname{vol} B_r(y) \leq C \operatorname{vol} B_r(\varphi(x)) \leq C \operatorname{vol} B_{2ar}(x).$$

From the assumption $r < R/8a^2$, we have that $4ar < d(o, x)$. Thus the condition (VD) has been applicable to the ball $B_{2ar}(x)$ and hence

$$\operatorname{vol} B_r(y) \leq C \operatorname{vol} B_{2a\alpha r}(x).$$

Using the conditions $(VD)_{loc}$, $(VC)_{loc}$ and the definition of the rough isometry again, we have

$$\operatorname{vol} B_r(y) \leq C \operatorname{vol} B_{2a^2\alpha r}(\varphi(x)) \leq C \operatorname{vol} B_{2a^2\alpha r}(y). \qquad \square$$

Now we prove that the Poincaré inequality (P) holds on the end E. By using the program of Coulhon and Saloff-Coste [6], we can prove the following lemma. But in (P), each ball must be contained in the end D. Thus the result of Coulhon and Saloff-Coste is not fully available in our setting. So we need some modifications for the program of Coulhon and Saloff-Coste [6].

LEMMA 3.3. *Let D and E be ends of complete Riemannian manifolds M and N, respectively, and D satisfy the condition (P). Let $\varphi : D \to E$ be a rough isometry. Then there exists constant $C < \infty$ such that for any point $y \in \partial B_R(o') \cap E$, any $0 < r < R/2$ and all $f \in C^\infty(B_r(y))$,*

$$\int_{B_{r/100n^{10}}(y)} |f - \overline{f}|^2 \leq Cr^2 \int_{B_r(y)} |\nabla f|^2,$$

where $\overline{f} = (\operatorname{vol} B_{r/100n^{10}}(y))^{-1} \int_{B_{r/100n^{10}}(y)} f$.

From Lemma 3.2 and Lemma 3.3, we have immediately the Harnack inequality on every ball $B_r(y)$ in E. We have only to prove that there exists a sequence of hypersurfaces $\{H_R\}$ in E such that $H_R \cap E$ is covered by finitely many balls, whose union is connected.

REMARK. In the condition (FC), since $\bigcup_{i=1}^m B_{\alpha R}(o)$ is connected, we may assume that for each $i = 2, 3, \cdots, m$,

$$B_{\alpha R}(x_i) \cap B_{\alpha R}(x_{i-1}) \neq \emptyset.$$

Therefore, by adding some balls centered on $\partial C_R(o)$, we can rephrased the condition (FC) as follows: For given $0 < \alpha < 1/4$ and for sufficiently large $R > 0$, there is an integer $m' = m'(\alpha)$ and $x_1, x_2, \cdots, x_{m'} \in \partial C_R(o) \cap D$ such that

$$(3.1) \qquad B_{\alpha R/2}(\partial C_R(o) \cap D) \subset \bigcup_{i=1}^{m'} B_{\alpha R}(x_i)$$

and $\bigcup_{i=1}^{m'} B_{\alpha R}(x_i)$ is connected. In particular, $d(x_i, x_{i-1}) \leq \alpha R$ for each $i = 2, 3, \cdots, m'$.

Choose a smallest integer $n \in \mathbb{N}$ such that $n \geq 8a^2/\alpha$, and a finite sequence $\{R_j \mid j = 0, 1, 2, \cdots, n\}$ such that

$$R_0 = R/2a, \quad R_j = R_{j-1} + \alpha R/4a, \quad \text{for} \quad j = 1, 2, \cdots, n.$$

By the condition (3.1), there exist points $x_1^j, x_2^j, \cdots, x_{m'}^j \in \partial C_{R_j}(o)$ for each $j = 0, 1, 2, \cdots, n$ such that

$$B_{\alpha R_j/2}(\partial C_{R_j}(o)) \subset \bigcup_{i=1}^{m'} B_{\alpha R_j}(x_i^j)$$

and $\bigcup_{i=1}^{m'} B_{\alpha R_j}(x_i^j)$ is connected. Therefore, we obtain

$$(3.2) \quad (B_{a(R+\tau+b)}(o) \setminus \overline{B}_{(R-\tau-b)/a}(o)) \cap E \subset \bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{a\alpha R}(x_i^j) \cup \bigcup_{j=0}^{n} A_j,$$

where A_j is the union of bounded components of $E \setminus C_{R_j}(o)$. Obviously, $\bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{a\alpha R}(x_i^j)$ is also connected.

LEMMA 3.4. *Let D and E be ends of complete Riemannian manifolds M and N, respectively, and D satisfy the condition (FC). Let $\varphi : D \to E$ be a rough isometry. Then there exists a sequence of hypersurfaces $\{H_R\}$ in E such that $d(o', H_R) \to \infty$ as $R \to \infty$ and H_R divides E into two parts A_R and U_R, where A_R is a bounded subset of E and U_R is the unbounded component of E. In particular,*

$$H_R \cap E \subset \bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{3a^2\alpha R}(\varphi(x_i^j))$$

and $\bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{3a^2\alpha R}(\varphi(x_i^j))$ is connected, where the points $\{x_i^j \mid i = 1, 2, \cdots, m', j = 0, 1, 2, \cdots, n\}$ are given in (3.2).

Proof. Since $d(x_i^j, x_{i-1}^j) \leq \alpha R$ for each $i = 2, 3, \cdots, m'$ and $j = 0, 1, 2, \cdots, n$, we have

$$d(\varphi(x_i^j), \varphi(x_{i-1}^j)) \leq 2a\alpha R + b \leq 3a\alpha R.$$

Thus $\bigcup_{i=1}^{m'} B_{3a\alpha R}(\varphi(x_i^j))$ is connected. On the other hand, since

$$d(\partial C_{R_j}(o), \partial C_{R_{j-1}}(o)) \leq \alpha R/2a$$

for each $j = 1, 2, \cdots, m'$, there exist points $w_j \in \partial C_{R_j}(o)$ such that $d(w_j, w_{j-1}) \leq \alpha R/2a$ for each $j = 1, 2, \cdots, m'$. By the definition of the rough isometry, $d(\varphi(w_j), \varphi(w_{j-1})) \leq \alpha R/2 + b$ for each $j = 1, 2, \cdots, m'$. Therefore, we have the consequence that

$$\bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{3a^2\alpha R}(\varphi(x_i^j))$$

is connected.

Suppose that there exist no sequences of hypersurfaces $\{H_R\}$ satisfying the above statement. Then we may assume that for some sufficiently large $R > 0$, there exists a point $y \in (\partial C_R(o') \cap E) \setminus (\bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{3a^2\alpha R}(\varphi(x_i^j)))$. And for such a point $y \in \partial C_R(o') \cap E$, we can choose an arclength parametrized curve $\gamma : [0, \infty) \to E \setminus B_R(o')$ such that $\gamma(0) = y$, $\gamma(t) \to \infty$ as $t \to \infty$, and $B_c(\gamma[0, \infty)) \cap (\bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{2a^2\alpha R}(\varphi(x_i^j))) = \emptyset$, where $c > 5a^2(\tau + a + b)$. Then from the definition of the rough isometry, we can choose a sequence $\{z_n\}$ in $D \setminus B_{R/2a}(o)$ such that $d(\varphi(z_n), \gamma(n)) \leq \tau$ for all $n \in \mathbb{N}$. Therefore, there exists a curve $\sigma : [0, \infty) \to D \setminus B_{R/2a}(o)$ such that $\sigma(t) \to \infty$ as $t \to \infty$, and $\sigma[0, \infty) \subset \bigcup_{n=0}^{\infty} B_{a(2\tau+b+1)}(z_n)$. It is easy to prove that $\varphi(\sigma[0, \infty)) \subset B_{4a^2(\tau+a+b)}(\gamma[0, \infty))$. Since $\sigma(0) \in B_{2aR}(0)$ and $\sigma(t) \to \infty$ as $t \to \infty$, $\sigma[0, \infty) \cap (\partial C_{2aR}(o) \cap D) \neq \emptyset$. But since $B_\tau(\varphi(\sigma[0, \infty))) \subset B_c(\gamma[0, \infty))$ and

$$B_\tau(\varphi(\bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{\alpha R}(x_i^j))) \subset \bigcup_{j=0}^{n} \bigcup_{i=1}^{m'} B_{3a^2\alpha R}(\varphi(x_i^j)),$$

this is a contradiction. □

From Theorem 3.1 and the maximum principle, we have the consequence that every positive harmonic function f on each end E being roughly isometric to an end satisfying the conditions (VD), (P) and (FC) is asymptotically constant at infinity of E, i.e., there exists a constant $0 \leq c \leq \infty$ such that

$$\lim_{x \to \infty, x \in E} f(x) = c.$$

We follow the program of Li and Tam [12], [13], [14] and [15] in proving our main theorem.

THEOREM 3.5 (MAIN THEOREM). *Let N be a complete Riemannian manifold roughly isometric to a complete Riemannian manifold M which satisfies the volume doubling condition (VD), the Poincaré inequality (P) and the finite covering condition (FC) at infinity of each end. Suppose that M has l-nonparabolic ends and s-parabolic ends, respectively. Then the dimension of the space spanned by harmonic functions which are bounded on one side at infinity of each end of N equals $l + s$. And the dimension of the space of bounded harmonic functions on N equals l. Furthermore, every bounded harmonic function on M has the finite Dirichlet integral. If M has at least one nonparabolic end, then the dimension of the space spanned by positive harmonic functions on N equals $l + s$. In particular, if M has only parabolic ends, then every positive harmonic function on N is constant.*

Proof. From Lemma 2.1, Lemma 2.2 and Lemma 2.3, N has l-nonparabolic ends and s-parabolic ends, respectively. Let $E_1, E_2, \cdots$, E_l be nonparabolic ends of N and $e_1, e_2, \cdots, e_s$ be parabolic ends of N, respectively.

From Theorem 3.1, the nonparabolicity and the maximum principle, for each $i = 1, 2, \cdots, l$, there exists a unique positive harmonic function f_i on N satisfying the followings:

(1) $\lim_{x \to \infty, x \in E_i} f_i(x) = 1$ and

(2) $\lim_{x \to \infty, x \in E_k} f_i(x) = 0$, where $k = 1, 2, \cdots, l$ and $k \neq i$.

In particular, each f_i has the finite Dirichlet integral, i.e., $\int_N |\nabla f_i|^2 < \infty$. And for each $j = 1, 2, \cdots, s$, there exists a positive and bounded harmonic function h_j on N satisfying the followings:

(1) $\displaystyle\lim_{x \to \infty, x \in e_j} h_j(x) = \infty$,

(2) $\displaystyle\lim_{x \to \infty, x \in E_i} h_j(x) = 0$, where $i = 1, 2, \cdots, l$ and

(3) $0 \le h_j \le c < \infty$ on $N \setminus e_j$ for some constant $c > 0$.

Moreover, from Theorem 3.1, the parabolicity and the maximum principle, we have that each h_j is unique up to a positive scalar multiple. (For detail, see [12].) It is easy to prove that the set $\{f_i, h_j | i = 1, 2, \cdots, l$ and $j = 1, 2, \cdots, s\}$ is linearly independent.

From Theorem 3.1 and the maximum principle, we have that for any nonnegative harmonic function f on N, either

$$\lim_{x \to \infty, x \in E} f(x) = \infty \quad \text{or} \quad \lim_{x \to \infty, x \in E} f(x) = c < \infty,$$

for each end E of N. On the other hand, from Theorem 3.1 and the definition of the nonparabolicity, we can construct a harmonic function u_i on each nonparabolic end $E_i \setminus B_{R_0}(o')$ satisfying the followings:

(3.3)
$$\begin{cases} \Delta u_i = 0 & \text{on } E_i \setminus B_{R_0}(o'); \\ u_i = 0 & \text{on } \partial B_{R_0}(o') \cap E_i; \\ \displaystyle\lim_{x \to \infty} u_i(x) = 1, & (x \in E_i), \end{cases}$$

where $i = 1, 2, \cdots, l$. Let f be a nonnegative harmonic function on N, then by (3.3) and the maximum principle, there exists a constant d_i such that

$$\lim_{x \to \infty, x \in E_i} f(x) = d_i < \infty,$$

for $i = 1, 2, \cdots, l$. And using the parabolicity and the maximum principle, we can prove that any two bounded harmonic functions with same data at infinity of each nonparabolic end are equal to each other. Thus if f is a bounded harmonic function on N, then

$$f = \sum_{i=1}^{l} d_i f_i,$$

i.e., the space of all bounded harmonic functions is generated by $\{f_i|\ i = 1, 2, \cdots, l\}$.

In the case that f is a positive harmonic function, we still have

$$\lim_{x \to \infty, x \in E_k} \left(f - \sum_{i=1}^{l} d_i f_i \right)(x) = 0,$$

for each $k = 1, 2, \cdots, l$, and $f - \sum_{i=1}^{l} d_i f_i$ is bounded below on N. Assume that $h = f - \sum_{i=1}^{l} d_i f_i$ is unbounded only on parabolic ends $e_1, e_2, \cdots, e_t$ for some $1 \le t \le s$.

We will prove that there exist constants $0 < c_j < \infty$ such that $h - c_j h_j$ is bounded on e_j for each $j = 1, 2, \cdots, t$. Then $h - \sum_{j=1}^{t} c_j h_j$ is bounded on N and still

$$\lim_{x \to \infty, x \in E_k} \left(h - \sum_{j=1}^{t} c_j h_j \right)(x) = 0$$

for all $k = 1, 2, \cdots, l$. From Theorem 3.1, the parabolicity and the maximum principle, we have $h = \sum_{j=1}^{t} c_j h_j$ on N.

From Theorem 3.1 and the maximum principle, we can choose a constant $0 < c_j < \infty$ such that

(3.4) $h \ge c_j h_j \ \text{ or } \ h \le c_j h_j \ \text{ on } e_j$

for each $j = 1, 2, \cdots, t$. First, in the case that $c_j h_j - h \ge 0$ on e_j, we will prove our claim. Set $\underline{c}_j = \inf\{c_j \,|\, c_j h_j \ge h \text{ on } e_j\}$. Then $0 < \underline{c}_j < \infty$ because $h > 0$.

If $\underline{c}_j h_j - h \ge 0$ is still unbounded on e_j, then, by (3.4) and the definition of $\underline{c}_j$, there exists a constant $0 < c_j' < \infty$ such that

$$(\underline{c}_j - c_j') h_j \le h.$$

Set $\overline{c}_j = \sup\{c_j \,|\, c_j h_j \le h \text{ on } e_j\}$. Since $\underline{c}_j h_j - h \ge 0$ is unbounded on e_j, by the strong maximum principle, we have

$$\underline{c}_j h_j - h > 0 \ \text{ and } \ h - \overline{c}_j h_j > 0 \ \text{ on } e_j.$$

By (3.4), there exists a constant $0 < c_j'' < \infty$ such that

$$\underline{c}_j h_j - h \geq c_j''(h - \bar{c}_j h_j) \ \ \text{or} \ \ \underline{c}_j h_j - h \leq c_j''(h - \bar{c}_j h_j) \ \text{on} \ e_j,$$

i.e.,

$$\frac{\underline{c}_j + c_j'' \bar{c}_j}{c_j'' + 1} h_j \geq h \ \ \text{or} \ \ \frac{\underline{c}_j + c_j'' \bar{c}_j}{c_j'' + 1} h_j \leq h \ \text{on} \ e_j.$$

From the definition of $\underline{c}_j$ and $\bar{c}_j$, we have $\underline{c}_j = \bar{c}_j$, i.e., $\underline{c}_j h_j = h = \bar{c}_j h_j$ on e_j. This is a contradiction. Thus $h - \underline{c}_j h_j$ must be bounded on e_j, where $0 < \underline{c}_j < \infty$. The remains are same.

Suppose that N has only parabolic ends $e_1, e_2, \cdots, e_s$. Then there exist harmonic functions $k_2, k_3, \cdots, k_s$ on N such that $k_j - b_j G(o',)$ is bounded on e_1 for some $b_j > 0$, $k_j + G(o',)$ is bounded on e_j, and k_j is bounded on any other ends, where $j = 2, 3, \cdots, s$ and $G(o',)$ is a symmetric Green's function on N. (See [13].) It is easy to prove that $\{1, k_1, k_2, \cdots, k_s\}$ is linearly independent.

Let h be a harmonic function on N which is bounded on one side at infinity of each end. Then for each $j = 2, 3, \cdots, s$, there exists a positive harmonic function v_j on e_j such that $v_j = 0$ on $\partial B_{R_0}(o') \cap e_j$ and $h - \epsilon_j v_j$ is bounded on e_j, where ϵ is $+1$ or -1 depending on whether h is bounded above or below on e_j. Using Theorem 3.1 and the maximum principle, we can choose a constant c_j such that $v_j = c_j g_j$ for each $j = 2, 3, \cdots, s$. Thus $h - (\epsilon_2 c_2 k_2 + \epsilon_3 c_3 k_3 + \cdots + \epsilon_s c_s k_s)$ is bounded on one side on N. Since N is parabolic, there exists a constant c such that $h = c + \epsilon_2 c_2 k_2 + \epsilon_3 c_3 k_3 + \cdots + \epsilon_s c_s k_s$.

In particular, by the maximum principle and the parabolicity, every positive harmonic function must be constant. $\qquad \square$

COROLLARY 3.6. *Let N be a complete Riemannian manifold roughly isometric to a complete Riemannian manifold M which has nonnegative Ricci curvature outside a compact set and the finite first Betti number. Suppose that M has l-nonparabolic ends and s-parabolic ends, respectively. Then the dimension of the space spanned by harmonic functions which are bounded on one side at infinity of each end of N equals $l + s$. And the dimension of the space spanned by bounded harmonic functions on N equals l. Furthermore, every bounded harmonic function on M has the finite Dirichlet integral. If M has at*

least one nonparabolic end, then the dimension of the space spanned
by positive harmonic functions on N equals $l + s$. In particular, if M
has only parabolic ends, then every positive harmonic function on N
is constant.

COROLLARY 3.7. *Let N be a complete Riemannian manifold rough-
ly isometric to a complete Riemannian manifold M which is a con-
nected sum of complete Riemannian manifolds with nonnegative Ricci
curvature. Suppose that M has l-nonparabolic ends and s-parabolic
ends, respectively. Then the dimension of the space spanned by har-
monic functions which are bounded on one side at infinity of each end
of N equals $l + s$. And the dimension of the space spanned by bounded
harmonic functions on N equals l. Furthermore, every bounded har-
monic function on M has the finite Dirichlet integral. If M has at
least one nonparabolic end, then the dimension of the space spanned
by positive harmonic functions on N equals $l + s$. In particular, if M
has only parabolic ends, then every positive harmonic function on N
is constant.*

4. Applications

Recently, Kendall [11] obtained a remarkable result that if a do-
main manifold supports no nonconstant bounded harmonic functions,
then every bounded harmonic map from the domain manifold into a
regular geodesic ball is a constant map. Later, using the estimate in
[1], Sung, Tam and Wang [17] generalized Kendall's result in such a
way that if every bounded harmonic function on a domain manifold
is asymptotically constant at infinity on each nonparabolic end, then
every harmonic map with bounded image is asymptotically constant
at infinity on each nonparabolic end, when the image is contained in a
geodesic ball, which lies inside the cut locus of the center of the ball.
Applying these facts to our case, we have the following theorem:

THEOREM 4.1. *Let M be a complete Riemannian manifold being
roughly isometric to a complete Riemannian manifold satisfying the
conditions (VD), (P) and (FC) on each nonparabolic end. Let N
be a complete Riemannian manifold with sectional curvature bounded
from above by $K > 0$. Let $U : M \to N$ be a harmonic map such that
$U(M) \subset B_r(0)$, where $B_r(0)$ is the geodesic ball of radius r centered at*

0 in N. If $B_r(0)$ lies inside the cut locus of 0, then U is asymptotically constant at infinity of its nonparabolic ends.

In particular, if N is a simply connected complete Riemannian manifold with nonpositive curvature, then any harmonic map from M to N with bounded image must be asymptotically constant at infinity of each nonparabolic end.

Moreover, if M has only parabolic ends, then the harmonic map U is a constant map.

COROLLARY 4.2. *Let M be a complete Riemannian manifold being roughly isometric to a complete Riemannian manifold which has the nonnegative Ricci curvature outside a compact set and the finite first Betti number. Let N be a complete Riemannian manifold with sectional curvature bounded from above by $K > 0$. Let $U : M \to N$ be a harmonic map such that $U(M) \subset B_r(0)$ where $B_r(0)$ is the geodesic ball of radius r centered at 0 in N. If $B_r(0)$ lies inside the cut locus of 0, then U is asymptotically constant at infinity of each large end.*

In particular, if N is a simply connected complete Riemannian manifold with nonpositive curvature, then any harmonic map from M to N with bounded image must be asymptotically constant at infinity of each large end.

COROLLARY 4.3 (The Liouville Theorem). *Let M be a complete Riemannian manifold being roughly isometric to a complete Riemannian manifold with nonnegative Ricci curvature. Let N be a complete manifold with sectional curvature bounded from above by $K > 0$. Let $U : M \to N$ be a harmonic map such that $U(M) \subset B_r(0)$, where $B_r(0)$ is the geodesic ball of radius r centered at 0 in N. If $B_r(0)$ lies inside the cut locus of 0, then U is constant.*

In particular, if N is a simply connected complete Riemannian manifold with nonpositive curvature, then any harmonic map from M to N with bounded image must be constant.

Using the estimate in [1], we have the existence and the uniqueness theorem for bounded harmonic map as follows:

THEOREM 4.4. *Let M be a complete Riemannian manifold being roughly isometric to a complete Riemannian manifold satisfying the conditions (VD), (P) and (FC) on each nonparabolic end and let $E_1, E_2, \cdots, E_l$ be the nonparabolic ends of M. Let N be a complete*

Riemannian manifold with sectional curvature bounded from above by
$K > 0$ *and let* $B_r(0)$ *be the geodesic ball of radius* r *centered at* 0 *in* N.
If $B_r(0)$ *lies inside the cut locus of* 0, *then for any points* $p_1, p_2, \cdots, p_l$
in $B_r(0)$, *there exists a unique harmonic map* $U : M \to N$ *with finite*
total energy such that $U(M) \subset B_r(0)$ *and* $\lim_{x \to \infty, x \in E_i} U(x) = p_i$ *for all*
$i = 1, 2, \cdots, l$.

From Theorem 4.1 and Theorem 4.4, we have immediately the following corollary:

COROLLARY 4.5. *Let* M *be a complete Riemannian manifold being*
roughly isometric to a manifold satisfying the conditions (VD), (P)
and (FC) *on each nonparabolic end. Let* N *be a complete Riemannian*
manifold with sectional curvature bounded from above by $K > 0$ *and*
let $B_r(0)$ *be the geodesic ball of radius* r *centered at* 0 *in* N *which*
lies inside the cut locus of 0. *Then every bounded harmonic map*
$U : M \to B_r(0)$ *has the finite total energy.*

References

[1] P. Avilés, H. I. Choi and M. Micallef, *Boundary Behavior of Harmonic Maps on Non-smooth Domains and Complete Negatively Curved Manifolds*, J. Funct. Anal. **99** (1991), 293-331.

[2] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. Ec. Norm. Sup. Paris **15** (1982), 213-230.

[3] S. Y. Cheng, *The Liouville theorem for harmonic maps*, Proc. Sympos. Pure Math. **36** (1980), 147-151.

[4] H. I. Choi, *On the Liouville Theorem for Harmonic Maps*, Proc. Amer. Math. Soc. **85** (1982), 91-94.

[5] H. I. Choi and Y. H. Lee, *Rough Isometry and Harnack Inequality*, J. Korean Math. Soc. **33** (1996), no. 2, 455-468.

[6] Th. Coulhon and L. Saloff-Coste, *Variétés riemanniennes isométriques á l'infini*, Rev. Mat. Iberoamericana **11** (1995), 687-726.

[7] A. A. Grigor'yan, *The heat equation on noncompact Riemannian manifolds*, Mat. Sb. **128** (1985), 354-363: English tlansl. in Math. USSR-Sb. **56** (1987) 349-358.

[8] M. Kanai, *Analytic inequalities, and rough isometries between noncompact Riemannian manifolds*, Lecture Notes in Math. **1201**, Springer-Verlag, Berlin, Heidelberg, New York, (1981), 122-137.

[9] ――――, *Rough isometries, and combinatorial approximations of geometries of non-compact riemannian manifolds*, J. Math. Soc. Japan **37** (1985), no. 3, 391-413.

[10] _____, *Rough isometries and the parabolicity of riemannian manifolds*, J. Math. Soc. Japan **38** (1986), no. 2, 227-238.

[11] W. S. Kendall, *Probability, convexity, and harmonic maps with small image* I: *uniqueness and fine existence*, Proc. London Math. Soc. **61** (1990), 158-181.

[12] P. Li and L. F. Tam, *Positive harmonic functions on complete Riemannian manifolds with non-negative curvature outside a compact set*, Ann. of Math. **125** (1987), 171-207.

[13] _____, *Symmetric Green's functions on complete manifolds*, Amer. J. Math. **109** (1987), 1129-1154.

[14] _____, *Harmonic functions and the structure of complete manifolds*, J. Differential Geom. **35** (1992), 359-383.

[15] _____, *Green's functions, Harmonic functions, and Volume Comparison*, J. Differential Geom. **41** (1995), 277-318.

[16] Z-D. Liu, *Ball covering on manifolds with nonnegative Ricci curvature near infinity*, preprint.

[17] C. J. Sung, L. F. Tam and J. Wang, *Bounded Harmonic Maps on a Class of Manifolds*, preprint.

[18] N. Varopoulos, *Potential theory and diffusion on Riemannian manifolds*, Conf. on Harmonic Analysis in Honor of Antoni Zygmund, vol. I, II, Wadaworth Math. Ser. Wadaworth, Belmont, CA, 1983, 821-837.

[19] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201-228.

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