J. Korean Math. Soc. **36** (1999), No. 1, pp. 37–49

THE STRONG LAWS OF LARGE NUMBERS FOR WEIGHTED SUMS OF PAIRWISE QUADRANT DEPENDENT RANDOM VARIABLES

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ABSTRACT. We derive the almost sure convergence for weighted sums of random variables which are either pairwise positive quadrant dependent or pairwise negative quadrant dependent and then apply this result to obtain the almost sure convergence of weighted averages. We also extend some results on the strong law of large numbers for pairwise independent identically distributed random variables established in Petrov to the weighted sums of pairwise negative quadrant dependent random variables.

1. Introduction

Many recent papers have been concerned with concepts of positive dependence and negative dependence for families of random variables (see for example Karlin and Rinott (1980), Block and Ting (1981), Ebrahimi and Ghosh (1981), Block, Savits and Shaked (1982) and the references therein). Lehmann (1966) introduced the notions of positive quadrant dependence and negative quadrant dependence: A sequence $\{X_i : i \geq 1\}$ of random variables is called pairwise positive quadrant dependent(pairwise PQD) if for any real r_i, r_j and $i \neq j$

$$(1) \qquad P\{X_i > r_i, X_j > r_j\} \geq P\{X_i > r_i\}\ P\{X_j > r_j\}$$

and it is called pairwise negative quadrant dependent (pairwise NQD) if for any real r_i, r_j and $i \neq j$

$$(2) \qquad P\{X_i > r_i, X_j > r_j\} \leq P\{X_i > r_i\}\ P\{X_j > r_j\}.$$

Received July 25, 1997.

1991 Mathematics Subject Classification: 60F15, 60F99.

Key words and phrases: pairwise positive quadrant dependent, pairwise negative quadrant dependent, almost sure convergence, weighted sums.

This paper was supported by WonKwang University Grant in 1998.

Let $\{X_i \,:\, i \geq 1\}$ be a sequence of random variables, assumed throughout this article to be nondegenerate, and let $\{w_i \,:\, i \geq 1\}$ be a sequence of positive numbers. Define $S_n = \sum_{i=1}^{n} w_i\, X_i$ and $W_n = \sum_{i=1}^{n} w_i$.

Etemadi (1983, b) already has studied the almost sure convergence of $(S_n - ES_n) \,/\, W_n$ to zero as $n \longrightarrow \infty$, under restrictions (a) $\sup_{i\geq 1} EX_i < \infty$ and (b) $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} [\, w_i\, w_j\, Cov^{+}(X_i,\, X_j)] \,/\, W_j^2 < \infty$, for the case where $\{X_i \,:\, i \geq 1\}$ is a sequence of nonnegative random variables with finite second moments and $\{w_i \,:\, i \geq 1\}$ is a sequence of positive numbers satisfying

$$(3) \qquad w_n \,/\, W_n \longrightarrow 0 \textit{ and } W_n \longrightarrow \infty \textit{ as } n \longrightarrow \infty.$$

In this paper we prove the almost sure convergence of $(S_n - ES_n) \,/\, W_n$ to zero as $n \longrightarrow \infty$, for the case where the random variables are either pairwise positive quadrant dependent (PQD) or pairwise negative quadrant dependent (NQD). We also obtain analogues of "Kolmogorov's theorem" for weighted averages of pairwise positive (negative) quadrant random variables. These will ensure that the work of Etemadi (1983, b) on the almost sure convergence of weighted averages of pairwise independent random variables remains valid if the assumption of pairwise independence is replaced by either pairwise positive quadrant dependence or pairwise negative quadrant dependence.

In Section 2 we study the strong law of large number for weighted sums of pairwise positively correlated nonnegative random variables and apply this result to the pairwise PQD random variables and in Section 3 we obtain a similar result for the weighted sums of pairwise negatively correlated nonnegative random variables and apply it to the pairwise NQD random variables. Finally we extend some results on the strong laws of large numbers for the pairwise independent identically distributed random variables established in Petrov (1996) to the weighted sums of pairwise NQD random variables with the common distribution.

2. PQD random variables

The following lemma is a modified version of Theorem 1 of Etemadi (1983, b) to a sequence of pairwise nonnegatively correlated nonnegative random variables:

LEMMA 2.1. *Let* $\{w_i \ : \ i \geq 1\}$ *be a sequence of positive numbers satisfying* (3) *and let* $\{X_i \ : \ i \geq 1\}$ *be a sequence of pairwise nonnegatively correlated nonnegative random variables with finite second moments. Let* $S_n = \sum_{i=1}^{n} w_i \ X_i$. *Assume*

$$(a) \quad \sup_{i \geq 1} \ EX_i < \infty,$$

$$(b) \quad \sum_{i=1}^{\infty} \ w_i \ Cov(X_i, \ S_i) \ / \ W_i^2 < \infty.$$

Then as $n \to \infty$, $(S_n - ES_n) \ / \ W_n \to 0$ *a. s.*

Proof. First note that

$$\begin{aligned} \sum_{j=1}^{\infty}\sum_{i=1}^{j} w_i w_j Cov(X_i, X_j)^+/W_j^2 &= \sum_{j=1}^{\infty}\sum_{i=1}^{j}[w_i w_j Cov(X_i, X_j)]/W_j^2 \\ &= \sum_{j=1}^{\infty} w_j(X_j, S_j)/W_j^2 \\ &< \infty \end{aligned}$$

since $Cov(X_i, X_j)^+ = Cov(X_i, X_j) \geq 0$ for all $i \neq j$. Thus by Theorem 1 of Etemadi (1983,b) Lemma 2.1 is proved. □

THEOREM 2.1. *Let* $\{w_i \ : \ i \geq 1\}$ *be a sequence of positive numbers satisfying* (3) *and let* $\{X_i \ : \ i \geq 1\}$ *be a sequence of pairwise PQD random variables with finite second moments. Let* $S_n = \sum_{i=1}^{n} w_i X_i$. *Assume*

$$(a) \quad \sup_{i \geq 1} \ E|X_i - EX_i| < \infty,$$

$$(b) \quad \sum_{i=1}^{\infty} w_i Cov(X_i, S_i)/W_i^2 < \infty.$$

Then as $n \to \infty$, $(S_n - ES_n)/W_n \to 0$ *a. s.*

Proof. An equivalent condition to (1) is that

$$(4) \qquad Cov(f(X_i), \ g(X_j)) \ \geq \ 0$$

for all nondecreasing (nonincreasing) functions f and g such that the covariance exists (see Lemma 1 of Lehmann (1966)). Hence the random variables $X_i - EX_i$, $i \geq 1$, are pairwise PQD and we may assume that for $i \geq 1, EX_i = 0$. It clear that $\{w_i X_i \ : \ i \geq 1\}$ is a sequence of pairwise PQD random variables. We consider the sequence $\{w_iX_i^+ \ : \ i \geq 1\}$ and its corresponding sum $S_n^* = \sum_{i=1}^n w_iX_i^+$. For real t put $f(t) = \max\{0, t\}$, $g(t) = t$. Then f, $g - f$ and $g + f$ are nondecreasing. According to (4) there hold

$$0 \leq Cov(f(w_iX_i), f(w_jX_j))$$

and

$$\begin{aligned} 0 &\leq \frac{1}{2}Cov((g-f)(w_iX_i),\ (g+f)(w_jX_j)) \\ &+ \frac{1}{2}Cov((g+f)(w_iX_i),\ (g-f)(w_jX_j)) \\ &= Cov(g(w_iX_i),\ g(w_jX_j)) - Cov(f(w_iX_i),\ f(w_jX_j)), \end{aligned}$$

which yields

$$0 \leq Cov(w_iX_i^+,\ w_jX_j^+) \leq Cov(w_iX_i,\ w_jX_j). \tag{5}$$

It follows from (a), (b) here and (5) that $\{w_iX_i^+ \ : \ i \geq 1\}$ is a sequence of pairwise nonnegatively correlated nonnegative random variables and that satisfies the conditions (a) and (b) of Lemma 2.1. Thus as $n \to \infty$, $(S_n^* - ES_n^*)/W_n \to 0$ *a. s.* A similar consideration for negative parts, say $S_n^{**} = \sum_{i=1}^n w_iX_i^-$, and the fact that $ES_n^* - ES_n^{**} = 0$ complete the proof of Theorem 2.1. □

REMARK. The strong law of large number for pairwise PQD random variables established in Birkel (1989) is the case where the w_i's are identically one (see Theorem 1 of Birkel (1989)).

From Theorem 2.1 Corollary 2.1 can be obtained:

COROLLAY 2.1. (Etemadi, 1983,b) *Let $\{w_i : i \geq 1\}$ be a sequence of positive number satisfying* (3) *and let $\{X_i : i \geq 1\}$ be a sequence of pairwise independent random variables with finite second moments. Assume*

$$(a) \ \sup_{i\geq 1} E|X_i - EX_i| < \infty,$$

$$(b) \ \sum_{i=1}^{\infty} w_i^2 VarX_i \ / \ W_i^2 \ < \ \infty.$$

Let $S_n = \sum_{i=1}^n w_iX_i$. Then as $n \to \infty$, $(S_n - ES_n) \ / \ W_n \to 0$ a. s.

Next, we introduce two applications of Theorem 2.1. The following corollaries are the almost sure convergence of weighted averages and weighted logarithmic averages for pairwise PQD random variables respectively.

COROLLAY 2.2. *Let $\{X_i : i \geq 1\}$ be a sequence of pairwise PQD random variables with finite second moments and let $S_n = \sum_{i=1}^n X_i$. Assume*

$$
\begin{aligned}
&(a) \quad EX_i > 0 \text{ for all } i,\\
&(b) \quad \{EX_i : i \geq 1\} \text{ satisfies (3)},\\
&(c) \quad \sup_{i\geq 1} E|X_i - EX_i| < \infty,\\
&(d) \quad \sum_{i=1}^{\infty} \frac{Cov(X_i, S_i)}{(ES_i)^2} < \infty.
\end{aligned}
$$

Then as $n \to \infty$, $S_n/ES_n \to 1$ a. s.

Proof. Let $Y_n = X_n \,/\, EX_n$ and $w_n = EX_n$. Then $\{Y_i : i \geq 1\}$ is a sequence of pairwise PQD random variables with $EY_i = 1$ and $EY_i^2 < \infty$ and $\{w_i : i \geq 1\}$ is a sequence of positive numbers satisfying (3) and $W_n = ES_n$. Let $T_n = \sum_{i=1}^n w_i Y_i$. Then $T_n = S_n = \sum_{i=1}^n X_i$. Thus from (d) we obtain

$$\sum_{i=1}^{\infty} w_i Cov(Y_i,\ T_i) \,/\, W_i^2 < \infty,$$

which yields, as $n \to \infty$ $(T_n - ET_n) \,/\, W_n = (S_n - ES_n) \,/\, ES_n \to 0$ a. s. according to Theorem 2.1. This completes the proof. □

COROLLAY 2.3. *Let $\{X_i : i \geq 1\}$ be a sequence of pairwise PQD random variables with finite second moments and let $S_n = \sum_{i=1}^n X_i$. Assume*

$$
\begin{aligned}
&(a) \quad EX_i > 0 \text{ for all } i,\\
&(b) \quad \{EX_i : i \geq 1\} \text{ satisfies (3)},\\
&(c) \quad \sup_{i\geq 1} E|X_i - EX_i| < \infty,\\
&(d) \quad \sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{Cov(X_i, X_j)}{ES_i ESj(\log ES_j)^2} < \infty.
\end{aligned}
$$

Then as $n \to \infty$, $(\log ES_n)^{-1} \sum_{i=1}^{n} (X_i/ES_i) \to 1$ *a. s.*

Proof. Let $Y_n = X_n \,/\, EX_n$ and $w_n = EX_n \,/\, ES_n$. Then $\{Y_i : i \geq 1\}$ is a sequence of pairwise PQD random variables and $\{w_i : i \geq 1\}$ is a sequence of positive numbers satisfying (3). From (b) it follows that $W_n \sim \log ES_n$(see the proof of Corollary 3 of Etemadi (1983, b)). Let $T_n = \sum_{i=1}^{n} w_i Y_i$. Then it follows from (d) that

$$\sum_{j=1}^{\infty} \sum_{i=1}^{j} Cov(w_i Y_i,\ w_j Y_j) \,/\, W_j^2 = \sum_{j=1}^{\infty} w_j Cov(Y_j,\ T_j) \,/\, W_j^2 < \infty. \qquad \square$$

Now use Theorem 2.1 to get the desired result.

3. NQD random variables

The following lemma is an application of the strong law of large number for nonnegative random variables in Etemadi (1983, a) to the weighted sums of pairwise nonpositively correlated and nonnegative random variables.

LEMMA 3.1. *Let* $\{w_i : i \geq 1\}$ *be a sequence of positive numbers satisfying* (3) *and let* $\{X_i : i \geq 1\}$ *be a sequence of pairwise nonpositively correlated nonnegative random variables with finite second moments. Assume*

$$(a) \quad \sup_{i \geq 1} EX_i < \infty,$$

$$(b) \quad \sum_{i=1}^{\infty} w_i^2 Var\ X_i \,/\, W_i^2 < \infty.$$

Let $S_n = \sum_{i=1}^{n} w_i X_i$. *Then as* $n \to \infty$, $(S_n - ES_n) \,/\, W_n \to 0$ *a. s.*

Proof. Let $a > 1$ and for each $k \geq 1$ set $n_k = \inf\{n : W_n \geq a^k\}$. Since $W_n \,/\, W_{n+1} \to 1$ as $n \to \infty$ it follows that $W_n \sim a^k$ for all large k. Therefore for some $c > 0$ and every $i = 1, 2, 3, \ldots,$ $\{k : n_k \geq i\} \subset \{k : W_{n_k} \geq W_i\} \subset \{k : ca^k \geq W_i\}$ (see the proof of Theorem 1 in Etemadi (1983, b)). By Chebyshev's inequality and by using proof of Theorem 1 in Etemadi (1983, a)

$$
\begin{aligned}
\sum_{k=1}^{\infty} P\{|S_{n_k} - ES_{n_k}| \ / \ W_{n_k} > \epsilon\} &\leq b \sum_{k=1}^{\infty} Var S_{n_k} \ / \ W_{n_k}^2 \\
&\leq b \sum_{k=1}^{\infty} [\sum_{i=1}^{n_k} w_i^2 Var X_i] \ / \ a^{2k} \qquad (6) \\
&\leq b \sum_{i=1}^{\infty} w_i^2 Var X_i \ / \ W_i^2
\end{aligned}
$$

for every $\epsilon > 0$ since $Cov(X_i, \ X_j) \leq 0$ for all $i \neq j$. Thus by the Borel-Cantelli lemma as $k \to \infty$,

$$(S_{n_k} - ES_{n_k}) \ / \ W_{n_k} \to 0 \ a. \ s. \tag{7}$$

Now given n, a positive integer, for $n_k \leq n < n_{k+1}$

$$\left| \frac{S_n - ES_n}{W_n} \right| \leq \left| \frac{S_{n_{k+1}} - ES_{n_{k+1}}}{W_{n_{k+1}}} \right| \frac{W_{n_{k+1}}}{W_{n_k}} + \frac{ES_{n_{k+1}} - ES_{n_k}}{W_{n_k}} \tag{8}$$

by the monotonicity of S_n it follows from (a), (7) and (8) that

$$\limsup(|S_n - ES_n| \ / \ W_n) \leq \sup_{i \geq 1}(EX_i)(a-1) \tag{9}$$

for every $a > 1$ which concludes the proof. □

LEMMA 3.2. (Birkel, 1992) *Let X be a random variable with the finite second moment. Let $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$. Then*

$$(a) \quad Var(X^+) \leq Var(X),$$
$$(b) \quad Var(X^-) \leq Var(X).$$

THEOREM 3.1. *Let $\{w_i \ : \ i \geq 1\}$ be a sequence of positive numbers satisfying* (3) *and let $\{w_i \ : \ i \geq 1\}$ be a sequence of pairwise NQD random variables with finite second moments. Assume*

$$(a) \quad \sup_{i \geq 1} E|X_i - EX_i| < \infty,$$
$$(b) \quad \sum_{i=1}^{\infty} w_i^2 Var X_i \ / \ W_i^2 < \infty.$$

Let $S_n = \sum_{i=1}^n w_i X_i$. *Then as* $n \to \infty$, $(S_n - ES_n) \ / \ W_n \to 0$ *a. s.*

Proof. First note that $\{w_i X_i \ : \ i \le 1\}$ is a sequence of pairwise NQD random variable since $w_i \ge 0$. An equivalent condition to (2) is that

$$Cov(f(X_i),\ g(X_j)) \le 0 \tag{10}$$

for all nondecreasing (nonincreasing) functions f and g such that the covariance exists (see Lemma 1 of Lehmann (1966)). Hence the random variables $X_i - EX_i$, $i \ge 1$, are pairwise NQD and we may assume that, for $i \ge 1$, $EX_i = 0$. We consider the sequence $\{w_i X_i^+ \ : \ i \ge 1\}$ and its corresponding sum S_n^*. For real t, put $f(t) = \max\{0, t\}$. Clearly, $f(t)$ is a nondecreasing function. Thus according to (10) there holds $Cov(w_i X_i^+,\ w_j X_j^+) \le 0$, i.e., $\{w_i X_i^+ \ : \ i \ge 1\}$ is a sequence of nonpositively correlated nonnegative random variables. Since for $i = 1, 2, 3, \cdots$, $Var(w_i X_i^+) \le Var(w_i X_i)$, according to Lemma 3.2, our assumptions (a) and (b) together with Lemma 3.1 imply that as $n \to \infty$, $(S_n^* - ES_n^*) \ / \ W_n \to 0$ *a. s.* A similar consideration for negative parts, say $S_n^{**} = \sum_{i=1}^n X_i^-$, and the fact that $ES_n^* - ES_n^{**} = 0$ complete the proof. □

Now we consider the almost sure convergence of weighted averages for pairwise NQD random variables as an application of Theorem 3.1.

COROLLAY 3.1. *Let* $\{X_i \ : \ i \ge 1\}$ *be a sequence of pairwise NQD random variables with finite second moments and let* $S_n = \sum_{i=1}^n X_i$. *Assume*

$$\begin{aligned}
&(a) \quad EX_i > 0 \textit{ for all } i \ge 1,\\
&(b) \quad \{EX_i \ : \ i \ge 1\} \textit{ satisfies } (3),\\
&(c) \quad \sup_{i \ge 1} E|X_i - EX_i| < \infty,\\
&(d) \quad \sum_{i=1}^{\infty} \frac{VarX_i}{(ES_i)^2} < \infty.
\end{aligned}$$

Proof. Let $Y_n = X_n \ / \ EX_n$ and $w_n = EX_n$, and use the proof of Corollary 2.2 and Theorem 3.1. Then the proof of Corollary 3.1 is complete. □

COROLLAY 3.2. *Let* $\{X_i : i \geq 1\}$ *be a sequence of pairwise NQD random variables with finite second moments and let* $S_n = \sum_{i=1}^n X_i$. *Assume*

$$\begin{aligned}
&(a) \quad EX_i > 0 \text{ for all } i \geq 1,\\
&(b) \quad \{EX_i : i \geq 1\} \text{ satisfies } (3),\\
&(c) \quad \sup_{i\geq 1} E|X_i - EX_i| < \infty,\\
&(d) \quad \sum_{i=1}^{\infty} \frac{Var X_i}{(ES_i)^2 (\log S_i)^2} < \infty.
\end{aligned}$$

Then as $n \to \infty$,

$$\frac{1}{\log ES_n}\left(\sum_{i=1}^{n} \frac{X_i}{ES_i}\right) \to 1 \ a.\ s. \tag{11}$$

Proof. Let $Y_n = X_n \ / \ EX_n$ and $w_n = EX_n \ / \ ES_n$. Clearly $\{Y_i : i \geq 1\}$ is a sequence of pairwise NQD random variables and $W_n \sim \log ES_n$ as in the proof of Corollary 2.3. Hence from (d) we have

$$\begin{aligned}
\sum_{i=1}^{\infty} Var\left(\frac{X_i}{ES_i}\right) / (\log ES_i)^2 &= \sum_{i=1}^{\infty} \frac{Var(w_i Y_i)}{W_i^2}\\
&= \sum_{i=1}^{\infty} w_i^2 Var(Y_i)/W_i^2 < \infty.
\end{aligned} \tag{12}$$

Let $T_n = \sum_{i=1}^n w_i Y_i$. Then from (12) as $n \to \infty$ $(T_n - ET_n)/W_n \to 0$ *a. s.* according to Theorem 3.1. Since $T_n = \sum_{i=1}^n w_i Y_i = \sum_{i=1}^n (X_i/ES_i)$ and $ET_n = \sum_{i=1}^n w_i = W_n$ (11) follows. □

Finally we extend some results on the strong law of large numbers of pairwise independent identically distributed random variables established by Petrov (1996) to the weighted sums of pairwise NQD random variables with the same distribution and derive necessary conditions of almost sure convergence of S_n/W_n to zero. First we recall the following version of the Borel-Cantelli lemma (cf. Petrov (1975)).

LEMMA 3.3. (Matula, 1992) *Let* $\{A_n\}$ *be a sequence of events. If* $\sum_{n=1}^{\infty} P(A_n) < \infty$, *then* $P(A_n \text{ } i.o) = 0$, *if* $\sum_{n=1}^{\infty} P(A_n) = \infty$, *and* $P(A_i \cap A_j) \le P(A_j)P(A_j)$ *for* $i \ne j$, *then* $P(A_n \text{ } i.o.) = 1$.

LEMMA 3.4. (Lehmann, 1966) *Let* $\{X_i : i \ge 1\}$ *be a sequence of pairwise NQD random variables and let* $\{g_n\}$ *be a sequence of nondecreasing (nonincreasing) functions* $g_n : R \to R$. *Then* $\{g_n(X_n\}$ *is also a sequence of pairwise NQD random variables.*

Let $\{X_i : i \ge 1\}$ be a sequence of random variables and let $\{w_i : i \ge 1\}$ be a sequence of positive numbers called weight. Define $S_n = \sum_{i=1}^{n} w_i X_i$ and $W_n = \sum_{i=1}^{n} w_i$. We assume

$$W_n/w_n \uparrow \infty \quad \text{and} \quad W_n \to \infty \quad \text{as} \quad n \to \infty. \tag{13}$$

In the following lemma we derive a necessary condition for almost sure convergence of weighted sums of pairwise NQD random variables with same distribution.

LEMMA 3.5. *Let* $\{X_i : i \ge 1\}$ *be a sequence of pairwise NQD random variables and let* $\{w_i : i \ge 1\}$ *be a sequence of positive numbers satisfying* (13). *Let* $S_n = \sum_{i=1}^{n} w_i X_i$. *If*

$$S_n/W_n \to 0 \ a.\ s., \tag{14}$$

then

$$\sum_{n=1}^{\infty} P(|w_n X_n| \ge W_n) < \infty. \tag{15}$$

Proof. From (14) and the equality

$$\frac{w_n X_n}{W_n} = \frac{S_n}{W_n} - \frac{S_{n-1}}{W_{n-1}} \cdot \frac{W_{n-1}}{W_n}$$

it follows that

$$\frac{w_n X_n}{W_n} \to 0 \ a.\ s. \tag{16}$$

since $W_{n-1}/W_n \to 1$. Put $X_n^+ = \max(0,\ X_n)$ and $X_n^- = \max(0,\ -X_n)$. Then it follows from (16) that $w_n X_n^+/W_n \to 0$ $a.\ s.$ and $w_n X_n^-/W_n \to 0$ $a.\ s.$ Since X_n^+ is a nondecreasing function of X_n, $\{X_n^+\}$ is a sequence of pairwise NQD random variables by Lemma 3.4. Similarly, from the fact that $\{-X_n\}$ is a sequence of pairwise NQD random variables $\{X_n^-\}$ is also a sequence of pairwise NQD random variables according to Lemma 3.4. Now define the events, for all $n \geq 1$, $A_n = \{w_n X_n^+ > \frac{1}{3} W_n\}$, $B_n = \{w_n X_n^- > \frac{1}{3} W_n\}$. Then for $i \neq j$, we have

$$P(A_i \cap A_j) \leq P(A_i)P(A_j),$$
$$P(B_i \cap B_j) \leq P(B_i)P(B_j).$$

We apply Lemma 3.3: if $\sum_{n=1}^{\infty} P(w_n X_n^+ \geq \frac{1}{3} W_n) = \infty$ then $P(A_n\ i.o.) = 1$ contrary to almost sure convergence of $w_n X_n^+ \ / \ W_n$ to 0. Thus we get $\sum_{n=1}^{\infty} P(w_n X_n^+ \geq \frac{1}{3} W_n) < \infty$. The similar consideration for $\{X_n^-\}$ yields $\sum_{n=1}^{\infty} P(w_n X_n^- \geq \frac{1}{3} W_n) < \infty$. Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} P(|w_n X_n| \geq W_n) &= \sum_{n=1}^{\infty} P(w_n X_n^+ + w_n X_n^- \geq W_n) \\
&\leq \sum_{n=1}^{\infty} P(w_n X_n^+ \geq \frac{1}{3} W_n) + \sum_{n=1}^{\infty} P(w_n X_n^- \geq \frac{1}{3} W_n) \\
&< \infty,
\end{aligned}$$

and the proof is complete. □

Let $f(x)$ be an even continuous function that is positive and strictly increasing in the region $x > 0$ and satisfying the condition $f(x) \to \infty$ as $x \to \infty$. We put

$$f^{-1}(n) = W_n / w_n. \tag{17}$$

THEOREM 3.2. *Let $\{X_i \ : \ i \geq 1\}$ be a sequence of pairwise NQD random variables with same distribution. If* (14) *holds then*

$$E(f(X_1)) < \infty. \tag{18}$$

Proof. Assumptions (14) and (17) with Lemma 3.5 imply the relation

$$\begin{aligned}\sum_{n=1}^{\infty} P(|w_n X_n| \geq W_n) &= \sum_{n=1}^{\infty} P(|X_1| \geq W_n/w_n) \\ &= \sum_{n=1}^{\infty} P(|X_1| \geq f^{-1}(n)) < \infty. \qquad (19)\end{aligned}$$

From (19) we obtain

$$\sum_{n=1}^{\infty} P(f(X_1) \geq n) < \infty. \qquad (20)$$

For an arbitrary random variable Y the conditions $\sum_{n=1}^{\infty} P(|Y| \geq n) < \infty$ and $E|Y| < \infty$ are equivalent. Therefore it follows from (20) that (18) holds. Thus the proof is complete. □

ACKNOWLEDGEMENTS. The author wishes to thank the referee for the helpful comments to improve the draft.

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