J. Korean Math. Soc. **36** (1999), No. 1, pp. 15–36

# THE DIMENSION OF THE RECTANGULAR PRODUCT OF LATTICES

Deok Rak Bae

Abstract. In this paper, we determine the dimension of the rectangular product of certain finite lattices. In fact, if $L_1$ and $L_2$ be finite lattices which satisfy the some conditions, then we have $\dim(L_1 \square L_2) = \dim(L_1) + \dim(L_2) - 1$.

## 1. Introduction

We define an *ordered set* $P$ to be a pair $(P, R)$, where $P$ is a nonempty set and $R$ is an order-relation on $P$. An order $R$ on a set is called an *extension* of another order $S$ on the same set if $S \subseteq R$. For $a, b \in P$, we usually write $a \leq b$ for $(a, b) \in R$ and also $a < b$ when $a \leq b$ and $a \neq b$. For elements $a > b$ in an ordered set $P$, we write $a \succ b$ or $b \prec a$ ($a$ *covers* $b$ or $b$ is *covered by* $a$) if $a \geq c > b$ implies $a = c$ for every element $c$ of $P$. A *linear extension* of an ordered set $P$ is a linear order $E : x_1 \prec x_2 \prec \cdots \prec x_n$ containing the order of $P$. E. Szpilarjn [10] shows that any order has a linear extension. It then follows that the intersection of all linear extensions of a partial order is the partial order itself. B. Dushnik and E. Miller [3] later defined the *dimension* of an ordered set $P$, denoted by $\dim(P)$, to be the minimum cardinality of a family of its linear extensions whose intersection is the order itself. The following alternative definition is often credited to O. Ore [7], but appeared earlier in Hiraguchi [4]: The dimension of an ordered set $P$ is the minimum size of a family of chains whose direct product embedded $P$. From this we can easily see that, for $P$ and $Q$ be arbitrary finite

Received July 19, 1997.
1991 Mathematics Subject Classification: 06A07.
Key words and phrases: order-dimension, rectangular product, Ferrers relation, Ferrers-dimension, join-cover.

ordered sets,

$$\max\{\dim(P), \dim(Q)\} \leq \dim(P \times Q) \leq \dim(P) + \dim(Q).$$

Then we know that the dimension of the product turns out to be always close to the upper bound. W. T. Trotter [11] obtained the following nontrivial result: For positive integers $n \geq 3$, then

$$\dim(S_n \times S_n) = 2n - 2,$$

where $S_n$ is the so-called $n$-dimensional standard ordered set. C. Lin [6] obtained the following nontrivial result: For positive integers $m, n \geq 3$, then

$$\dim(S_m \times S_n) = m + n - 2.$$

A lattice is called *bounded* if it has both the least element 0 and the greatest element 1. M. K. Bennett [2] defined the *rectangular product* of two bounded lattices $L_1$ and $L_2$, denoted by $L_1 \Box L_2$, to be the set

$$\{(x, y) \mid (x, y) \in L_1 \times L_2 \text{ with } x \neq 0 \text{ and } y \neq 0\} \cup \{(0, 0)\}$$

with the order induced from the direct product $L_1 \times L_2$, which is also a bounded lattice.

Let $J(L)$ be the set of all *join-irreducible* elements of the finite lattice $L(a \in J(L)$ iff $a = \bigvee S$ implies $a \in S)$. The set $M(L)$ of all *meet-irreducible* elements is defined dually. An *atom* is any element which covers the least element and a *dual atom* is any element which is covered by the greatest element. Let us denote by $A(L)$ and $DA(L)$ the sets of all atoms and dual atoms of $L$, respectively. We shall compute the dimension of the rectangular product of the certain finite lattices. To do this we need a concept introduced by R. Wille [12]. Let $G$ and $M$ be the sets and let $I$ be a binary relation between $G$ and $M$. We defined a *context* as a triple $(G, M, I)$ and we define a concept of the context $(G, M, I)$, which is a pair $(A, B)$ with the following properties:

$$A \subseteq G, \; B \subseteq M, \; A' = B \text{ and } A = B'$$

where $A' = \{m \in M \mid gIm \; \forall g \in A\}$ and $B' = \{g \in G \mid gIm \; \forall m \in B\}$. Put

$$\mathcal{B}(G, M, I) = \{(A, B) \mid A \subset G, B \subset M\}$$

with the order relation in $\mathcal{B}(G, M, I)$ as follows:

$$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2.$$

Then $(\mathcal{B}(G,M,I),\leq)$ is a complete lattice, which is called the *concept lattice* of $(G,M,I)$. A relation $F\subseteq G\times M$ is called a *Ferrers relation* if $g_1Fm_1$ and $g_2Fm_2$ implies $g_1Fm_2$ or $g_2Fm_1$ for all $g_1,g_2\in G$ and $m_1,m_2\in M$. The *Ferrers dimension* of a context $(G,M,I)$, denoted by $\text{fdim}(G,M,I)$, is defined to be the smallest number of Ferrers relations $F_1,F_2,\cdots,F_n$ with $I=\bigcap F_i$. Observe that the complement of a Ferrers relation $F$ is again a Ferrers relation in $G\times M-I$. Therefore, one can alternatively define $\text{fdim}(G,M,I)$ as the minimum number of Ferrers relations $F_1,F_2,\cdots,F_n$ with $F_i\subseteq G\times M-I$ such that $G\times M-I=\bigcup F_i$. Let $L$ be arbitrary finite lattice. Then it is known that $(L,L,\leq)$ and $(J(L),M(L),\leq_{J(L)\times M(L)})$ are contexts and that

$$\dim(L)=\text{fdim}(L,L,\leq)=\text{fdim}(J(L),M(L),\leq_{J(L)\times M(L)}).$$

We assume throughout in this paper as follows: $F$ is a Ferrers relation in $J(L)\times M(L)$ is the same meaning as $F$ is a Ferrers relation in $J(L)\times M(L)-I$ for any lattice $L$.

Our main result in this paper is the following.

THEOREM. *Let $L_1$ and $L_2$ be finite lattices with $\dim(L_1)=s$ and $\dim(L_2)=t$. Suppose that there are Ferrers relations $F_i(1\leq i\leq s)$ and $G_j(1\leq j\leq t)$ such that $\bigcup_{i=1}^{s}F_i=J(L_1)\times M(L_1)-I_1$ and $\bigcup_{j=1}^{t}G_j=J(L_2)\times M(L_2)-I_2$. If $J(L_i)=A(L_i),M(L_i)=DA(L_i)(i=1,2)$ and $\bigcup_{i=1}^{s}c(F_i)=J(L_1)$ or $\bigcup_{j=1}^{t}c(G_j)=J(L_2)$, then we have*

$$\dim(L_1\Box L_2)=\dim(L_1)+\dim(L_2)-1.$$

## 2. Preliminaries

To prove the main theorem we need the following lemmas.

LEMMA 1. *Let $L_1$ and $L_2$ be finite lattices with $J(L_i)=A(L_i)$ and $M(L_i)=DA(L_i)$ for $i=1,2$. Then we have*

$$J(L_1\Box L_2)=A(L_1\Box L_2)=A(L_1)\times A(L_2),$$
$$M(L_1\Box L_2)=DA(L_1\Box L_2)=DA(L_1)\times\{1\}\cup\{1\}\times DA(L_2).$$

An incomparable pair $(a,b)$ in an ordered set $P$ is called a *critical pair* if $x<a$ implies $x<b$ and $x>b$ implies $x>a$, then $\text{Crit}(P)$ denotes the set of all critical pairs in $P$ and $\text{Crit}(y)$ denotes the set of all elements $x \in P$ with $(x, y) \in \operatorname{Crit}(P)$. For $A \subseteq P$, let $A^u = \{x \in P \mid (\forall a \in A)\ a \leq x\}$, $A^l = \{x \in P \mid (\forall a \in A)\ a \geq x\}$ and $DM(P) = \{A \subseteq P \mid A^{ul} = A\}$. Then $(DM(P), \subseteq)$ is a complete lattice, known as the *Dedekind-MacNeille completion* of $P$.

LEMMA 2. *For any two elements $a$ and $b$ of a finite lattice $L$, $(a, b)$ is a critical pair of $L$ if and only if $a \wedge b$ is a unique dual cover of $a$ and $a \vee b$ is a unique cover of $b$.*

*Proof.* Let $(a, b)$ be a critical pair of a lattice $L$. If $x < a$ in $L$, then $x < b$ in $L$ and so $x < a \wedge b$ in $L$. Hence $a \wedge b$ is a unique dual cover of a in $L$. By duality, there is a unique element $a \vee b$ in $L$ such that $a \vee b$ is covers $b$ in $L$. Conversely, if $x < a$ in $L$, then $x < a \wedge b$ in $L$ and so $x < a \wedge b < b$ in $L$. If $y > b$ in $L$, then $y > a \vee b > a$ in $L$. Hence $y > a$ in $L$. Thus $(a, b)$ is a critical pair of $L$. □

A family $\mathcal{R} = \{E_1, E_2, \cdots, E_t\}$ of linear extensions of an ordered set $(P, \leq)$ is called a *realizer* of $P$ (also, we say that $\mathcal{R}$ *realizes* $P$) if $(P, \leq) = \bigcap_{i=1}^{t} E_i$.

LEMMA 3. [8] *Let $P$ be an ordered set and let $\mathcal{R}$ be the family of linear extensions of $P$. Then the following statements are equivalent:*

(1) *$\mathcal{R}$ is a realizer of $P$.*
(2) *For all critical pair $(x, y)$ of $P$, there is a linear extension $E \in \mathcal{R}$ such that $y < x$ in $E$.*

By Lemma 2, for any finite lattice $L$, we have $\operatorname{Crit}(L) \subseteq J(L) \times M(L)$. In particular, for any two finite complemented modular lattices $L_1$ and $L_2$, it is known that

$$J(L_i) = A(L_i) \text{ and } M(L_i) = DA(L_i)$$

and hence we have the following properties:

$$\operatorname{Crit}(L_i) = J(L_i) \times M(L_i) \cap \mathcal{I}(L_i),$$

where $\mathcal{I}(L_i)$ is the set of all incomparable pairs in $L_i$ for $i = 1, 2$. Now, by Lemma 1 and Lemma 2, we have the following lemma.

LEMMA 4. *Let $L_1$ and $L_2$ be finite lattices with $J(L_i) = A(L_i)$ and $M(L_i) = DA(L_i)$ for all $i = 1, 2$. Then we have*

$$\begin{aligned} \operatorname{Crit}(L_1 \square L_2) &= \{((a, c), (b, 1)) \mid (a, b) \in \operatorname{Crit}(L_1) \text{ and } c \in A(L_2)\} \cup \\ &\quad \{((a, c), (1, d)) \mid a \in A(L_1) \text{ and } (c, d) \in \operatorname{Crit}(L_2)\}. \end{aligned}$$

By Lemma 4, we have

$$\mathrm{Crit}(b,1) = \mathrm{Crit}(b) \times A(L_2) \text{ and } \mathrm{Crit}(1,d) = A(L_1) \times \mathrm{Crit}(d).$$

Furthermore, we have

$$\mathrm{Crit}(b,1) \cap \mathrm{Crit}(1,d) = \mathrm{Crit}(b) \times \mathrm{Crit}(d).$$

For any two subordered sets $A$ and $B$ of the finite lattice $L$, we define $A < B$ if $a < b$ for all $a \in A$ and $b \in B$. Suppose that $J(L) = \{a_1, a_2, \cdots, a_n\}$ and $M(L) = \{b_1, b_2, \cdots, b_k\}$. For each Ferrers relation $F_i$ in $J(L) \times M(L)$, we defined the sets $c(F_i), C(F_i), r(F_i)$ and $R(F_i)$ as follows: $c(F_i)$ is the set of first coordinate of the shortest column of $F_i$, $C(F_i)$ is the set of first coordinate of the longest column of $F_i$, $r(F_i)$ is the set of second coordinate of the shortest row of $F_i$ and $R(F_i)$ is the set of second coordinate of the longest row of $F_i$. In fact, there are finite sequences $\{a_{in_i}\}$ in $J(L)$ and $\{b_{ik_i}\}$ in $M(L)$ such that

$$F_i(a_{i1}) \supseteq F_i(a_{i2}) \supseteq \cdots \supseteq F_i(a_{in_i}) \quad (\neq \emptyset) \text{ and}$$
$$F_i(b_{i1}) \supseteq F_i(b_{i2}) \supseteq \cdots \supseteq F_i(b_{ik_i}) \quad (\neq \emptyset)$$

where $F_i(a) = \{b \mid (a,b) \in F_i\}$ and $F_i(b) = \{a \mid (a,b) \in F_i\}$. In this case, we have

$$c(F_i) = F_i(b_{ik_i}),\ C(F_i) = F_i(b_{i1}),\ r(F_i) = F_i(a_{in_i}) \text{ and } R(F_i) = F_i(a_{i1}).$$

For any Ferrers relation $F_i$ in $J(L) \times M(L)$ with $F_i(b_{i1}) \supseteq F_i(b_{i2}) \supseteq \cdots \supseteq F_i(b_{ik_i})$, we have a partial linear extension $E_i$ from $F_i$ as following:

$$E_i : \{b_{i1}\} < F^*_{i1} < \{b_{i2}\} < F^*_{i2} < \cdots < \{b_{i(k_i-1)}\} < F^*_{i(k_i-1)} < \{b_{ik_i}\} < F_{ik_i},$$

where $F^*_{iu} = F_i(b_{iu}) - F_i(b_{i(u-1)})$ for all $u = 1, 2, \cdots, k_i - 1$, that is, a Ferrers relation of ordered set induces a partial linear extension of the ordered set. If $\{F_1, F_2, \cdots, F_s\}$ is a family of Ferrers relations in $J(L) \times M(L)$ with $\bigcup_{F \in \mathcal{F}} F = J(L) \times M(L) - I$, then every critical pair of $L$ is reversed in $\bigcup_{E_i \in \mathcal{R}} E_i$, that is, $\mathcal{R} = \{E_1, E_2, \cdots, E_s\}$ is a realizer of $L$. Further, the Ferrers relations are need not disjoint.

We say that the Ferrers relation $F$ is a *saturated* in $J(L) \times M(L)$ if there is no Ferrers relation $F'$ in $J(L) \times M(L)$ such that $F \subset F'$. Let $\mathcal{F}$ be a family of Ferrers relations in $J(L) \times M(L)$ and let $\mathcal{E}$ be a subfamily of $\mathcal{F}$. We say that $\mathcal{E}$ is a *join-cover* (resp, *meet-cover*) if $\bigcup_{E \in \mathcal{E}} c(E) = J(L)$ (resp, $\bigcup_{E \in \mathcal{E}} r(E) = M(L)$).

LEMMA 5. *Let $L$ be a finite lattice with $\dim(L) = s$ and let $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ be a family of Ferrers relations in $J(L) \times M(L)$ such that $\bigcup_{i=1}^{s} F_i = J(L) \times M(L) - I$. Then we have following properties:*

(1) *If $\bigcup_{i=1}^{s} c(F_i) = J(L)$, then $c(F_i)$ and $c(F_j)$ are distinct subsets in $J(L)$ for all $i, j$ with $i \neq j$.*

(2) *If $\bigcup_{i=1}^{s} r(F_i) = M(L)$, then $r(F_i)$ and $r(F_j)$ are distinct subsets in $M(L)$ for all $i, j$ with $i \neq j$.*

*Proof.* (1) Suppose not, that is, $c(F_j) = c(F_{j_0})$ for some $j$ and $j_0$ with $j \neq j_0$. Since $\bigcup_{i=1}^{s} c(F_i) = J(L)$, it follows that $\bigcup_{i=1}^{s} c(F_i) - c(F_{j_0}) = J(L)$. For all $1 \leq i \leq s$ with $i \neq j_0$, let

$$E_i = F_i \cup \{(a,b) \in F_{j_0} \mid a \in c(F_i)\}.$$

Hence each $E_i$ is a saturated Ferrers relation in $J(L) \times M(L)$ and $F_{j_0} \subseteq \bigcup_{i=1, i\neq j_0}^{s} E_i$ and hence $\mathcal{E} = \{E_1, E_2, \cdots, E_s\} - \{E_{j_0}\}$ is the family of Ferrers relations with $\bigcup_{E \in \mathcal{E}} E = J(L) \times M(L) - I$ and $|\mathcal{E}| \leq s-1$, which is a contradiction as $\dim(L) = s$.

(2) Symmetrically, we obtain from (1). □

LEMMA 6. *Let $L$ be a finite lattice with $\dim(L) = s$ and let $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ be a family of Ferrers relations in $J(L) \times M(L)$ such that $\bigcup_{i=1}^{s} F_i = J(L) \times M(L) - I$. If there is a subfamily $\mathcal{E}$ of $\mathcal{F}$ such that $\bigcup_{E \in \mathcal{E}} c(E) = J(L)$ or $\bigcup_{E \in \mathcal{E}} r(E) = M(L)$, then $|\mathcal{F}| = |\mathcal{E}|$.*

*Proof.* Let $\mathcal{E}$ be the subfamily of $\mathcal{F}$ with $\bigcup_{E \in \mathcal{E}} c(E) = J(L)$ and let $F_0 \in \mathcal{F} - \mathcal{E}$ with $F_0 \neq \emptyset$. For all $E \in \mathcal{E}$, let

$$E^* = E \cup \{(a,b) \in F_0 \mid a \in c(E)\}.$$

Since $c(E) \times r(E)$ is a rectangular Ferrers relation in $J(L) \times M(L)$ and $\bigcup_{E \in \mathcal{E}} c(E) = J(L)$, it follows that $E^*$ is also a Ferrers relation in $J(L) \times M(L)$ and $F_0 \subset \bigcup_{E \in \mathcal{E}} E^*$. Hence we have

$$\bigcup_{F \in \mathcal{F} - (\mathcal{E} \cup \{F_0\})} F \cup \bigcup_{E \in \mathcal{E}} E^* = J(L) \times M(L) - I$$

and $|(\mathcal{F} - (\mathcal{E} \cup \{F_0\})) \cup \{E^* \mid E \in \mathcal{E}\}| = |\mathcal{F}| - 1$, which is a contradiction. □

LEMMA 7. *Let $L$ be a finite lattice. If there are disjoint families $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_w$ of Ferrers relations in $J(L) \times M(L)$ such that $\bigcup_{F \in \mathcal{F}_i} c(F) = J(L)$ $(i = 1, 2, \cdots, w)$ with $\mathcal{F} = \bigcup_{i=1}^{w} \mathcal{F}_i$, then we have the followings:*

(1) $\bigcup_{F\in\mathcal{F}} F = J(L) \times M(L) - I$ *implies that* $\dim(L) \leq |\mathcal{F}| - w + 1$

(2) $[J(L)\times M(L)-I]-\bigcup_{F\in\mathcal{F}} F \neq \emptyset$ *implies that* $\dim(L) \leq |\mathcal{F}|-w+k$, *where* $k = \dim([J(L) \times M(L) - I] - \bigcup_{F\in\mathcal{F}} F)$.

*Proof.* (1) Note that $|\mathcal{F}_i| \geq 2$ for all $i = 1, 2, \cdots, w$. Now we construct new Ferrers relations from $\mathcal{F}$ as follows:

$$\mathcal{F}_2^* = \{F_2^* \mid F_2 \in \mathcal{F}_2\} \text{ with } F_2^* = F_2 \cup \{(a,b) \in F_1 \mid a \in c(F_2)\}$$
$$\mathcal{F}_3^* = \{F_3^* \mid F_3 \in \mathcal{F}_3\} \text{ with } F_3^* = F_3 \cup \{(a,b) \in E_2^* \mid a \in c(F_3)\}$$
$$\cdots$$
$$\mathcal{F}_w^* = \{F_w^* \mid F_w \in \mathcal{F}_w\} \text{ with } F_w^* = F_w \cup \{(a,b) \in E_{w-1}^* \mid a \in c(F_w)\}$$
$$\mathcal{F}^* = (\mathcal{F}_1 - \{F_1\}) \cup \bigcup_{i=2}^{w-1} (\mathcal{F}_i^* - \{E_i^*\}) \cup \mathcal{F}_w^*$$

for some $F_1 \in \mathcal{F}_1$ and $E_i^* \in \mathcal{F}_i^*$ $(i = 2, 3, \cdots, w-1)$. Then $\mathcal{F}^*$ is the set of Ferrers relations in $J(L) \times M(L)$ and $\bigcup_{F\in\mathcal{F}^*} F = J(L) \times M(L) - I$ with $|\mathcal{F}^*| = |\mathcal{F}| - w + 1$. Hence we conclude that $\dim(L) \leq |\mathcal{F}| - w + 1$.

(2) Suppose that $[J(L) \times M(L) - I] - \bigcup_{F\in\mathcal{F}} F \neq \emptyset$. Then there is a Ferrers relation $F_0$ in $[J(L) \times M(L) - I] - \bigcup_{F\in\mathcal{F}} F$ with $F_0 \neq \emptyset$. Now we construct Ferrers relations from $\mathcal{F}$ as follows:

$$\mathcal{F}_1^* = \{F_1^* \mid F_1 \in \mathcal{F}_1\} \text{ with } F_1^* = F_1 \cup \{(a,b) \in F_0 \mid a \in c(F_1)\}$$
$$\mathcal{F}_2^* = \{F_2^* \mid F_2 \in \mathcal{F}_2\} \text{ with } F_2^* = F_2 \cup \{(a,b) \in E_1^* \mid a \in c(F_2)\}$$
$$\mathcal{F}_3^* = \{F_3^* \mid F_3 \in \mathcal{F}_3\} \text{ with } F_3^* = F_3 \cup \{(a,b) \in E_2^* \mid a \in c(F_3)\}$$
$$\cdots$$
$$\mathcal{F}_w^* = \{F_w^* \mid F_w \in \mathcal{F}_w\} \text{ with } F_w^* = F_w \cup \{(a,b) \in E_{w-1}^* \mid a \in c(F_w)\}$$
$$\mathcal{F}^* = \bigcup_{i=1}^{w-1} (\mathcal{F}_i^* - \{E_i^*\}) \cup \mathcal{F}_w^*$$

for some $E_i^* \in \mathcal{F}_i^*$ $(i = 1, 2, \cdots, w-1)$. Then $\mathcal{F}^*$ is the set of Ferrers relations in $J(L) \times M(L)$ and $\bigcup_{F\in\mathcal{F}^*} F = \bigcup_{F\in\mathcal{F}} F \cup \{F_0\}$ with $|\mathcal{F}^*| = |\mathcal{F}| - w$. Hence we conclude that $\dim(L) \leq |\mathcal{F}| - w + k$, where $k = \dim([J(L) \times M(L) - I] - \bigcup_{F\in\mathcal{F}} F)$. □

REMARK. By Lemma 7, we know that if $[J(L) \times M(L) - I] - \bigcup_{F\in\mathcal{F}} F \neq \emptyset$, then we obtains that $\dim([J(L) \times M(L) - I] - \bigcup_{F\in\mathcal{F}} F) \geq \dim(L) - |\mathcal{F}| + w$. In particular, if $[J(L) \times M(L) - I] - \bigcup_{F\in\mathcal{F}} F \neq \emptyset$ and $|\mathcal{F}| \leq \dim(L)$, then we have $\dim([J(L) \times M(L) - I] - \bigcup_{F\in\mathcal{F}} F) \geq w$.

Consider the finite lattices $L_1$ and $L_2$ with dim $(L_1) = s$ and dim $(L_2) = t$. Then there are Ferrers relations $F_1, F_2, \cdots, F_s$ such that $J(L_1)\times M(L_1)-I_1 = \bigcup_{i=1}^{s} F_i$ and there are Ferrers relations $G_1, G_2, \cdots, G_t$ such that $J(L_2) \times M(L_2) - I_2 = \bigcup_{j=1}^{t} G_j$. For $A$ and $C$ with $A \subset J(L_1)$ and $C \subset J(L_2)$, we have the following properties:

$$\begin{aligned} \dim(J(L_1) \times C, M(L_1) \times \{1\}, I) &= \dim(J(L_1 \Box L_2), M(L_1) \times \{1\}, I) \\ &= \dim(J(L_1), M(L_1), I_1) = s \\ \dim(A \times J(L_2), \{1\} \times M(L_2), I) &= \dim(J(L_1 \Box L_2), \{1\} \times M(L_2), I) \\ &= \dim(J(L_2), M(L_2), I_2) = t. \end{aligned}$$

EXAMPLE. Consider the complemented modular lattices $L_1$ and $L_2$ with $L_1 = L_2 = M_3$. Then $J(M_3) = \{a, b, c\} = M(M_3)$ and $\dim(M_3) = 2$. Then there are two Ferrers relations $\mathcal{F} = \{F_1, F_2\}$ such that $F_1 = \{(a, b), (a, c), (b, c)\}$ and $F_2 = \{(b, a), (c, a), (c, b)\}$. But we have $J(L) \neq c(F_1) \cup c(F_2) = \{a, c\}$. Further, we know that $\dim(L_1 \Box L_2) = \dim(L_1) + \dim(L_2) = 4$ and so $\dim(L_1 \Box L_2) \neq \dim(L_1) + \dim(L_2) - 1$.

## 3. Proof of Theorem

Let $L_1$ and $L_2$ be finite lattices with $\dim(L_1) = s$ and $\dim(L_2) = t$ and let $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ and $\mathcal{G} = \{G_1, G_2, \cdots, G_t\}$ be the set of Ferrers relations in $J(L_1) \times M(L_1)$ and $J(L_2) \times M(L_2)$, respectively. Without loss of generality, we may assume that $\bigcup_{i=1}^{s} F_i = J(L_1) \times M(L_1) - I_1$ and $\bigcup_{i=1}^{s} c(F_i) = J(L_1)$ and that $\bigcup_{j=1}^{t} G_j = J(L_2) \times M(L_2) - I_2$. For each $F_i \in \mathcal{F}$ and $G_j \in \mathcal{G}$, let

$$\begin{aligned} F_i^* &= \{((a, c), (b, 1)) \mid (a, b) \in F_i \text{ and } c \in J(L_2)\} \\ G_j^* &= \{((a, c), (1, d)) \mid a \in J(L_1) \text{ and } (c, d) \in G_j\}. \end{aligned}$$

Then $F_i^*$ and $G_j^*$ are Ferrers relations in $J(L_1 \Box L_2) \times M(L_1 \Box L_2)$. In particular, for some $G_j \in \mathcal{G}$, let

$$U_{ij} = \{((a, c), (1, d)) \in G_j^* \mid (a, c) \in c(F_i^*)\}.$$

Then each $F_i^* \cup U_{ij} (i = 1, 2, \cdots, s)$ is also a Ferrers relation in $J(L_1 \Box L_2) \times M(L_1 \Box L_2)$. Since $\bigcup_{i=1}^{s} c(F_i) = J(L_1)$, it follows that $G_{j_0}^* \subset \bigcup_{i=1}^{s}(F_i^* \cup$

$U_{ij_0}$) for some $j_0$ with $1 \le j_0 \le t$ and hence

$$\bigcup_{i=1}^{s}(F_i^* \cup U_{ij_0}) \cup \bigcup_{j=1, j\neq j_0}^{t} G_j^* = J(L_1\Box L_2) \times M(L_1\Box L_2) - I.$$

Hence we have

$$\dim(L_1 \times L_2) \le \dim(L_1) + \dim(L_2) - 1.$$

Let $\mathcal{H}$ be a set of the saturated Ferrers relations in $J(L_1\Box L_2) \times M(L_1\Box L_2)$ with $|\mathcal{H}| \le \dim(L_1) + \dim(L_2) - 1 = s + t - 1$ such that $\bigcup_{H\in\mathcal{H}} H = J(L_1\Box L_2) \times M(L_1\Box L_2) - I$ and let $\mathcal{X} = \{H \in \mathcal{H} \mid r(H) = B' \times \{1\}\}$ and $\mathcal{Y} = \{H \in \mathcal{H} \mid r(H) = \{1\} \times D'\}$ for some $B' \subset M(L_1)$ and $D' \subset M(L_2)$.

Claim 1. $\mathcal{X} \cup \mathcal{Y}$ is a partition of $\mathcal{H}$.

Suppose not, that is, there is a Ferrers relation $H \in \mathcal{H}$ such that $\{(b,1),(1,d)\} \subseteq r(H)$. Then

$$\begin{aligned} H(b,1) = H(1,d) &= \{((a,c),(b,1)) \mid a \not\le b \text{ in } L_1 \text{ and } c \in J(L_2)\} \cap \\ &\quad \{((a,c),(1,d)) \mid a \in J(L_1) \text{ in and } c \not\le d \text{ in } L_2\}. \end{aligned}$$

Hence $H \cup \{((a,c),(b,1)) \mid a \not\le b \text{ in } L_1 \text{ and } c \in J(L_2)\}$ is also a Ferrers relation in $J(L_1\Box L_2) \times M(L_1\Box L_2)$ and $\{((a,c),(b,1)) \mid a \not\le b \text{ in } L_1$ and $c \in J(L_2)\} \not\subseteq H$, which is a contradiction. Hence we have $\mathcal{X} \cup \mathcal{Y}$ is a partition of $\mathcal{H}$.

Consider the projection mappings $\pi_1, \pi_2$ as follows:
Define $\pi_1 : \mathcal{H} \longrightarrow J(L_1) \times M(L_1)$ by

$$\pi_1(H) = \{(a,b) \mid (\{a\} \times J(L_2)) \times \{(b,1)\} \subseteq H\}$$

for $H \in \mathcal{H}$. Similarly, define $\pi_2 : \mathcal{H} \longrightarrow J(L_2) \times M(L_2)$ by

$$\pi_2(H) = \{(c,d) \mid (J(L_1) \times \{c\}) \times \{(1,d)\} \subseteq H\}$$

for $H \in \mathcal{H}$. Let $X$ be an arbitrary element of $\mathcal{X}$. If $(a_1,b_1),(a_2,b_2) \in \pi_1(X)$, then $a_1 \not\le b_1$ and $a_2 \not\le b_2$ in $L_1$ and $((a_1,c),(b_1,1)),((a_2,c),(b_2,1)) \in X$ for all $c \in J(L_2)$. Hence we know that $((a_1,c),(b_2,1)) \in X$ or $((a_2,c),(b_1,1)) \in X$ for all $c \in J(L_2)$ and hence $(a_1,b_2) \in \pi_1(X)$ or $(a_2,b_1) \in \pi_1(X)$. Thus $\pi_1(X)$ is a Ferrers relation in $J(L_1) \times M(L_1)$ for all $X \in \mathcal{X}$. Similarly, we know that $\pi_2(Y)$ is a Ferrers relation in $J(L_2) \times M(L_2)$ for all $Y \in \mathcal{Y}$.

Define $p_1 : \mathcal{H} \longrightarrow J(L_1) \times M(L_1)$ by

$$p_1(H) = \{(a,b) \mid ((a,c),(b,1)) \in H\}$$

for $H \in \mathcal{H}$. Similarly, define $p_2 : \mathcal{H} \longrightarrow J(L_2) \times M(L_2)$ by

$$p_2(H) = \{(c',d') \mid ((a',c'),(1,d')) \in H\}$$

for $H \in \mathcal{H}$. For $X \in \mathcal{X}$, if $(a_1,b_1),(a_2,b_2) \in p_1(X)$, then $a_1 \not\leq b_1, a_2 \not\leq b_2$ in $L_1$ and $((a_1,c),(b_1,1)),((a_2,c'),(b_2,1)) \in X$ for some $c,c' \in J(L_2)$. Hence we know that $((a_1,c),(b_2,1)) \in X$ or $((a_2,c'),(b_1,1)) \in X$ for some $c,c' \in J(L_2)$ and hence $(a_1,b_2) \in p_1(X)$ or $(a_2,b_1) \in p_1(X)$. Thus $p_1(X)$ is a Férrers relation in $J(L_1) \times M(L_1)$ for all $X \in \mathcal{X}$. Similarly, we know that $p_2(Y)$ is also a Ferrers relation in $J(L_2) \times M(L_2)$ for all $Y \in \mathcal{Y}$. Hence, for $H \in \mathcal{H}$, we conclude that $\pi_i(H)$ and $p_i(H)$ are saturated Ferrers relation in $J(L_i) \times M(L_i)$ for $i = 1,2$.

Claim 2. $|\mathcal{X}| \geq s$ or $|\mathcal{Y}| \geq t$.

Suppose that $|\mathcal{X}| = s_1 < s$ and $|\mathcal{Y}| = t_1 < t$.

*Step* 1. For all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, we know that $p_1(X)$ and $p_2(Y)$ are saturated Ferrers relations in $J(L_1) \times M(L_1)$ and $J(L_2) \times M(L_2)$, respectively. Then there is at least one Ferrers relation $U_1$ in $J(L_1 \Box L_2) \times M(L_1 \Box L_2)$ such that

$$(\{a_1\} \times J(L_2)) \times \{(b_1,1)\} \subset U_1 - \bigcup_{X \in \mathcal{X}} X \subset \bigcup_{Y \in \mathcal{Y}} Y$$

for some $(a_1,b_1) \in J(L_1) \times M(L_1) - I_1$. Similarly, there is at least one Ferrers relation $V_1$ in $J(L_1 \Box L_2) \times M(L_1 \Box L_2)$ such that

$$(J(L_1) \times \{c_1\}) \times \{(1,d_1)\} \subset V_1 - \bigcup_{Y \in \mathcal{Y}} Y \subset \bigcup_{X \in \mathcal{X}} X$$

for some $(c_1,d_1) \in J(L_2) \times M(L_2) - I_2$. If $\{\pi_1(X) \mid X \in \mathcal{X}\}$ does not a join-cover of $L_1$ or $\{\pi_2(Y) \mid Y \in \mathcal{Y}\}$ does not a join-cover of $L_2$, then $U_1 \not\subset \bigcup_{H \in \mathcal{H}} H$ or $V_1 \not\subset \bigcup_{H \in \mathcal{H}} H$, which is a contradiction. Then we may assume that $\bigcup_{X \in \mathcal{X}} c(\pi_1(X)) = J(L_1)$ and $\bigcup_{X' \in \mathcal{X}'} c(\pi_1(X')) \neq J(L_1)$ and that $\bigcup_{Y \in \mathcal{Y}} c(\pi_2(Y)) = J(L_2)$ for all $\mathcal{X}' \subset \mathcal{X}$. Since $(\{a_1\} \times J(L_2)) \times \{(b_1,1)\} \subset U_1 \subset \bigcup_{Y \in \mathcal{Y}} Y$, it follows that $\dim([J(L_1 \Box L_2) \times (\{1\} \times M(L_2)) - I] - V_1 - \bigcup_{Y \in \mathcal{Y}} Y) \geq 1$ by Lemma 7. Hence there is a Ferrers relation $V_2$ in $J(L_1 \Box L_2) \times M(L_1 \Box L_2)$ such that

$$(J(L_1) \times \{c_2\} - c(U_1)) \times \{(1,d_2)\} \subset V_2 - V_1 - \bigcup_{Y \in \mathcal{Y}} Y \subset \bigcup_{X \in \mathcal{X}} X$$

for some $(c_2, d_2) \in J(L_2) \times M(L_2) - I_2$. But, since $(J(L_1) \times \{c_1\}) \times \{(1, d_1)\} \subset V_1 \subset \bigcup_{X \in \mathcal{X}} X$ and $\bigcup_{X' \in \mathcal{X}'} c(\pi_1(X')) \neq J(L_1)$ for all $\mathcal{X}' \subset \mathcal{X}$, it follows that $X \cap V_1 \neq \emptyset$ and $c(\pi_1(X)) \not\subset \bigcup_{A \in \mathcal{X} - \{X\}} c(\pi_1(A))$ for all $X \in \mathcal{X}$ and hence there is at least one element $a_0 \in c(\pi_1(X)) - \bigcup_{A \in \mathcal{X} - \{X\}} c(\pi_1(A))$ such that

$$\{((a_0, c), (1, d)) \mid ((a, c), (1, d)) \in V_1\} \subset X$$

for all $X \in \mathcal{X}$. Further, there are two distinct elements $((a_0, c'), (1, d')) \in V_1$ and $((a, c''), (1, d'')) \in V_2$ such that

$$c' < d'' \text{ and } c'' < d' \text{ in } L_2$$

for all $a \in J(L_1)$. Hence we have $((a_0, c''), (1, d'')) \notin X$ for all $X \in \mathcal{X}$. And hence $((a_0, c''), (1, d'')) \in V_2 \not\subset \bigcup_{X \in \mathcal{X}} X$, which is a contradiction as $V_2 \subset \bigcup_{X \in \mathcal{X}} X$. Similarly, it is impossible that $\bigcup_{Y \in \mathcal{Y}} c(\pi_2(Y)) = J(L_2)$ and $\bigcup_{Y' \in \mathcal{Y}'} c(\pi_2(Y')) \neq J(L_2)$ for all $\mathcal{Y}' \subset \mathcal{Y}$.

*Step* 2. By Step 1, there are proper subfamilies $\mathcal{X}_1$ and $\mathcal{Y}_1$ of $\mathcal{X}$ and $\mathcal{Y}$, respectively, with $\bigcup_{X \in \mathcal{X}_1} p_1(X) \cap p_1(U_1) = \emptyset$ and $\bigcup_{Y \in \mathcal{Y}_1} p_2(Y) \cap p_2(V_1) = \emptyset$ such that

$$\bigcup_{X \in \mathcal{X}_1} c(\pi_1(X)) = J(L_1) \text{ and } V_1 \subset \bigcup_{X \in \mathcal{X}_1} X,$$

$$\bigcup_{Y \in \mathcal{Y}_1} c(\pi_2(Y)) = J(L_2) \text{ and } U_1 \subset \bigcup_{Y \in \mathcal{Y}_1} Y.$$

Since $\{\pi_1(X) \mid X \in \mathcal{X}_1\}$ is a join-cover of $L_1$ and it is not a meet-cover of $L_1$, it follows that $\dim([J(L_1 \square L_2) \times (M(L_1) \times \{1\}) - I] - U_1 - \bigcup_{X \in \mathcal{X}} X) \geq 1$ by Lemma 7. Since each $p_1(X)(X \in \mathcal{X})$ is a saturated Ferrers relation in $J(L_1) \times M(L_1)$, it follows that there is a Ferrers relation $U_2$ in $J(L_1 \square L_2) \times M(L_1 \square L_2)$ such that

$$(\{a_2\} \times J(L_2) - c(V_1)) \times \{(b_2, 1)\} \subset U_2 - U_1 - \bigcup_{X \in \mathcal{X}} X,$$

$$p_1(U_2) \subset \bigcup_{X \in \mathcal{X}_1} p_1(X) \text{ and } U_2 \subset \bigcup_{Y \in \mathcal{Y} - \mathcal{Y}_1} Y$$

for some $(a_2, b_2) \in J(L_1) \times M(L_1) - I_1$. Similarly, since $\{\pi_2(Y) \mid Y \in \mathcal{Y}_1\}$ is the join-cover of $L_2$ and it is not a meet-cover of $L_2$, it follows that $\dim([J(L_1 \square L_2) \times (\{1\} \times M(L_2)) - I] - V_1 - \bigcup_{Y \in \mathcal{Y}} Y) \geq 1$ by Lemma 7. Since each $p_2(Y)(Y \in \mathcal{Y})$ is a saturated Ferrers relation in $J(L_2) \times M(L_2)$, it follows that there is a Ferrers relation $V_2$ in $J(L_1\Box L_2) \times M(L_1\Box L_2)$ such that

$$(J(L_1) \times \{c_2\} - c(U_1)) \times \{(1, d_2)\} \subset V_2 - V_1 - \bigcup_{Y\in\mathcal{Y}} Y,$$

$$p_2(V_2) \subset \bigcup_{Y\in\mathcal{Y}_1} p_2(Y) \text{ and } V_2 \subset \bigcup_{X\in\mathcal{X}-\mathcal{X}_1} X$$

for some $(c_2, d_2) \in J(L_2) \times M(L_2) - I_2$.

Consider the subordered sets $A'$ and $C'$ of $J(L_1)$ and $J(L_2)$, respectively, as follows:

$$A' = c(p_1(U_1)) \cup \bigcup_{X\in\mathcal{X}-\mathcal{X}_1} c(p_1(X)) \text{ and } C' = c(p_2(V_1)) \cup \bigcup_{Y\in\mathcal{Y}-\mathcal{Y}_1} c(p_2(Y)).$$

Suppose that $A' \subset J(L_1)$ or $C' \subset J(L_2)$ and that $a_1 < b_2, a_2 < b_1$ in $L_1$ and $c_1 < d_2, c_2 < d_1$ in $L_2$. Since $(\{a_1\} \times J(L_2)) \times \{(b_1, 1)\} \subset U_1 \subset \bigcup_{Y\in\mathcal{Y}_1} Y$ and $(J(L_1) \times \{c_1\}) \times \{(1, d_1)\} \subset V_1 \subset \bigcup_{X\in\mathcal{X}_1} X$, it follows that, for all $X \in \mathcal{X}_1$ and $Y \in \mathcal{Y}_1$,

$$[(J(L_1)\times\{c_1\})\times\{(1,d_1)\}]\cap X \neq \emptyset \text{ and } [(\{a_1\}\times J(L_2))\times\{(b_1,1)\}]\cap Y \neq \emptyset$$

and hence we obtain that

$$(J(L_1) \times \{c_2\}) \times \{(1, d_2)\} \cap \bigcup_{X\in\mathcal{X}_1} X = \emptyset \text{ and}$$

$$(\{a_2\} \times J(L_2)) \times \{(b_2, 1)\} \cap \bigcup_{Y\in\mathcal{Y}_1} Y = \emptyset.$$

Thus we have $U_2 \not\subset \bigcup_{Y\in\mathcal{Y}_1} Y$ and $V_2 \not\subset \bigcup_{X\in\mathcal{X}_1} X$. Further, since $A' \subset J(L_1)$ or $C' \subset J(L_2)$, it follows that

$$(\{a_2\} \times (J(L_2) - C')) \times \{(b_2, 1)\} \not\subset \bigcup_{X\in\mathcal{X}_1} X \cup \bigcup_{Y\in\mathcal{Y}-\mathcal{Y}_1} Y$$

or

$$((J(L_1) - A') \times \{c_2\}) \times \{(1, d_2)\} \not\subset \bigcup_{Y\in\mathcal{Y}_1} Y \cup \bigcup_{X\in\mathcal{X}-\mathcal{X}_1} X.$$

Since $p_1(U_2) \subset \bigcup_{X\in\mathcal{X}_1} p_1(X)$ and $p_2(V_2) \subset \bigcup_{Y\in\mathcal{Y}_1} p_2(Y)$, it follows that

$$p_1(U_2) \not\subset \bigcup_{X\in\mathcal{X}-\mathcal{X}_1} p_1(X) \text{ and } p_2(V_2) \not\subset \bigcup_{Y\in\mathcal{Y}-\mathcal{Y}_1} p_2(Y).$$

Hence we have

$$(\{a_2\} \times (J(L_2) - C')) \times \{(b_2, 1)\} \not\subset \bigcup_{H\in\mathcal{H}} H$$

or

$$((J(L_1) - A') \times \{c_2\}) \times \{(1, d_2)\} \not\subset \bigcup_{H \in \mathcal{H}} H,$$

which is a contradiction. Hence we may assume that $A' = J(L_1)$ and $C' = J(L_2)$, that is, there is a subfamily $\mathcal{X}_2$ of $\mathcal{X} - \mathcal{X}_1$ such that

$$(J(L_1) \times \{c_2\} - c(U_1)) \times \{(1, d_2)\} \subset V_2 - V_1 - \bigcup_{Y \in \mathcal{Y}} Y,$$

$$J(L_1) - c(\pi_1(U_1)) \subseteq \bigcup_{X \in \mathcal{X}_2} c(\pi_1(X)) \text{ and } V_2 \subset \bigcup_{X \in \mathcal{X}_2} X$$

and there is a subfamily $\mathcal{Y}_2$ of $\mathcal{Y} - \mathcal{Y}_1$ such that

$$(\{a_2\} \times J(L_2) - c(V_1)) \times \{(b_2, 1)\} \subset U_2 - U_1 - \bigcup_{X \in \mathcal{X}} X,$$

$$J(L_2) - c(\pi_2(V_1)) \subseteq \bigcup_{Y \in \mathcal{Y}_2} c(\pi_2(Y)) \text{ and } U_2 \subset \bigcup_{Y \in \mathcal{Y}_2} Y.$$

*Step* 3. Let $k$ be a positive integer with $k \geq 2$. By step 2, we know that $\{\pi_1(X) \mid X \in \mathcal{X}_k\} \cup \{p_1(U_{k-1})\}$ is a join-cover of $L_1$ and it is not a meet-cover of $L_1$. By Lemma 7, we have $\dim([J(L_1 \square L_2) \times (M(L_1) \times \{1\}) - I] - \bigcup_{i=1}^{k} U_i - \bigcup_{X \in \mathcal{X}} X) \geq 1$. Hence there is a Ferrers relation $U_{k+1}$ in $J(L_1 \square L_2) \times M(L_1 \square L_2)$ such that

$$(\{a_{k+1}\} \times J(L_2) - c(V_k)) \times \{(b_{k+1}, 1)\} \subset U_{k+1} - \bigcup_{i=1}^{k} U_i - \bigcup_{X \in \mathcal{X}} X,$$

$$p_1(U_{k+1}) \subset \bigcup_{X \in \mathcal{X}_k} p_1(X) \text{ and } U_{k+1} \subset \bigcup_{Y \in \mathcal{Y} - \cup_{j=1}^{k} \mathcal{Y}_j} Y$$

for some $(a_{k+1}, b_{k+1}) \in J(L_1) \times M(L_1) - I_1$. Hence there is a subfamily $\mathcal{Y}_{k+1}$ of $\mathcal{Y} - \bigcup_{j=1}^{k} \mathcal{Y}_j$ such that

$$J(L_2) - c(\pi_2(V_k)) \subseteq \bigcup_{Y \in \mathcal{Y}_{k+1}} c(\pi_2(Y)) \text{ and}$$

$$(\{a_{k+1}\} \times J(L_2) - c(V_k)) \times \{(b_{k+1}, 1)\} \subset U_{k+1} \subset \bigcup_{Y \in \mathcal{Y}_{k+1}} Y.$$

Similarly, since $\{\pi_2(Y) \mid Y \in \mathcal{Y}_k\} \bigcup \{p_2(V_{k-1})\}$ is a join-cover of $L_2$ and it is not a meet-cover of $L_2$, then there is a Ferrers relation $V_{k+1}$ in $J(L_1 \Box L_2) \times M(L_1 \Box L_2)$ such that

$$(J(L_1) \times \{c_{k+1}\} - c(U_k)) \times \{(1, d_{k+1})\} \subset V_{k+1} - \bigcup_{j=1}^{k} V_j - \bigcup_{Y \in \mathcal{Y}} Y,$$

$$p_2(V_{k+1}) \subset \bigcup_{Y \in \mathcal{Y}_k} p_2(Y) \text{ and } V_{k+1} \subset \bigcup_{X \in \mathcal{X} - \cup_{i=1}^{k} \mathcal{X}_i} X$$

for some $(c_{k+1}, d_{k+1}) \in J(L_2) \times M(L_2) - I_2$. Hence there is a subfamily $\mathcal{X}_{k+1}$ of $\mathcal{X} - \bigcup_{i=1}^{k} \mathcal{X}_i$ such that

$$J(L_1) - c(\pi_1(U_k)) \subseteq \bigcup_{X \in \mathcal{X}_{k+1}} c(\pi_1(X)) \text{ and}$$

$$(J(L_1) \times \{c_{k+1}\} - c(U_k)) \times \{(1, d_{k+1})\} \subset V_{k+1} \subset \bigcup_{X \in \mathcal{X}_{k+1}} X.$$

By the finite repeating of the above same methods, without loss of generality, we may assume that there is a Ferrers relation $U_{m+1}$ in $J(L_1 \Box L_2) \times M(L_1 \Box L_2)$ and $V_{n+1}$ in $J(L_1 \Box L_2) \times M(L_1 \Box L_2)$ such that

$$(\{a_{m+1}\} \times J(L_2) - c(V_m)) \times \{(b_{m+1}, 1)\} \subset U_{m+1} - \bigcup_{i=1}^{m} U_i - \bigcup_{X \in \mathcal{X}} X,$$

$$p_1(U_{m+1}) \subset \bigcup_{X \in \mathcal{X}_m} p_1(X) \text{ and } \bigcup_{X \in \mathcal{X} - \cup_{i=1}^{m} \mathcal{X}_i} c(\pi_1(X)) \not\subseteq J(L_1)$$

and

$$(J(L_1) \times \{c_{n+1}\} - c(U_n)) \times \{(1, d_{n+1})\} \subset V_{n+1} - \bigcup_{j=1}^{n} V_j - \bigcup_{Y \in \mathcal{Y}} Y,$$

$$p_2(V_{n+1}) \subset \bigcup_{Y \in \mathcal{Y}_n} p_2(Y) \text{ and } \bigcup_{Y \in \mathcal{Y} - \cup_{j=1}^{n} \mathcal{Y}_j} c(\pi_2(Y)) \not\subseteq J(L_2).$$

for some $(a_{m+1}, b_{m+1}) \in J(L_1) \times M(L_1) - I_1$ and $(c_{n+1}, d_{n+1}) \in J(L_2) \times M(L_2) - I_2$. Note that for any two distinct Ferrers relations $U_i$ and $U_j$ of $\{U_1, U_2, \cdots, U_{m+1}\}$, $U_i \cup U_j$ is not a Ferrers relation and so there exist elements $(a_i, b_i) \in p_1(U_i)$ and $(a_j, b_j) \in p_1(U_j)$ such that $a_i < b_j$ and $a_j < b_i$ in $L_1$. Similarly, for any two distinct Ferrers relations $V_k$ and $V_l$ of $\{V_1, V_2, \cdots, V_{n+1}\}$, $V_k \cup V_l$ is not a Ferrers relation and so there exist elements $(c_k, d_k) \in p_2(V_k)$ and $(c_l, d_l) \in p_2(V_l)$ such that $c_k < d_l$ and

$c_l < d_k$ in $L_2$. Then $\mathcal{X}_i$ $(i = 1, 2, \cdots, m \leq n)$ and $\mathcal{Y}_j$ $(j = 1, 2, \cdots, n)$ have the following properties:

(i) $\mathcal{X}_1$ and $\mathcal{Y}_1$ are join-covers of $L_1 \Box L_2$ which do not contain a join-cover proper subfamily.

(ii) For all $i = 2, 3, \cdots, m$, $\mathcal{X}_i$ or $\mathcal{X}_i \cup \{U_i\}$ is a join-cover of $L_1 \Box L_2$ which does not contains a join-cover proper subfamily.

(iii) For all $j = 2, 3, \cdots, n$, $\mathcal{Y}_j$ or $\mathcal{Y}_j \cup \{V_j\}$ is a join-cover of $L_1 \Box L_2$ which does not contains a join-cover proper subfamily.

(iv) $J(L_1 \Box L_2) \times (M(L_1) \times \{1\}) - I = \bigcup_{X \in \mathcal{X}} X \cup \bigcup_{i=1}^{m+1} U_i$ and $U_i \not\subset \bigcup_{X \in \mathcal{X}} X$ for all $i = 1, 2, \cdots, m+1$.

(v) $J(L_1 \Box L_2) \times (\{1\} \times M(L_2)) - I = \bigcup_{Y \in \mathcal{Y}} Y \cup \bigcup_{j=1}^{n} V_j$ and $V_j \not\subset \bigcup_{Y \in \mathcal{Y}} Y$, $V_{j_0} \not\subset \bigcup_{Y \in \mathcal{Y}} Y$ for all $j = 1, 2, \cdots, m$ and for some $V_{j_0} \in \{V_{m+1}, V_{m+2}, \cdots, V_{n+1}\}$.

We let

$$\mathcal{U} = \{U_i \mid 1 \leq i \leq m+1\} \cup (\mathcal{X} - \bigcup_{i=1}^{m} \mathcal{X}_i),$$

$$\mathcal{V} = \{V_j \mid 1 \leq j \leq n+1\} \cup (\mathcal{Y} - \bigcup_{j=1}^{n} \mathcal{Y}_j).$$

By (iv) and (v), we know that

$$\bigcup_{i=1}^{m+1} U_i \subseteq \bigcup_{Y \in \mathcal{Y}} Y \text{ and } \bigcup_{j=1}^{m} V_j \cup V_{j_0} \subseteq \bigcup_{X \in \mathcal{X}} X$$

If $\mathcal{U}$ and $\mathcal{V}$ do not join-covers of $L_1 \Box L_2$, then we know that $U_{m+1} \not\subset \bigcup_{H \in \mathcal{H}} H$ and $V_{j_0} \not\subset \bigcup_{H \in \mathcal{H}} H$. Further, even if $\mathcal{U}$ and $\mathcal{V}$ are join-covers of $L_1 \Box L_2$, but, since $|\mathcal{U}| \leq |\mathcal{X}| + 1 \leq \dim(L_1)$ and $|\mathcal{V}| \leq |\mathcal{Y}| + 1 \leq \dim(L_2)$, we know that neither $\mathcal{U} - \{U\}$ nor $\mathcal{V} - \{V\}$ is join-cover of $L_1 \Box L_2$ for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Hence, if $U_{m+1} \subset \bigcup_{V \in \mathcal{V}} V$, then $\mathcal{U} - \{U_{m+1}\}$ is not a join-cover of $L_1 \Box L_2$ and hence $V_{j_0} \not\subset \bigcup_{H \in \mathcal{H}} H$, which is a contradiction. Similarly, if $V_{j_0} \subset \bigcup_{U \in \mathcal{U}} U$, then $U_{m+1} \not\subset \bigcup_{H \in \mathcal{H}} H$, which is a contradiction. Then the contradiction completes the proof of the second claim.

CLAIM 3. $|\mathcal{X}| \leq s$ and $|\mathcal{Y}| \leq t$.

Suppose that $|\mathcal{X}| > s$. Note that $|\mathcal{H}| \leq s + t - 1$. Without loss of generality, we may assume that $|\mathcal{X}| = s + k (1 \leq k \leq t - 1)$ and

$|\mathcal{Y}| = t-(k+2)$. Since $|\mathcal{Y}| = t-(k+2)$ and $\dim(J(L_1\Box L_2)\times(\{1\}\times M(L_1))) = t$, we have $\dim([J(L_2)\times M(L_2)-I_2]-\bigcup_{Y\in\mathcal{Y}} p_2(Y)) \geq k+2$ and hence there are $(k+2)$-distinct Ferrers relations $V_1, V_2, \cdots, V_{k+2}$ in $[J(L_1\Box L_2)\times M(L_1\Box L_2)-I]-\bigcup_{Y\in\mathcal{Y}} Y$ such that

$$\pi_2(V_j) = p_2(V_j),\ c(P_2(V_i)) \neq c(P_2(V_j)) \text{ and}$$

$$(J(L_1)\times\{c_j\})\times\{(1,d_j)\}\subset V_j \subset \bigcup_{X\in\mathcal{X}} X$$

for some $(c_j, d_j)\in J(L_2)\times M(L_2)-I_2$ and for all $1\leq i,\ j\leq k+2$ with $i\neq j$. Since $|\mathcal{Y}| = t-(k+2)$ and $\dim(J(L_1\Box L_2)\times(\{1\}\times M(L_1))) = t$, we have

$$c(p_2(V_j)) - \bigcup_{Y\in\mathcal{Y}} c(p_2(Y) \neq \emptyset$$

for all $j = 1, 2, \cdots, k+2$. Hence there are $(k+2)$-distinct subfamilies $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_{k+2}$ of $\mathcal{X}$ such that

$$\bigcup_{X_j\in\mathcal{X}_j} c(\pi_1(X_j)) = J(L_1) \text{ and } (J(L_1)\times\{c_j\})\times\{(1,d_j)\}\subset V_j\subset \bigcup_{X_j\in\mathcal{X}_j} X_j$$

for all $j = 1, 2, \cdots, k+2$. If $J(L_1\Box L_2)\times(M(L_1)\times\{1\})\subseteq\bigcup_{X\in\mathcal{X}} X$ or $\mathcal{X} = \bigcup_{i=1}^{k+2}\mathcal{X}_i$, then $\dim(L_1)\leq s+k-(k+1) = s-1$, which is a contradiction. Then we may assume that $J(L_1\Box L_2)\times(M(L_1)\times\{1\}) - \bigcup_{X\in\mathcal{X}} X\neq\emptyset$ and $\mathcal{X}\neq\bigcup_{i=1}^{k+2}\mathcal{X}_i$. Hence we know that $\dim(J(L_1\Box L_2)\times(M(L_1)\times\{1\})-\bigcup_{X\in\mathcal{X}} X)\geq 2$ by Lemma 7. Without loss of generality, we may assume that there are at least two distinct Ferrers relations $U_1$ and $U_2$ in $J(L_1\Box L_2)\times M(L_1\Box L_2)$ such that

$$(\{a_1\}\times J(L_2)-C_1)\times\{(b_1,1)\}\subset U_1 - U_2 - \bigcup_{X\in\mathcal{X}} X,$$

$$(\{a_2\}\times J(L_2)-C_2)\times\{(b_2,1)\}\subset U_2 - U_1 - \bigcup_{X\in\mathcal{X}} X,$$

$$p_1(U_1)\cup p_1(U_2)\subset \bigcup_{X\in\cup_{i=1}^{k+2}\mathcal{X}_i} p_1(X) \text{ and } U_1\cup U_2\subset\bigcup_{Y\in\mathcal{Y}} Y$$

for some $(a_i,b_i)\in J(L_1)\times M(L_1)-I_1 (i=1,2)$ with $a_1 < b_2,\ a_2 < b_1$ in $L_1$ and $C_1\cup C_2 = \bigcup_{j=1}^{k+2} c(V_j)$. If $J(L_1\Box L_2)\times(\{1\}\times M(L_2)) - I\subseteq\bigcup_{Y\in\mathcal{Y}} Y\cup\bigcup_{j=1}^{k+2} V_j$, then we have $U_1\not\subset\bigcup_{H\in\mathcal{H}} H$ or $U_2\not\subset\bigcup_{H\in\mathcal{H}} H$ by Lemma 6. Then we may assume that $[J(L_1\Box L_2)\times(\{1\}\times M(L_2))-I]-(\bigcup_{Y\in\mathcal{Y}} Y\cup$

$\bigcup_{j=1}^{k+2} V_j) \neq \emptyset$. Hence we know that these conditions are the same situation in Claim 2. Without loss of generality, we may assume that there are finite subfamilies $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_{k+2}, \mathcal{X}_{k+3}, \mathcal{X}_{k+4}, \cdots, \mathcal{X}_{k+2m-1}, \mathcal{X}_{k+2m}$ of $\mathcal{X}$ and non-empty Ferrers relations $U_1, U_2, U_3, U_4, \cdots, U_{2m-1}, U_{2m}$ such that

(i) For all $i = 1, 2, \cdots, k+2$, each $\mathcal{X}_i$ is a join-cover of $L_1 \square L_2$ which does not contain join-cover proper subfamily.

(ii) For all $l = k+3, k+5, \cdots, k+2m-3$, $\mathcal{X}_l$ or $\mathcal{X}_l \cup \{U_{l-k-2}\}$ and $\mathcal{X}_{l+1}$ or $\mathcal{X}_{l+1} \cup \{U_{l-k-1}\}$ are join-covers of $L_1 \square L_2$ which do not contain join-cover proper subfamily.

(iii) $p_1(U_1) \cup p_1(U_2) \subset \bigcup_{X \in \cup_{i=1}^{k+2} \mathcal{X}_i} p_1(X)$ and $p_1(U_i) \subset \bigcup_{X \in \mathcal{X}_{k+i}} p_1(X)$ for all $i = 3, 4, \cdots, 2m$.

(iv) $J(L_1 \square L_2) \times (M(L_1) \times \{1\}) - I = \bigcup_{X \in \mathcal{X}} X \cup \bigcup_{i=1}^{2m} U_i$ and $U_i \not\subset \bigcup_{X \in \mathcal{X}} X$ for all $i = 1, 2, \cdots, 2m$.

Similarly, there are subfamilies $\mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_{2n-1}, \mathcal{Y}_{2n}$ of $\mathcal{Y}$ and non-empty Ferrers relations $V_1, V_2, \cdots, V_{k+1}, V_{k+2}, \cdots, V_{k+2n+1}, V_{k+2n+2}$ with $m \leq n$ such that

(i)′ For all $j, j'$ with $1 \leq j, j' \leq k+2$ and $j \neq j'$, $c(p_2(V_j)) - \bigcup_{Y \in \mathcal{Y}} c(p_2(Y)) \neq \emptyset$ and $c(p_2(V_j)) \neq c(p_2(V_{j'}))$.

(ii)′ For all $j = 1, 3, 5, \cdots, 2n-1$, $p_2(V_{k+j+2}) \subset \bigcup_{Y \in \mathcal{Y}_j} p_2(Y)$ and $p_2(V_{k+j+3}) \subset \bigcup_{Y \in \mathcal{Y}_{j+1}} p_2(Y)$.

(iii)′ $\mathcal{Y}_1$ or $\mathcal{Y}_1 \cup \{V_{j_1}, V_{j_2}, \cdots, V_{j_u}\}$ and $\mathcal{Y}_2$ or $\mathcal{Y}_2 \cup \{V_{j_u+1}, V_{j_u+2}, \cdots, V_{j_{k+2}}\}$ are join-covers of $L_1 \square L_2$ which do not contain a join-cover proper subfamily for some rearrangement $\{j_1, j_2, \cdots, j_{k+2}\}$ of $\{1, 2, \cdots, k+2\}$

(iv)′ For all $j = 3, 5, \cdots, 2n-3$, $\mathcal{Y}_j$ or $\mathcal{Y}_j \cup \{V_{j+k}\}$ and $\mathcal{Y}_{j+1}$ or $\mathcal{Y}_{j+1} \cup \{V_{j+k+1}\}$ are join-covers of $L_1 \square L_2$ which do not contain a join-cover proper subfamily.

(v)′ $J(L_1 \square L_2) \times (\{1\} \times M(L_2)) - I = \bigcup_{Y \in \mathcal{Y}} Y \cup \bigcup_{j=1}^{2n+2} V_j$ and $V_j \not\subset \bigcup_{Y \in \mathcal{Y}} Y, V_{j_0} \not\subset \bigcup_{Y \in \mathcal{Y}} Y$, $V'_{j_0} \not\subset \bigcup_{Y \in \mathcal{Y}} Y$ for all $j = 1, 2, \cdots, 2m$ and for some $V_{j_0}$ and $V'_{j_0}$ of $\{V_{2m+1}, V_{2m+2}, \cdots, V_{2n+2}\}$.

We let

$$\mathcal{U} = \{U_i \mid 1 \leq i \leq 2m\} \cup (\mathcal{X} - \bigcup_{i=1}^{k+2m} \mathcal{X}_i) \text{ and}$$

$$\mathcal{V} = \{V_j \mid 1 \leq j \leq k+2n+2\} \cup (\mathcal{Y} - \bigcup_{j=1}^{2n} \mathcal{Y}_j).$$

By (iv) and (v)′, we know that

$$\bigcup_{i=1}^{2m} U_i \subseteq \bigcup_{Y\in\mathcal{Y}} Y \text{ and } \bigcup_{j=1}^{2m} V_j \cup V_{j_0} \cup V'_{j_0} \subseteq \bigcup_{X\in\mathcal{X}} X.$$

If $\mathcal{U}$ and $\mathcal{V}$ do not join-covers of $L_1\Box L_2$, then we have

$$U_{2m-1} \cup U_{2m} \not\subset \bigcup_{H\in\mathcal{H}} H \text{ and } V_{j_0} \cup V'_{j_0} \not\subset \bigcup_{H\in\mathcal{H}} H.$$

Further, we know that, if $\mathcal{U}-\{U\}$ or $\mathcal{V}-\{V\}$ is join-cover of $L_1\Box L_2$ for some $U\in\mathcal{U}$ and $V\in\mathcal{V}$, then we have $\dim(L_1)\leq s-1$ or $\dim(L_2)\leq t-1$. Hence we conclude that neither $\mathcal{U}-\{U\}$ nor $\mathcal{V}-\{V\}$ is join-cover of $L_1\Box L_2$ for all $U\in\mathcal{U}$ and $V\in\mathcal{V}$. Then, even if $\mathcal{U}$ and $\mathcal{V}$ are join-covers of $L_1\Box L_2$, but we know that neither $\mathcal{U}-\{U\}$ nor $\mathcal{V}-\{V\}$ is join-cover of $L_1\Box L_2$ for all $U\in\mathcal{U}$ and $V\in\mathcal{V}$. Hence, if $U_{2m-1}\cup U_{2m}\subset \bigcup_{V\in\mathcal{V}} V$, then $\mathcal{U}-\{U_{2m-1},U_{2m}\}$ is not a join-cover of $L_1\Box L_2$ and hence $V_{j_0}\cup V'_{j_0}\not\subset\bigcup_{H\in\mathcal{H}} H$, which is a contradiction. Similarly, if $V_{j_0}\subset\bigcup_{U\in\mathcal{U}} U$, then $U_{2m-1}\not\subset\bigcup_{H\in\mathcal{H}} H$ and $U_{2m}\not\subset\bigcup_{H\in\mathcal{H}} H$, which is a contradiction. Then the contradiction completes the proof of the third claim.

Claim 4. If $|\mathcal{X}|=s$, then there is at most one Ferrers relation $Y\in\mathcal{Y}$ such that $Y\subset\bigcup_{X\in\mathcal{X}} X$.

Clearly, there is one Ferrers relation $Y_0\in\mathcal{Y}$ such that $Y_0\subset\bigcup_{X\in\mathcal{X}} X$. Note that, for all $Y\in\mathcal{Y}$ and for all $a_i\in J(L_1)$,

$$Y\cap(\{a_i\}\times J(L_2),\{1\}\times M(L_2),I)\neq\emptyset$$

and that there is a pair $(c,d)\in J(L_2)\times M(L_2)-I_2$ such that $(J(L_1)\times\{c\})\times\{(1,d)\}\subseteq Y$. By Lemma 6, if $J(L_1\Box L_2)\times(M(L_1)\times\{1\})\subseteq\bigcup_{X\in\mathcal{X}} X$, then there exist at most one Ferrers relation $Y\in\mathcal{Y}$ such that $Y\subset\bigcup_{X\in\mathcal{X}} X$. Hence we have $|\mathcal{Y}|\geq t-1$ and hence $|\mathcal{H}|\geq s+t-1$.

Then we enough to show that $J(L_1\Box L_2)\times(M(L_1)\times\{1\})\subseteq\bigcup_{X\in\mathcal{X}} X$. Suppose that $J(L_1\Box L_2)\times(M(L_1)\times\{1\})\not\subseteq\bigcup_{X\in\mathcal{X}} X$ and that $|\mathcal{X}|=s$ and $|\mathcal{Y}|=t-2$. Note that neither $\mathcal{X}'$ nor $\mathcal{Y}'$ is a meet-cover of $L_1\Box L_2$ for all $\mathcal{X}'\subseteq\mathcal{X}$ and $\mathcal{Y}'\subseteq\mathcal{Y}$ and that $p_1(X)$ and $p_2(Y)$ are also saturated Ferrers relations in $J(L_1)\times M(L_1)$ and $J(L_2)\times M(L_2)$, respectively, for all $X\in\mathcal{X}$ and $Y\in\mathcal{Y}$. But, these conditions are the similar to the case in Claim 3 for $k=0$. By the similar methods in Claim 3, we obtains a contradiction. Then it is impossible that $J(L_1\Box L_2)\times(M(L_1)\times\{1\})\not\subseteq\bigcup_{X\in\mathcal{X}} X$ with $|\mathcal{X}|=s$ and $|\mathcal{Y}|=t-2$. Hence the proof of claim 4 is completed.

Consider the given $\mathcal{H}$ be the set of the saturated Ferrers relations $J(L_1\Box L_2) \times M(L_1\Box L_2) - I$ with $|\mathcal{H}| \leq \dim(L_1) + \dim(L_2) - 1$ such that $\bigcup_{H\in\mathcal{H}} H = J(L_1\Box L_2) \times M(L_1\Box L_2) - I$ and let $\mathcal{X} = \{H \in \mathcal{H} \mid r(H) = B' \times \{1\}\}$ and $\mathcal{Y} = \{H \in \mathcal{H} \mid r(H) = \{1\} \times D'\}$ for some $B' \subset M(L_1)$ and $D' \subset M(L_2)$. Further, we know that $\mathcal{X} \cup \mathcal{Y}$ is a partition of $\mathcal{H}$ and that $|\mathcal{X}| = s$ or $|\mathcal{Y}| = t$ by Claims 2 and 3. By Claim 4, there is at most one Ferrers relation $Y_0 \in \mathcal{Y}$ such that $Y_0 \subset \bigcup_{X\in\mathcal{X}} X$. Furthermore, for all $X \in \mathcal{X}$, there is $(a_i, b_i) \in J(L_1) \times M(L_1) - I_1$ such that $(\{a_i\} \times J(L_2)) \times \{(b_i, 1)\} \subseteq X$. Then, by Lemma 6, there does not exist Ferrers relation $X_0 \in \mathcal{X}$ such that $X_0 \subseteq \bigcup_{Y\in\mathcal{Y}-\{Y_0\}} Y$. Hence $|\mathcal{H}| = |\mathcal{X}| + |\mathcal{Y}| \geq \dim(L_1) + \dim(L_2) - 1$. Hence we conclude that $\dim(L_1\Box L_2) = \dim(L_1) + \dim(L_2) - 1$. □

## 4. Concluding remarks

In the main theorem, we have the following:

$$\begin{aligned} |\mathcal{X}| &= s \text{ iff } |\mathcal{Y}| = t - 1 \text{ for } \bigcup_{i=1}^{s} c(F_i) = J(L_1) \\ |\mathcal{X}| &= s - 1 \text{ iff } |\mathcal{Y}| = t \text{ for } \bigcup_{j=1}^{t} c(G_j) = J(L_2). \end{aligned}$$

By $(L_1\Box L_2) \cong (L_2\Box L_1)$ and Claims 1 and 2, if $|\mathcal{X}| = s - 1$, then there is no Ferrers relation $Y \in \mathcal{Y}$ such that $Y \subset \bigcup_{X\in\mathcal{X}} X$. If $|\mathcal{X}| = s$, then there is at most one Ferrers relation $Y \in \mathcal{Y}$ such that $Y \subset \bigcup_{X\in\mathcal{X}} X$. Hence even if $\bigcup_{i=1}^{s} c(F_i) = J(L_1)$ and $\bigcup_{j=1}^{t} c(G_j) = J(L_2)$, then $|\mathcal{X}| = s$ or $|\mathcal{X}| = s - 1$. Whenever $|\mathcal{X}| = s$ or $|\mathcal{Y}| = t$, then $\bigcup_{H\in\mathcal{H}} H$ is the largest subset of $J(L_1\Box L_2) \times M(L_1\Box L_2) - I$ and $|\mathcal{H}| = |\mathcal{X}| + |\mathcal{Y}| = s + t - 1$.

For any finite complemented modular lattice $L$, we known that $J(L) = A(L)$, $M(L) = DA(L)$ and $|J(L)| = |A(L)| = |M(L)| = |DA(L)|$. Then we have the following.

I. Let $L_1$ and $L_2$ be finite complemented modular lattices with $\dim(L_1) = s$ and $\dim(L_2) = t$. Suppose that $\mathcal{F} = \{F_1, F_2, \cdots, F_s\}$ is the set of Ferrers relations in $J(L_1) \times M(L_1)$ and $\mathcal{G} = \{G_1, G_2, \cdots, G_t\}$ is the set of Ferrers relations in $J(L_2) \times M(L_2)$. If $J(L_1) = \bigcup_{i=1}^{s} c(F_i)$ or $J(L_2) = \bigcup_{j=1}^{t} c(G_j)$, then we have

$$\dim(L_1 \square L_2) = \dim(L_1) + \dim(L_2) - 1.$$

Further, we know that $\mathbf{2}^2 \square \mathbf{2}^2 \cong C_4 \cup \{0,1\}$ with $\dim(\mathbf{2}^2 \square \mathbf{2}^2) = \dim(C_4 \cup \{0,1\}) = \dim(C_4) = 3$.

Furthermore, for any natural numbers $m$ and $n(\geq 2)$, we define an ordered set $R_n^m$ as follows:

$$R_n^m = J(\mathbf{2}^n \square M_m) \cup M(\mathbf{2}^n \square M_m).$$

Then, for all integers $i, k$ and $j, l$ with $1 \leq i, k \leq n$ and $1 \leq j, l \leq m$, we define the ordered set as follows:

$$A(R_n^m) = \{a_{ij}\},\ DA(R_n^m) = \{b_{k1}, b_{0l}\},$$

$$a_{ij} < b_{k1} \text{ iff } i \neq k \text{ and } a_{ij} < b_{0l} \text{ iff } j = l.$$

In fact, we know that for any natural number $n(\geq 2)$, there are infinitely many ordered sets $R_n^m$ with $\dim(R_n^m) = n+1$. In particular, $R_n^2$ is an irreducible ordered set with $\dim(R_n^2) = n+1$ and

$$\dim(\mathbf{2}^2 \square \mathbf{2}^2) = \dim(\mathbf{2}^2) + \dim(\mathbf{2}^2) - 1 = 3,$$

$$\dim(R_n^m) = \dim(\mathbf{2}^n \square M_m) = \dim(\mathbf{2}^n) + \dim(M_m) - 1 = n+1.$$

II. Note that, for all integers $i, k$ and $j, l$ with $1 \leq i, k \leq n$ and $1 \leq j, l \leq 2$,

$$A(R_n^2) = \{a_{ij}\} \text{ and } DA(R_n^2) = \{b_{k1}, b_{0l}\};$$

$$a_{ij} < b_{k1} \text{ iff } i \neq k \text{ and } a_{ij} < b_{0l} \text{ iff } j = l.$$

For any element $a_{kl} \in J(R_n^2)$, for all $i = 1, 2, \cdots, n$ with $i \neq k$, let

$$F_i = \begin{cases} \{(a_{i1}, b_{i1}), (a_{i2}, b_{i1}), (a_{il}, b_{02})\} & \text{if } l = 1 \\ \{(a_{i1}, b_{i1}), (a_{i2}, b_{i1}), (a_{il}, b_{01})\} & \text{if } l = 2 \end{cases}$$

and

$$G = \begin{cases} \{(a_{i2}, b_{01}) \mid i = 1, 2, \cdots, n\} & \text{if } l = 1 \\ \{(a_{i1}, b_{02}) \mid i = 1, 2, \cdots, n\} & \text{if } l = 2. \end{cases}$$

Then we have

$$\bigcup_{i=1, i\neq k} F_i \cup G = J(R_n^2) \times M(R_n^2) - I.$$

Further, we know that there is a subordered set $R$ of $R_n^2 - \{a_{kl}\}$ such that $R \cong S_n$. Hence we have $\dim(R_n^2 - \{a_{kl}\}) = n$. Clearly, we know that $\dim(R_n^2 - \{b_{kl}\}) = n$ and $\dim(R_n^2 - \{b_{0l}\}) = n$. We conclude that $R_n^2$ is an irreducible ordered set with $\dim(R_n^2) = n + 1$.

For a natural number $m$, we may assume that, for all $i$ with $1 \leq i \leq m$, $a_i$ and $b_i$ are incomparable in $\mathbf{2}^m$ and that $J(\mathbf{2}^m) = \{a_1, a_2, \cdots, a_m\}$ and $M(\mathbf{2}^m) = \{b_1, b_2, \cdots, b_m\}$ with $\dim(\mathbf{2}^m) = m$ and $\bigcup_{i=1}^m F_i(b_i) = J(\mathbf{2}^m)$, where $F_i = \{(a_i, b_i)\}$. Then we have the following.

III. For any finite lattice $L$ with $J(L) = A(L)$ and $M(L) = DA(L)$, then we have

$$\dim(\mathbf{2}^m \square L) = \dim(\mathbf{2}^m) + \dim(L) - 1.$$

IV. Let $m$ be natural number and let $L$ be a finite complemented modular lattice. Then we have

$$\dim(\mathbf{2}^m \square L) = \dim(\mathbf{2}^m) + \dim(L) - 1.$$

V. Let $n_1, n_2, \cdots, n_k$ be the natural numbers. By the main theorem, we have $\dim(\mathbf{2}^{n_1} \square \mathbf{2}^{n_2}) = n_1 + n_2 - 1$. By Lemma 1, we have

$$J(\mathbf{2}^{n_1} \square \mathbf{2}^{n_2}) = J(\mathbf{2}^{n_1}) \times J(\mathbf{2}^{n_2}) = A(\mathbf{2}^{n_1}) \times A(\mathbf{2}^{n_2}) = A(\mathbf{2}^{n_1} \square \mathbf{2}^{n_2})$$

and

$$M(\mathbf{2}^{n_1} \square \mathbf{2}^{n_2}) = (M(\mathbf{2}^{n_1}) \times \{1\}) \cup (\{1\} \times M(\mathbf{2}^{n_2})) = DA(\mathbf{2}^{n_1} \square \mathbf{2}^{n_2}).$$

Then we have

$$\dim(\mathbf{2}^{n_1} \square \mathbf{2}^{n_2} \square \mathbf{2}^{n_3}) = n_1 + n_2 + n_3 - 2.$$

By the induction on $k$, we conclude that

$$\dim(\mathbf{2}^{n_1} \square \mathbf{2}^{n_2} \square \cdots \square \mathbf{2}^{n_k}) = n_1 + n_2 + \cdots + n_k - k + 1.$$

A *partition* of a set $A$ is a set $\pi$ of nonempty pairwise disjoint subsets of $A$ whose union is $A$. The members of $\pi$ are called the *blocks* of $\pi$. If $a$ and $b$ $(a, b \in A)$ belong to the same block we write $a \equiv b(\pi)$.

Part$(A)$ will denote the set of all partitions of $A$ is an ordered set if

$$\pi_1 \leq \pi_2 \text{ iff } x \equiv y\,(\pi_1) \text{ implies } x \equiv y(\pi_2).$$

In particular, if $|A| = n$, then $(\mathrm{Part}(L_1), \leq)$ is denote by $\Pi_n$. Hence $\Pi_n$ is a simple geometric lattice and that $J(\Pi_n) = A(\Pi_n)$ and $M(\Pi_n) = DA(\Pi_n)$. Then we have following:

VI. Let $m$ be natural number and let $\Pi_n$ be a finite partition lattice. Then we have

$$\dim(\mathbf{2}^m \square \Pi_n) = \dim(\mathbf{2}^m) + \dim(\Pi_n) - 1.$$

ACKNOWLEDGEMENT. I am grateful to Professor J. G. Lee for his valuable suggestions.

## References

[1] K. A. Baker, *Dimension, join-independence and breadth in partially ordered sets*, Unpublished, 1961.
[2] M. K. Bennett, *Rectangular products of lattices*, Discrete Math. **79** (1990), 235–249.
[3] B. Dushnik and E. Miller, *Partially ordered sets*, Amer. Math. Soc. **63** (1941), 600–610.
[4] Hiraguchi, *On the dimension of partially ordered sets*, Sci. Rep. Kanazawa. Univ. **4** (1955), 77–94.
[5] D. Kelly, *On the dimension of partially ordered sets*, Discrete Math. **35** (1981), 135–156.
[6] C. Lin, *The dimension of the Cartesian product of posets*, Discrete Math. **88** (1991), 79–92.
[7] O. Ore, *Theory of graphs*, vol. 38, Amer. Math. Soc. Colloq. Pub., 1930.
[8] I. Rabinovitch and I. Rival, *The Rank of Distributive Lattice*, Discrete Math. **25** (1979), 275–279.
[9] K. Reuter, *On the dimension of the Cartesian product of relation and orders*, Order **6** (1989), 277–293.
[10] E. Szpilrajn, *Sur l'extension de l'ordre partiel*, Fund. Math. **16** (1930), 386–389.
[11] W. T. Trotter, *The dimension of the Cartesian product of partial orders*, Discrete Math. **53** (1985), 255–263.
[12] R. Wille, *Tensorial Decomposition of Concept Lattices*, Order **2** (1985), 81–95.

Department of Mathematics
Sogang University
Seoul 121-742, Korea
*E-mail*: bae@math.sogang.ac.kr