J. Korean Math. Soc. **36** (1999), No. 2, pp. 403–419

LINEAR STABILITY OF A PERIODIC ORBIT OF TWO-BALL LINEAR SYSTEMS

DONG PYO CHI AND SUNBOK SEO

ABSTRACT. We introduce a Hamiltonian system which consists of two balls in the vertical line colliding elastically with each other and the floor. Wojtkowski proved that for the system of two linear balls with a linear potential (with gravity), there is a periodic orbit which becomes linearly stable if $m_1 < m_2$ where m_1 is the mass of a lower particle and m_2 is that of an upper particle.

For our system having a quadratic potential, we find an appropriate coordinate to obtain symplectic collision maps, obtain a periodic orbit and prove conclusively that the periodic orbit is linearly stable without the mass condition.

1. Introduction

Hamiltonian systems with many degrees of freedom are likely to exhibit strong mixing behavior produced by exponential divergence of nearby orbits. Although strict integrability is easily destroyed by perturbations, the KAM theory guarantees its survival on some exotic subsets. This interplay of integrability and nonintegrability is still a great challenge for the theory. We will consider the linear stability of a periodic orbit in a specific Hamiltonian system.

Our system consists of some particles in the half line colliding elastically with each other and the floor (see Figure 1).

Wojtkowski (Wojtkowski [3]) has introduced a Hamiltonian system with arbitrary number of degrees of freedom for which he can establish the nonvanishment of at least one Lyapunov exponent almost everywhere. It is a system of n particles in a line which fall down with a constant acceleration toward a hard floor. In that paper (Wojtkowski

Received October 3, 1998.
1991 Mathematics Subject Classification: 70F35.
Key words and phrases: system of falling balls; symplectic map; collision map; linear stability of a periodic orbit.

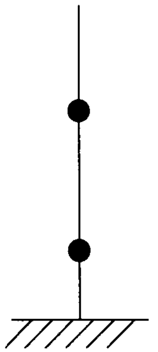

FIGURE 1. A Hamiltonian system

[3]), it is proved that for the system of two linear balls with a linear potential (with gravity), there is a periodic orbit which becomes linearly stable if $m_1 < m_2$ where m_1 is the mass of a lower particle and m_2 is that of an upper particle.

In this paper, we introduce a Hamiltonian system which is a system of two particles in a line such that all the particles are connected to the floor with the springs having same spring constant. The particles are all under the influence of an external potential fields with same potential. We find an appropriate coordinate to obtain symplectic collision maps, obtain a periodic orbit and prove conclusively that the periodic orbit is linearly stable (without the mass condition) in our systems. Although general scheme of the proof is very similar to Wojtkowski (Wojtkowski [3]), there are many difference in details and technicality because potential in our system is what Wojtkowski left.

In Section 2 we describe our systems. In Section 3 we introduce a useful coordinate and obtain formulas for the derivatives of collision maps for our system. In Section 4 we prove the existence of a periodic orbit in our system and its linear stability.

2. Description of the system

Let us consider two point masses m_1, m_2 on a vertical line. We will refer to them as particles. We denote by q_1, q_2 the positions of the particles and by v_1, v_2 their velocities. The particles are all under the influence of an external potential field with a potential $V(q) = \frac{q^2}{2}$. They collide elastically with each other and with the rigid floor $q = 0$. Hence the dynamics between collisions is described by the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{2}\frac{p_i^2}{m_i} + \frac{1}{2}\sum_{i=1}^{2} q_i^2$$

where $p_i = m_i v_i$ $(i = 1, 2)$ are the momenta. At a collision of the 1^{st} and the 2^{nd} particles there is an instantaneous change of their velocities;

$$\begin{aligned} v_1^+ &= \gamma_{12} v_1^- + (1 - \gamma_{12}) v_2^-, \\ v_2^+ &= (1 + \gamma_{12}) v_1^- - \gamma_{12} v_2^- \end{aligned} \tag{1}$$

where $\gamma_{12} = \dfrac{m_1 - m_2}{m_1 + m_2}$, the $-$ sign in the superscripts refers to velocities before the collision and $+$ sign in the superscripts to velocities after the collision.

At the collision of the 1^{st} particle with the floor, we have

$$v_1^+ = -v_1^-. \tag{2}$$

If the particles have equal masses our system is completely integrable. Our system is a Hamiltonian flow with collisions as defined in (Wojtkowski [3]) (Section 1). Indeed we consider the Hamiltonian system

$$\begin{cases} \dot{q}_i = \dfrac{p_i}{m_i}, \\ \dot{p}_i = -q_i, \qquad i = 1, 2. \end{cases} \tag{3}$$

We let $\phi^t : N \longrightarrow N$ be the flow defined by the formulas (3) on the submanifold

$$N = \left\{(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2 | H(q, p) = \frac{1}{2}\right\}$$

and ϕ^t preserves the Liouville measure ν. Further we let $M \subset N$,

$$M = \{(q, p) \in N | 0 < q_1 < q_2\}.$$

The boundary ∂M of M is piecewise smooth and on an open subset ∂M_r, regular part of the boundary, $\subset \partial M$ the boundary is smooth and the flow ϕ^t is transversal to it which is defined below. The regular part ∂M_r of the boundary of M is the union of two submanifolds, i.e., $\partial M_r = \partial M_0 \cup \partial M_1$ where

$$\partial M_0 = \{(q,p) \in N | 0 = q_1 < q_2, v_1 \neq 0\},$$
$$\partial M_1 = \{(q,p) \in N | 0 < q_1 = q_2, v_1 - v_2 \neq 0\}.$$

Furthermore we have $\partial M_r^{\pm} = \partial M_0^{\pm} \cup \partial M_1^{\pm}$ where

$$\partial M_0^{\pm} = \{(q,p) \in \partial M_0 | \pm v_1 > 0\},$$
$$\partial M_1^{\pm} = \{(q,p) \in \partial M_1 | \pm (v_1 - v_2) < 0\}.$$

The singular part $\partial M_s = \partial M \backslash \partial M_r$ of the boundary of M is a finite union of submanifolds of N of codimension at least two, so the orbits of the flow ϕ^t which pass through ∂M_s form a set of Liouville measure zero.

The flow ϕ^t enters M in ∂M^+ and leaves it in ∂M^-. The collision rules (1) and (2) define the collision map $\Phi : \partial M^- \to \partial M^+$. The collision map is symplectic with respect to the canonical symplectic structure on ∂M_r. Indeed in the both cases of the collision with the floor and the collision between two particles, Φ is the restriction of a symplectic linear map of the ambient space $\mathbb{R}^2 \times \mathbb{R}^2$. That is we have

$$\Phi|_{\partial M_i^-} = S_i|_{\partial M_i^-}$$

$i = 0, 1$ and $S_1 = \begin{pmatrix} R & 0 \\ 0 & R^* \end{pmatrix}$ with $R = \begin{pmatrix} \gamma_{12} & 1 - \gamma_{12} \\ 1 + \gamma_{12} & -\gamma_{12} \end{pmatrix}$ where $\gamma_{12} = \dfrac{m_1 - m_2}{m_1 + m_2}$. Then S_1 commutes with the flow, i.e., $H \circ S_1 = H$. And we have $S_0 = \begin{pmatrix} I_1 & 0 \\ 0 & I_1 \end{pmatrix}$ where $I_1 = \operatorname{diag}(-1, 1)$ is the diagonal matrix but S_0 does not commute with the flow.

In this setup we introduce the Hamiltonian flow with collisions $\psi^t : M \cup \partial M^+ \to M \cup \partial M^+$ defined by

$$\psi^t(x) = \phi^t(x)$$

for $0 \leq t < \tau(x)$ where $\tau(x)$ is the first return time when ϕ^t reaches ∂M^- and

$$\psi^{\tau(x)} = \Phi \phi^{\tau(x)}(x).$$

The flow ψ^t preserves the Liouville measure ν and is differentiable almost everywhere in M. The collision map is not defined at multiple collisions where more than two q's assume the same value. Multiple collisions belong to the singular part of the boundary. It is useful to consider the quotients of the operators $D\psi^t$ by the one dimensional subspace spanned by the velocity vector of the flow. We will denote this quotient by L_x^t. We can describe $L_x^t : \mathcal{T}_x \to \mathcal{T}_{\psi^t x}$ in the following way where $\mathcal{T}_y$, $y \in N$ is the quotient of $T_y N$ by the 1-dimensional subspace of $T_y N$ spanned by Hamiltonian vector field.

Let $\psi^u x$ be well defined for $0 \le u \le t$ and suppose that $\tau(x) < t < \tau(\psi^\tau x)$, i.e., x has only one collision time $\tau(x)$ in the time interval $[0, t]$. If we choose representations of $\mathcal{T}_y$, $y \in M$ as subspaces of $T_y M$ transversal to the flow (3) we can write

$$L_x^t = \pi_1 \circ D_{\psi^\tau x}\phi^{t-\tau} \circ D_{\phi^\tau x}\Phi \circ \pi_0 \circ D_x\phi^\tau|_{\mathcal{T}_x} \tag{4}$$

where

$$\pi_0 : T_{\phi^\tau x}N \to T_{\phi^\tau x}(\partial M^-)$$

and

$$\pi_1 : T_{\psi^t x}M \to \mathcal{T}_{\psi^t x}$$

are the linear projections along the flow (see Figure 2).

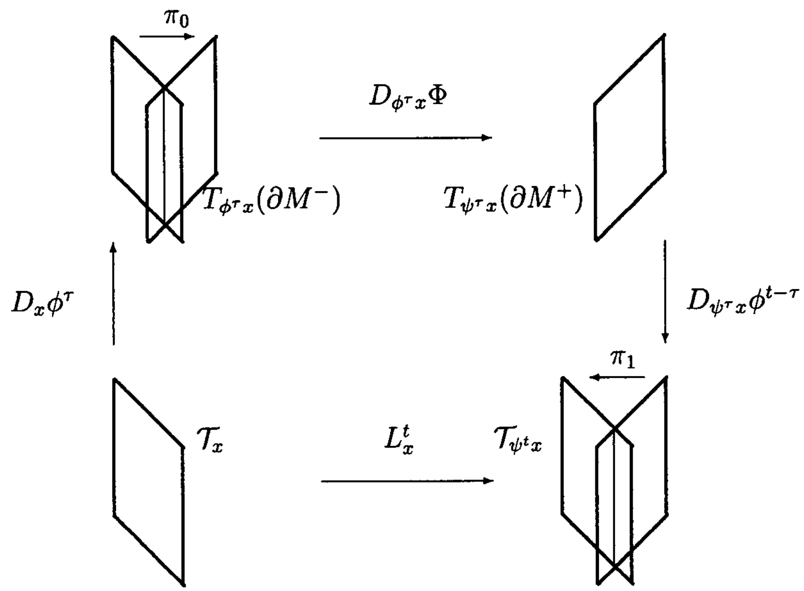

FIGURE 2. The quotient linear operator $L_x^t : \mathcal{T}_x \to \mathcal{T}_{\psi^t x}$

Note that the formula (4) makes perfect sense also for the collision time $t = \tau(x)$, i.e., $L_x^{\tau(x)}$ is well defined. Clearly then for ν almost all $x \in M \cup \partial M_r^+$, L_x^t is well defined for all $t \in \mathbb{R}$. Let us further assume that all $x \in M$ (ν almost all) are bound to leave M under ϕ^t. We define then a map $\Psi : \partial M_r^+ \to \partial M_r^+$ by $\Psi x = \Phi \phi^{\tau(x)} x$ where $\tau(x)$ is the first collision time of x. Ψ is a piecewise differentiable map. We will call Ψ the standard section map of the flow $\{\psi^t\}$. The derivative $D_x\Psi$ of Ψ at $x \in \partial M_r^+$ is equal to $L_x^{\tau(x)}$ under the natural identification of $\mathcal{T}_x$ with $T_x(\partial M_r^+)$.

3. Collision maps

We introduce as coordinates the energies of individual particles and proper angles:

$$\begin{cases} h_i = \dfrac{1}{2} m_i v_i^2 + \dfrac{1}{2} q_i^2, \\ \theta_i = \sqrt{m_i} \arctan \dfrac{\sqrt{m_i} v_i}{q_i}, \qquad i = 1, 2. \end{cases}$$

Then we have linear symplectic coordinates in the tangent spaces to the phase space $\mathbb{R}^2 \times \mathbb{R}^2$ by the formulas

$$\delta h_i = \frac{p_i}{m_i} \delta p_i + q_i \delta q_i = m_i v_i \delta v_i + q_i \delta q_i,$$
$$\delta \theta_i = \frac{q_i}{2h_i} \delta p_i - \frac{p_i}{2h_i} \delta q_i = \frac{m_i q_i}{2h_i} \delta v_i - \frac{m_i v_i}{2h_i} \delta q_i, \qquad i = 1, 2.$$

where $(\delta q, \delta p)$ are natural linear symplectic coordinates in $\mathbb{R}^2 \times \mathbb{R}^2$. In these coordinates, we obtain $\omega = \delta q \wedge \delta p = \delta h \wedge \delta \theta$ and the velocity vector field of the flow is $(0, -1)$, i.e., $\delta h = 0$, $\delta \theta = -1$.

Hence the flow ϕ^t for the velocity vector fields $(0, -1)$ acts on the linear manifold $N = \{h_1 + h_2 = \frac{1}{2}\}$ by translations and its derivative $D\phi^t$ is the identity operator. Our goal is to describe the derivative of the standard section map Ψ which is the map from a collision to the next collision. As it was explained in Section 2, $D\Psi$ coincides with the quotient of $D\psi^t$ by the velocity vector fields $(0, -1)$ if we identify the quotient of the tangent space to M by the flow with the tangent space to ∂M_r^+.

We form coordinates in the quotient (by the velocity vector field) of the tangent space to M by choosing a codimension one subspace $\mathcal{T}$ in the tangent space to M,

$$\mathcal{T} = \{(\delta h, \delta\theta) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \delta h_1 + \delta h_2 = 0, m_1\delta\theta_1 + m_2\delta\theta_2 = 0\}.$$

M being an open subset in the linear manifold N allows for identification of all its tangent spaces.

The part of the collision manifold corresponding to the collision of the 1^{st} and the 2^{nd} particles is

$$\partial M_{12} = \{(h,v) \in N \mid h_1 - \frac{1}{2}m_1v_1^2 = h_2 - \frac{1}{2}m_2v_2^2\}.$$

The collision map $\Phi_{12} : \partial M_{12}^- \to \partial M_{12}^+$ describing the collision of the 1^{st} and the 2^{nd} particles in the (h,v) coordinates is given by

$$\Phi_{12}(h^-,v^-) = (h^+,v^+) \quad \text{for} \quad (h^-,v^-) \in \partial M_{12}^-,\ (h^+,v^+) \in \partial M_{12}^+$$

where

$$\begin{aligned} h_1^+ &= h_1^- - \Delta, \\ h_2^+ &= h_2^- + \Delta, \\ \tan\frac{\theta_1^+}{\sqrt{m_1}} &= \gamma_{12}\tan\frac{\theta_1^-}{\sqrt{m_1}} + (1-\gamma_{12})\frac{\sqrt{m_1}}{\sqrt{m_2}}\tan\frac{\theta_2^-}{\sqrt{m_2}}, \\ \tan\frac{\theta_2^+}{\sqrt{m_2}} &= (1+\gamma_{12})\frac{\sqrt{m_2}}{\sqrt{m_1}}\tan\frac{\theta_1^-}{\sqrt{m_1}} - \gamma_{12}\tan\frac{\theta_2^-}{\sqrt{m_2}} \end{aligned} \tag{5}$$

where $\Delta = \dfrac{2m_1m_2}{m_1+m_2}(v_1^- - v_2^-)\dfrac{m_1v_1^- + m_2v_2^-}{m_1+m_2}$ and $\gamma_{12} = \dfrac{m_1-m_2}{m_1+m_2}$.
The equalities for the angle-coordinate in Eqs. (5) are obtained by Eqs. (1). We assume here that the 1^{st} particle is below the 2^{nd} particle.

Differentiation Eqs. (5), we obtain $D\Phi_{12} : (\delta h^-, \delta\theta^-) \mapsto (\delta h^+, \delta\theta^+)$ given by

$$\begin{aligned} \delta h_1^+ &= (1-a)\delta h_1^- - b\delta h_2^- - c\delta\theta_1^- - d\delta\theta_2^-, \\ \delta h_2^+ &= a\delta h_1^- + (1+b)\delta h_2^- + c\delta\theta_1^- + d\delta\theta_2^-, \\ \delta\theta_1^+ &= \gamma_{12}\frac{h_1^-}{h_1^+}\delta\theta_1^- + (1+\gamma_{12})\frac{h_2^-}{h_1^+}\delta\theta_2^-, \\ \delta\theta_2^+ &= (1-\gamma_{12})\frac{h_1^-}{h_2^+}\delta\theta_1^- - \gamma_{12}\frac{h_2^-}{h_2^+}\delta\theta_2^- \end{aligned} \tag{6}$$

where

$$a = \frac{2m_1m_2}{(m_1+m_2)^2}\Big(\frac{m_1 {v_1^-}^2}{h_1^-} + (m_2 - m_1)\frac{v_1^- v_2^-}{2h_1^-}\Big),$$
$$b = \frac{2m_1m_2}{(m_1+m_2)^2}\Big(- \frac{m_2 {v_2^-}^2}{h_2^-} + (m_2 - m_1)\frac{v_1^- v_2^-}{2h_2^-}\Big),$$
$$c = \frac{2m_1m_2}{(m_1+m_2)^2}\Big(2v_1^- q_1 + (m_2 - m_1)\frac{v_2^- q_1}{m_1}\Big),$$
$$d = \frac{2m_1m_2}{(m_1+m_2)^2}\Big(- 2v_2^- q_1 + (m_2 - m_1)\frac{v_1^- q_1}{m_2}\Big).$$

LEMMA 3.1. *When the* 1^{st} *and the* 2^{nd} *particles collide, we have the linear operator* $D\Psi_{12} : \mathcal{T} \to \mathcal{T}$ *defined by* $D\Psi_{12}(\delta h_1^-, \delta h_2^-, \delta\theta_1^-, \delta\theta_2^-) = (\bar{\delta h}_1^+, \bar{\delta h}_2^+, \bar{\delta\theta}_1^+, \bar{\delta\theta}_2^+)$;

$$D\Psi_{12} = \begin{pmatrix} 1 - a - \frac{U}{h_1^-} & -b + \frac{U}{h_2^-} & K & -K \\ a + \frac{U}{h_1^-} & 1 + b - \frac{U}{h_2^-} & -K & K \\ \frac{m_2 V}{h_1^-} & -\frac{m_2 V}{h_2^-} & \frac{m_2 W + m_1}{m_1 + m_2} & \frac{-m_2 W + m_2}{m_1 + m_2} \\ -\frac{m_1 V}{h_1^-} & -\frac{m_1 V}{h_2^-} & \frac{-m_1 W + m_1}{m_1 + m_2} & \frac{m_1 W + m_2}{m_1 + m_2} \end{pmatrix}$$

where

$$U = \frac{q_1(c+d)}{2(v_1^- - v_2^-)}, \quad K = \frac{v_2^- c + v_1^- d}{v_1^- - v_2^-},$$

$$A = \gamma_{12}\frac{h_1^-}{h_1^+}, \quad B = (1+\gamma_{12})\frac{h_2^-}{h_1^+}, \quad C = (1-\gamma_{12})\frac{h_1^-}{h_2^+}, \quad D = \gamma_{12}\frac{h_2^-}{h_2^+},$$

$$V = \frac{q_1(D+B-C+A)}{2(v_1^- - v_2^-)(m_1+m_2)}, \quad W = \frac{v_2^-(C-A) - v_1^-(D+B)}{(v_1^- - v_2^-)}.$$

Proof. We will construct $D\Psi_{12}$ like this,

$$D\Psi_{12} : \mathcal{T} \xrightarrow{\pi_0} T_{(h,\theta)}\partial M_{12}^{-} \xrightarrow{D\Phi_{12}} T_{(h,\theta)}\partial M_{12}^{+} \xrightarrow{\pi_1} \mathcal{T}$$
$$(\delta h^-, \delta\theta^-) \mapsto (\delta h^-, \delta\theta^- + \lambda) \mapsto (\delta h^+, \delta\theta^+) \mapsto (\bar{\delta h}^+, \bar{\delta\theta}^+) = (\delta h^+, \delta\theta^+ + \sigma)$$

where π_0 and π_1 are linear projections along the flow of our system.

Given $(\delta h^-, \delta\theta^-) \in \mathcal{T}$, i.e., $\delta h_1^- + \delta h_2^- = 0$, $m_1\delta\theta_1^- + m_2\delta\theta_2^- = 0$, we project the vector into the tangent space $T_{(h,\theta)}(\partial M_{12}^-)$ along the velocity vector of the flow $(0, 0, -1, -1)$. Since

$$T_{(h,\theta)}(\partial M_{12}^{\pm}) = \{(\delta h^\pm, \delta\theta^\pm) | \delta h_1^\pm + \delta h_2^\pm = 0,$$

$$\frac{q_1}{2h_2^\pm}\delta h_2^\pm - v_2^\pm\delta\theta_2^\pm - \frac{q_1}{2h_1^\pm} + v_1^\pm\delta\theta_1^\pm = 0\},$$

we have $(\delta h^-, \delta\theta^-) \mapsto (\delta h^-, \delta\theta^- + \lambda)$ where

$$\frac{q_1}{2h_2^-} - v_2^-(\delta\theta_2^- + \lambda) - \frac{q_1}{2h_1^-} + v_1^-(\delta\theta_1^- + \lambda) = 0$$

or

$$\lambda = \frac{1}{v_1^- - v_2^-}\left(\frac{q_1}{2h_2^-} - v_2^-\delta\theta_2^- - \frac{q_1}{2h_1^-} + v_1^-\delta\theta_1^-\right).$$

Then we apply the derivative $D\Phi_{12}$ of the collision map given by Eqs. (6). Let $(\delta h^+, \delta\theta^+) \in T_{(h,\theta)}(\partial M_{12}^+)$, then it follows that

$$\begin{aligned}
\delta h_1^+ &= (1-a)\delta h_1^- - b\delta h_2^- - c(\delta\theta_1^- + \lambda) - d(\delta\theta_2^- + \lambda), \\
\delta h_2^+ &= a\delta h_1^- + (1+b)\delta h_2^- + c(\delta\theta_1^- + \lambda) + d(\delta\theta_2^- + \lambda), \\
\delta\theta_1^+ &= \gamma_{12}\frac{h_1^-}{h_1^+}(\delta\theta_1^- + \lambda) + (1+\gamma_{12})\frac{h_2^-}{h_1^+}(\delta\theta_2^- + \lambda), \\
\delta\theta_2^+ &= (1-\gamma_{12})\frac{h_1^-}{h_2^+}(\delta\theta_1^- + \lambda) - \gamma_{12}\frac{h_2^-}{h_2^+}(\delta\theta_2^- + \lambda).
\end{aligned}$$

Finally we apply the projection onto $\mathcal{T}$ along the velocity vector $(0, 0, -1, -1)$, i.e.,

$$(\delta h^-, \delta\theta^-) \mapsto (\delta h^+, \delta\theta^+ + \sigma) = (\bar{\delta h}^+, \bar{\delta\theta}^+)$$

or

$$m_1\bar{\delta\theta}_1^+ + m_2\bar{\delta\theta}_2^+ = m_1(\delta\theta_1 + \sigma) + m_2(\delta\theta_2 + \sigma) = m_1\delta\theta_1^- + m_2\delta\theta_2^-.$$

Hence

$$\sigma = \left(\frac{m_1 - m_1\gamma_{12}\dfrac{h_1^-}{h_1^+} - m_2(1-\gamma_{12})\dfrac{h_1^-}{h_2^+}}{m_1+m_2} \right) \delta\theta_1^-$$
$$+ \left(\frac{m_2 - m_1(1+\gamma_{12})\dfrac{h_2^-}{h_1^+} + m_2\gamma_{12}\dfrac{h_2^-}{h_2^+}}{m_1+m_2} \right) \delta\theta_2^-$$
$$-\lambda \left(\frac{m_1(\gamma_{12}\dfrac{h_1^-}{h_1^+} + (1+\gamma_{12})\dfrac{h_2^-}{h_1^+}) + m_2((1-\gamma_{12})\dfrac{h_1^-}{h_2^+} - \gamma_{12}\dfrac{h_2^-}{h_2^+})}{m_1+m_2} \right)$$

which immediately yields the result. □

This operator $D\Psi_{12}$ has a useful extension DS_{12} from $\mathcal{T}$ to $\mathbb{R}^2 \times \mathbb{R}^2$.

LEMMA 3.2. *$D\Psi_{12} : \mathcal{T} \longrightarrow \mathcal{T}$ has a symplectic extension DS_{12} : $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2$, that is, the restriction of DS_{12} to $\mathcal{T}$ coincides with the operator $D\Psi_{12}$, in particular, $DS_{12}(\mathcal{T})=\mathcal{T}$.*
Furthermore, the quotient of DS_{12} by the invariant subspace $\mathcal{T}$ is the identity operator on the quotient space.
$DS_{12} : (\delta h_1^-, \delta h_2^-, \delta\theta_1^-, \delta\theta_2^-) \mapsto (\widehat{\delta h}_1^+, \widehat{\delta h}_2^+, \widehat{\delta\theta}_1^+, \widehat{\delta\theta}_2^+)$ is given by

$$DS_{12} = \begin{pmatrix} 1-a-\dfrac{U}{h_1^-}+f_1 & -b+\dfrac{U}{h_2^-}+f_1 & K & -K \\ a+\dfrac{U}{h_1^-}-f_1 & 1+b-\dfrac{U}{h_2^-}-f_1 & -K & K \\ \dfrac{m_2V}{h_1^-}+\dfrac{f_2}{m_1} & -\dfrac{m_2V}{h_2^-}+\dfrac{f_2}{m_1} & \dfrac{m_2W+m_1}{m_1+m_2} & \dfrac{-m_2W+m_2}{m_1+m_2} \\ -\dfrac{m_1V}{h_1^-}-\dfrac{f_2}{m_2} & -\dfrac{m_1V}{h_2^-}-\dfrac{f_2}{m_2} & \dfrac{-m_1W+m_1}{m_1+m_2} & \dfrac{m_1W+m_2}{m_1+m_2} \end{pmatrix}$$

where

$$f_1 = \frac{Wm_2(a-b-1)}{m_1+m_2} + \frac{U}{m_1+m_2}\left(\frac{m_1}{h_1^-} - \frac{m_2}{h_2^-}\right)$$
$$+ m_2\left(\frac{1}{h_1}+\frac{1}{h_2}\right)\left(KV + \frac{WU}{m_1+m_2}\right) + \frac{m_1 a + m_2 b + m_2}{m_1+m_2},$$

$$f_2 = -\frac{Vm_1m_2(h_2^- m_1 - h_1^- m_2)}{h_1^- h_2^-(m_1+m_2)}.$$

Proof. We consider the following setting;

$$\widehat{\delta h_1^+} = \left(1 - a - \frac{U}{h_1^-}\right)\delta h_1^- + \left(-b + \frac{U}{h_2^-}\right)\delta h_2^- + K\delta\theta_1^- - K\delta\theta_2^-$$
$$+ f_1(\delta h_1^- + \delta h_2^-) + g_1(m_1\delta\theta_1^- + m_2\delta\theta_2^-),$$

$$\widehat{\delta h_2^+} = \left(a + \frac{U}{h_1^-}\right)\delta h_1^- + \left(1 + b - \frac{U}{h_2^-}\right)\delta h_2^- + -K\delta\theta_1^- + K\delta\theta_2^-$$
$$- f_1(\delta h_1^- + \delta h_2^-) - g_1(m_1\delta\theta_1^- + m_2\delta\theta_2^-),$$

$$\widehat{\delta\theta_1^+} = \frac{m_2V}{h_1^-}\delta h_1^- - \frac{m_2V}{h_2^-}\delta h_2^- + \frac{m_2W + m_1}{m_1+m_2}\delta\theta_1^- + \frac{-m_2W + m_2}{m_1+m_2}\delta\theta_2^-$$
$$+ \frac{f_2}{m_1}(\delta h_1^- + \delta h_2^-) + \frac{g_2}{m_1}(m_1\delta\theta_1^- + m_2\delta\theta_2^-),$$

$$\widehat{\delta\theta_2^+} = -\frac{m_1V}{h_1^-}\delta h_1^- - \frac{m_1V}{h_2^-}\delta h_2^- + \frac{-m_1W + m_1}{m_1+m_2}\delta\theta_1^- + \frac{m_1W + m_2}{m_1+m_2}\delta\theta_2^-$$
$$- \frac{f_2}{m_1}(\delta h_1^- + \delta h_2^-) - \frac{g_2}{m_1}(m_1\delta\theta_1^- + m_2\delta\theta_2^-).$$

This setting yields that $DS_{12}(\mathcal{T}) = \mathcal{T}$ and DS_{12} preserves the linear functionals $\delta H = \delta h_1 + \delta h_2$ and $\delta\Phi = m_1\delta\theta_1 + m_2\delta\theta_2$. We see that $DS_{12} = DS_{12}|_{\mathcal{T}} + DS_{12}/\mathcal{T}$ where $DS_{12}/\mathcal{T} : \mathbb{R}^2 \times \mathbb{R}^2/\mathcal{T} \to \mathbb{R}^2 \times \mathbb{R}^2/\mathcal{T}$. Note that $DS_{12}/\mathcal{T}$ (the quotient of DS_{12} by $\mathcal{T}$) is the identity operator. Indeed the linear functionals δH and $\delta\Phi$ can be used as coordinates in the quotient space $\mathbb{R}^2 \times \mathbb{R}^2/\mathcal{T}$.

It is well-known that when the matrix $\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, and D are real $n \times n$ matrices, the matrix Ψ is symplectic if and only if its inverse is of the form $\Psi^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$.

Using this fact for the symplecticity, we obtain $g_1 = g_2 = 0$ and f_1 and f_2 are defined as above. □

We will consider now the collision of a particle with the floor. The part of the collision manifold corresponding to the collision of the 1^{st} particle with the floor is

$$\partial M_{01}^{\pm} = \left\{ (h,\theta) \in N | \quad \theta_1^{\pm} = \pm \frac{\sqrt{m_1}\pi}{2} \right\}.$$

The collision map $\Phi_{01} : \delta M_{01}^- \mapsto \delta M_{01}^+$ is given by $\Phi_{01}(h^-, \theta^-) = (h^+, \theta^+)$ where

$$\theta_1^+ = -\theta_1^-$$

We find that $D\Phi_{01}$ given by $(\delta h^-, \delta\theta^-) \mapsto (\delta h^+, \delta\theta^+)$ change the $\delta\theta_1$-coordinate only and

$$T_{(h,\theta)}(\partial M_{01}^{\pm}) = \{(\delta h^{\pm}, \delta\theta^{\pm}) | \delta h_1^{\pm} + \delta h_2^{\pm} = 0, \quad \delta\theta_1^{\pm} = 0\}.$$

Hence we see that $D\Phi_{01}$ is equal to the identity operator.

LEMMA 3.3. *When the 1^{th} particle collides with the floor, the derivative of the standard section map $D\Psi_{01}$ is equal to the restriction to $\mathcal{T}$ of the linear operator $(\delta h^-, \delta\theta^-) \mapsto (\bar{\delta h}^+, \bar{\delta\theta}^+)$ given by*

$$\bar{\delta h}_k^+ = \delta h_k^-, \qquad \bar{\delta\theta}_k^+ = \delta\theta_k^- \quad \text{for all } k = 1, 2.$$

Proof. We will construct $D\Psi_{01}$ like this,

$$D\Psi_{01} : \mathcal{T} \xrightarrow{\pi_0} T_{(h,\theta)}\partial M_{01}^- \xrightarrow{D\Phi_{01}} T_{(h,\theta)}\partial M_{01}^+ \xrightarrow{\pi_1} \mathcal{T}$$

$$(\delta h^-, \delta\theta^-) \mapsto (\delta h^-, \delta\theta^- + \lambda) \mapsto (\delta h^+, \delta\theta^+) \mapsto (\bar{\delta h}^+, \bar{\delta\theta}^+) = (\delta h^+, \delta\theta^+ + \sigma)$$

where π_0 and π_1 are linear projections along the flow of our system.

Given $(\delta h^-, \delta\theta^-) \in \mathcal{T}$, we project the vector into the tangent space $T_{(h,\theta)}(\partial M_{01}^-)$ along the velocity vector of the flow $(0,0,-1,-1)$. Since

$$T_{(h,\theta)}(\partial M_{01}^{\pm}) = \{(\delta h^{\pm}, \delta\theta^{\pm}) | \delta h_1^{\pm} + \delta h_2^{\pm} = 0, \quad \delta\theta_1^{\pm} = 0\},$$

we have $(\delta h_1^-, \delta h_2^-, \delta\theta_1^-, \delta\theta_2^-) \mapsto (\delta h_1^-, \delta h_2^-, \delta\theta_1^-, \delta\theta_2^-) + \lambda(0,0,1,1)$ where $\lambda = -\delta\theta_1^-$.

Then we apply the identity map $D\Phi_{01}$, i.e., $(\delta h_1^+, \delta h_2^+, \delta\theta_1^+, \delta\theta_2^+) = (\delta h_1^-, \delta h_2^-, \delta\theta_1^- - \delta\theta_1^-, \delta\theta_2^- - \delta\theta_1^-)$. Finally we apply the projection onto $\mathcal{T}$ along the velocity vector $(0,0,-1,-1)$, i.e.,

$$\begin{aligned}(\bar{\delta h}_1^+, \bar{\delta h}_2^+, \bar{\delta\theta}_1^+, \bar{\delta\theta}_2^+) &= (\delta h_1^+, \delta h_2^+, \delta\theta_1^+, \delta\theta_2^+) + \sigma(0,0,1,1) \\ &= (\delta h_1^-, \delta h_2^-, \sigma, \delta\theta_2^- - \delta\theta_1^- + \sigma)\end{aligned}$$

where

$$m_1\delta\bar{\theta}_1^+ + m_2\delta\bar{\theta}_2^+ = m_1\sigma + m_2(\delta\theta_2^- - \delta\theta_1^- + \sigma) = m_1\delta\theta_1^- + m_2\delta\theta_2^-$$

or $\sigma = \delta\theta_1^-$ and then this completes the proof. □

We will simplify DS_{12} by the following symplectic change of coordinates

$$\begin{cases} \xi = P^{-1}\delta h & \text{for } \xi = (\xi_0, \xi_1) \\ \eta = P^*\delta\theta & \text{for } \eta = (\eta_0, \eta_1) \end{cases}$$

where P^* and P^{-1} are given by

$$P^* = \begin{pmatrix} m_1 & m_2 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} \dfrac{1}{m_1+m_2} & \dfrac{1}{m_1+m_2} \\ -\dfrac{m_2}{m_1+m_2} & \dfrac{m_1}{m_1+m_2} \end{pmatrix}.$$

In (ξ, η) coordinates, we have $\mathcal{T}=\{\xi_0 = 0, \eta_0 = 0\}$ so that $\mathcal{T}$ acquires the structure of the standard symplectic space $\mathbb{R} \times \mathbb{R}$. These coordinates allow the expression of the derivatives DS_{12} in a fairly simple form. Expressing $D\tilde{S}_{12}$ in the coordinates (ξ, η) we obtain

$$D\tilde{S}_{12} = \begin{pmatrix} P^{-1} & 0 \\ 0 & P^* \end{pmatrix} DS_{12} \begin{pmatrix} P & 0 \\ 0 & P^{*-1} \end{pmatrix}$$

or

$$D\tilde{S}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + b_{12} - a_{12} + (c_{12} + d_{12})\omega & 0 & \dfrac{v_2^- c_{12} + v_1^- d_{12}}{v_1^- - v_2^-} \\ 0 & 0 & 1 & 0 \\ 0 & (C_{12} - A_{12} - B_{12} - D_{12})\omega & 0 & \dfrac{v_2^-(C_{12} - A_{12}) - v_1^-(D_{12} + B_{12})}{v_1^- - v_2^-} \end{pmatrix}$$

where $\omega = \dfrac{q_1(h_1^- + h_2^-)}{2h_1^- h_2^-(v_2^- - v_1^-)}$.

4. The existence of a periodic orbit and its linear stability

THEOREM 4.1. *There is a periodic orbit in the system with potential* $\frac{q^2}{2}$.

Proof. We prefer to have a periodic orbit with a small period, so it is reasonable to require that total energies h_i of individual particles do not change in collisions. From the formula (1) we see that this is the case if the centre of mass of colliding particles is at rest, i.e., the sum of their momenta is zero. In such a case a collision results in reversing the momenta, i.e., $p_i^+ = -p_i^-$.

We will obtain the periodic orbit having the following behaviour (see Figure 3).

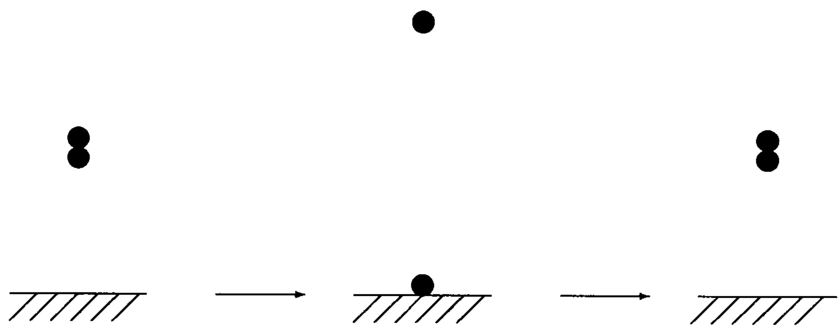

FIGURE 3. A periodic orbit when two particles are under the influence of an external potential field

By the Hamiltonian equation, for $0 \leq t \leq \tau$ (τ is a first collision time) we get $\ddot{q}(t) + \frac{q(t)}{m} = 0$ or equivalently we have

$$q(t) = q(0)\cos\left(\sqrt{\frac{1}{m}}t\right) + \sqrt{m}v^+(0)\sin\left(\sqrt{\frac{1}{m}}t\right),$$

$$v^-(t) = -\sqrt{\frac{1}{m}}q(0)\sin\left(\sqrt{\frac{1}{m}}t\right) + v^+(0)\cos\left(\sqrt{\frac{1}{m}}t\right).$$

We take initial conditions $(q_1(0), q_2(0), v_1(0), v_2(0))$ at $t = 0$ which describe the collision between the first and the second particle with their centers of mass at rest, i.e.,

$$q_1(0) = q_2(0), \qquad m_1 v_1^-(0) + m_2 v_2^-(0) = 0.$$

Furthermore, at time $t = t_0$ we want the first particle to hit the floor and the second particle to slow down to zero velocity.
Hence we obtain

$$\begin{aligned}
q_1(t_0) &= q_1(0)\cos\left(\sqrt{\frac{1}{m_1}}t_0\right) - \sqrt{m_1}v_1^-(0)\sin\left(\sqrt{\frac{1}{m_1}}t_0\right) = 0,\\
q_2(t_0) &= q_2(0)\cos\left(\sqrt{\frac{1}{m_2}}t_0\right) - \sqrt{m_2}v_2^-(0)\sin\left(\sqrt{\frac{1}{m_2}}t_0\right),\\
v_1^-(t_0) &= -\sqrt{\frac{1}{m_1}}q_1(0)\sin\left(\sqrt{\frac{1}{m_1}}t_0\right) - v_1^-(0)\cos\left(\sqrt{\frac{1}{m_1}}t_0\right),\\
v_2^-(t_0) &= -\sqrt{\frac{1}{m_2}}q_2(0)\sin\left(\sqrt{\frac{1}{m_2}}t_0\right) - v_2^-(0)\cos\left(\sqrt{\frac{1}{m_2}}t_0\right) = 0.
\end{aligned}$$

Here t_0 must satisfy $\tan\left(\sqrt{\frac{1}{m_1}}t_0\right)\tan\left(\sqrt{\frac{1}{m_2}}t_0\right) = \sqrt{\frac{m_1}{m_2}}$ since

$$\begin{aligned}
v_1^-(0) &= \sqrt{\frac{1}{m_1}}q_1(0)\cot\left(\sqrt{\frac{1}{m_1}}t_0\right)\\
v_2^-(0) &= -\sqrt{\frac{1}{m_2}}q_1(0)\tan\left(\sqrt{\frac{1}{m_2}}t_0\right)\\
m_1 v_1^-(0) &+ m_2 v_2^-(0) = 0\,.
\end{aligned} \tag{7}$$

Also we have $q_2(t_0) > 0$ and $v_1^+(t_0) < 0$ since $v_1^-(0) > 0$ and $v_2^-(0) < 0$.

The initial conditions $(q(0), v^-(0))$ obtained in such a way lead to a periodic orbit of period $2t_0$. Indeed we get

$$\begin{aligned}
q_1(2t_0) &= -\sqrt{m_1}v_1^-(t_0)\sin\left(\sqrt{\frac{1}{m_1}}t_0\right) = q_1(0),\\
q_2(2t_0) &= q_2(t_0)\cos\left(\sqrt{\frac{1}{m_2}}t_0\right) = q_2(0),\\
v_1^-(2t_0) &= -v_1^-(t_0)\cos\left(\sqrt{\frac{1}{m_1}}t_0\right) = v_1^-(0),\\
v_2^-(2t_0) &= -\sqrt{\frac{1}{m_2}}q_2(t_0)\sin\left(\sqrt{\frac{1}{m_1}}t_0\right) = v_2^-(0)
\end{aligned}$$

since $v_1^+(t_0) = -v_1^-(t_0)$ and Eqs. (7). □

DEFINITION 4.2. (Howard and Makay [2]) A periodic orbit is said to be *linearly stable* if, given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that all orbits of the tangent map initially within δ of 0 remain within ϵ of 0 for all forward time.

REMARK 4.3. A periodic orbit is linearly stable iff all eigenvalues of the tangent map have modulus less than or equal to 1 and all Jordan blocks corresponding to eigenvalues on the unit circle are one dimensional.

LEMMA 4.4. (Arnold [1]) *Let A be the matrix of a linear mapping of the plane to itself which preserves area* ($\det A = 1$). *Then the mapping A is stable if* $|\operatorname{tr} A| < 2$.

THEOREM 4.5. *The periodic orbit which is obtained is linearly stable.*

Proof. Its linear stability is described by the matrix

$$D\tilde{S}_{12} = \begin{pmatrix} 1+b-a+(c+d)\omega & \dfrac{v_2^- c + v_1^- d}{v_1^- - v_2^-} \\ (C-A-B-D)\omega & \dfrac{v_2^-(C-A) - v_1^-(D+B)}{v_1^- - v_2^-} \end{pmatrix}$$

where $\omega, a, b, c, d, A, B, C$ and D are defined as before, $v_2^- = -\dfrac{m_1 v_1^-}{m_2}$. By Lemma 4.4, the matrix is stable if and only if $-2 < \operatorname{tr}(D\tilde{S}_{12}) < 2$. By a direct computation, we have

$$\operatorname{tr}(D\tilde{S}_{12}) - 2 = -\frac{4m_1(m_1^2 v_1^2 + m_1 m_2 v_1^2 + 2m_2 q_1^2)^2}{(m_1+m_2)^2(m_1^2 v_1^2 + q_1^2 m_2)(m_1 v_1^2 + q_1^2)} < 0,$$

$$\operatorname{tr}(D\tilde{S}_{12}) + 2 = \frac{4q_1^2(m_1-m_2)^2(m_1^2 v_1^2 + m_1 m_2 v_1^2 + m_2 q_1^2)}{(m_1+m_2)^2(m_1^2 v_1^2 + m_2 q_1^2)(m_1 v_1^2 + q_1^2)} > 0.$$

This completes the proof. □

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Department of Mathematics
Seoul National University
Seoul 151-742, Korea
E-mail: dpchi@math.snu.ac.kr
E-mail: sunbok@math.snu.ac.kr