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ON LI-IDEALS AND PRIME LI-IDEALS OF LATTICE IMPLICATION ALGEBRAS

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ABSTRACT. As a continuation of the paper [3], in this paper we investigate the further properties on LI-ideals, and show that how to generate an LI-ideal by both an LI-ideal and an element. We define a prime LI-ideal, and give an equivalent condition for a proper LI-ideal to be prime. Using this result, we establish the extension property and prime LI-ideal theorem.

1. Introduction

In order to research the logical system whose propositional value is given in a lattice, Y. Xu [6] proposed the concept of lattice implication algebras, and discussed their some properties. Also, in [7], Y. Xu and K. Y. Qin discussed the properties of lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Y. Xu and K. Y. Qin [8] introduced the notion of filters in a lattice implication algebra, and investigated their properties. In [3], Y. B. Jun et al. defined an LI-ideal of a lattice implication algebra and showed that every LI-ideal is a lattice ideal. They gave an example that a lattice ideal may not be an LI-ideal, and showed that every lattice ideal is an LI-ideal in a lattice H implication algebra. They discussed the relationship between filters and LI-ideals, and studied how to generate an LI-ideal by a set. Moreover they constructed the quotient structure by using an LI-ideal, and studied the properties of LI-ideals related to implication homomorphisms.

This paper is a continuation of the paper [3]. In this paper we investigate the further properties on LI-ideals, and show that how to

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generate an LI-ideal by both an LI-ideal and an element. We define a prime LI-ideal, and give an equivalent condition for a proper LI-ideal to be prime. Using this result, we establish the extension property and prime LI-ideal theorem.

2. Preliminaries

By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation "$\rightarrow$" satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(I2) $x \rightarrow x = 1$,
(I3) $x \rightarrow y = y' \rightarrow x'$,
(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$.

Note that the conditions (L1) and (L2) are equivalent to the conditions

(L3) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$, and
(L4) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$, respectively.

A lattice implication algebra L is called a *lattice H implication algebra* if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in L$.

In the sequel the binary operation " $\rightarrow$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x \leq y$ if and only if $xy = 1$.

In a lattice implication algebra L, the following hold (see [6]):

(1) $0x = 1$, $1x = x$ and $x1 = 1$.
(2) $x' = x0$.
(3) $xy \leq (yz)(xz)$.
(4) $x \vee y = (xy)y$.
(5) $((yx)y')' = x \wedge y = ((xy)x')'$.
(6) $x \leq y$ implies $yz \leq xz$ and $zx \leq zy$.
(7) $x \leq (xy)y$.

In a lattice H implication algebra L, the following hold (see [7]):

(8) $x(xy) = xy$.

(9) $x(yz) = (xy)(xz)$.

3. LI-ideals

We begin with the following proposition.

PROPOSITION 3.1. *In a lattice implication algebra the following identity holds:*

(10) $((xy)y)y = xy$.

Proof. The inequality $xy \leq ((xy)y)y$ follows from (7). Now using (3), (I1) and (I2), we have

$$(((xy)y)y)(xy) \geq x((xy)y) = (xy)(xy) = 1$$

and hence $(((xy)y)y)(xy) = 1$, i.e., $((xy)y)y \leq xy$. This completes the proof. □

DEFINITION 3.2. (Y. B. Jun et al. [3]) Let L be a lattice implication algebra. An *LI-ideal* A of L is a non-empty subset of L such that

(LI1) $0 \in A$,

(LI2) $(xy)' \in A$ and $y \in A$ imply $x \in A$,

for all $x, y \in L$.

Under this definition $\{0\}$ and L are the trivial examples of LI-ideals. The following example shows that there is a proper LI-ideal in a lattice implication algebra.

EXAMPLE 3.3. (Y. B. Jun et al. [3, Example 2.1]) Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 1 as a partial ordering. Define a unary operation " $'$ " and a binary operation denoted by juxtaposition on L as follows (Tables 1 and 2, respectively):

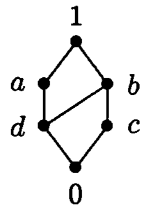

Figure 1

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

Table 1

	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table 2

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (xy)y,$$
$$x \wedge y := ((x'y')y')',$$

for all $x, y \in L$. Then L is a lattice implication algebra. It is easy to check that $A := \{0, c\}$ is an LI-ideal of L.

DEFINITION 3.4. (S. Burris et al. [1, Definition 8.2]) Let L be a lattice. An *ideal* I of L is a non-empty subset of L such that

(LI3) $x \in I$, $y \in L$ and $y \leq x$ imply that $y \in I$,

(LI4) $x, y \in I$ implies $x \vee y \in I$.

Throughout this paper we call this a *lattice ideal.*

Note that in a lattice H implication algebra, an LI-ideal and a lattice ideal coincide (see [3, Theorems 2.4 and 2.6]), also observe that a non-empty subset A of a lattice H implication algebra L is an LI-ideal of L if and only if for every $x, y \in L$,

(LI5) $x, y \in A \Leftrightarrow x \vee y \in A$.

PROPOSITION 3.5. *Let L be a lattice H implication algebra and $a \in L$. Then there is no proper LI-ideal of L containing a and a' simultaneously.*

Proof. Let A be a proper LI-ideal of L containing a and a' simultaneously. Then $1 = a \vee a' \in A$, and hence $A = L$ a contradiction. □

For any subsets A and B of a lattice H implication algebra L we set

$$A \wedge B = \{a \wedge b | a \in A \text{ and } b \in B\}.$$

PROPOSITION 3.6. *If A and B are LI-ideals of a lattice H implication algebra L, then so is $A \wedge B$.*

Proof. Let $x, y \in A \wedge B$. Then $x = a_1 \wedge b_1$ and $y = a_2 \wedge b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Since A is an LI-ideal, it follows from (LI5) that $a_1 \vee a_2 \in A$. Now

$$x \vee y = (a_1 \wedge b_1) \vee (a_2 \wedge b_2) = ((a_1 \wedge b_1) \vee a_2) \wedge ((a_1 \wedge b_1) \vee b_2).$$

Noticing that $(a_1 \wedge b_1) \vee a_2 = (a_1 \vee a_2) \wedge (b_1 \vee a_2) \leq a_1 \vee a_2$, we have $(a_1 \wedge b_1) \vee a_2 \in A$ by (LI3). Similarly we know that $(a_1 \wedge b_1) \vee b_2 \in B$. Hence

$$x \vee y = ((a_1 \wedge b_1) \vee a_2) \wedge ((a_1 \wedge b_1) \vee b_2) \in A \wedge B.$$

Conversely assume that $x \vee y \in A \wedge B$ for all $x, y \in L$. Then $x \vee y = a \wedge b$ for some $a \in A$ and $b \in B$. Observe that

$$x = x \wedge (x \vee y) = x \wedge (a \wedge b) \leq a$$

so from (LI3) that $x \in A$. Similarly, one can show that $x \in B$ and that $y \in A$ and $y \in B$. Thus $x, y \in A \wedge B$, ending the proof. □

PROPOSITION 3.7. *If A and B are LI-ideals of a lattice H implication algebra L, then $A \wedge B = A \cap B$.*

Proof. Let $x \in A \wedge B$. Then $x = a \wedge b$ for some $a \in A$ and $b \in B$. Observe that $x = a \wedge b \leq a$ (also b). It follows from (LI3) that $x \in A$ (also B) and hence $x \in A \cap B$. Conversely if $x \in A \cap B$ then $x = x \wedge x \in A \wedge B$. Therefore $A \wedge B = A \cap B$. □

PROPOSITION 3.8. *If A is an LI-ideal of a lattice H implication algebra L and $a \in L$, then the set $K := \{x \in L | x'a \in A\}$ is an LI-ideal of L.*

Proof. Let $x, y \in K$. Then $x'a \in A$ and $y'a \in A$. It follows from (L2) that $(x \vee y)'a = (x' \wedge y')a = (x'a) \vee (y'a) \in A$ so that $x \vee y \in K$. Conversely let $x, y \in L$ be such that $x \vee y \in K$. Then $(x \vee y)'a \in A$. Using (I3), (4) and (9), we have

$$\begin{aligned}(x \vee y)'a &= a'(x \vee y) = a'((xy)y) = (a'(xy))(a'y)\\ &= ((a'x)(a'y))(a'y) = (a'x) \vee (a'y) = (x'a) \vee (y'a).\end{aligned}$$

Thus $(x'a) \vee (y'a) \in A$ which implies that $x'a \in A$ and $y'a \in A$ because A is an LI-ideal. This means that $x \in K$ and $y \in K$, completing the proof. □

For any natural number n we define $n(x)y$ recursively as follows: $1(x)y = xy$ and $(n+1)(x)y = x(n(x)y)$.

Using (I1) repeatedly, we know that in a lattice implication algebra the following identity holds:

(11) $z(y_1(y_2(\cdots(y_n x)\cdots))) = y_1(y_2(\cdots(y_n(zx))\cdots))$.

As a special case of (11) we have

(12) $z(n(y)x) = n(y)(zx)$.

PROPOSITION 3.9. *For any elements $a, x, y_1, y_2, \cdots, y_n$ of a lattice implication algebra L we have*

(13) $y_1(y_2(\cdots(y_n(xa))\cdots)) = ((y_1(y_2(\cdots(y_n(xa))\cdots)))a)a$.

Proof. The inequality

$$y_1(y_2(\cdots(y_n(xa))\cdots)) \leq ((y_1(y_2(\cdots(y_n(xa))\cdots)))a)a$$

follows from (7). Using (10), (11), (I1) and (I2), we get

$$\begin{aligned}&(((y_1(y_2(\cdots(y_n(xa))\cdots)))a)a)(y_1(y_2(\cdots(y_n(xa))\cdots)))\\ &= y_1(y_2(\cdots(y_n(x((((y_1(y_2(\cdots(y_n(xa))\cdots)))a)a)a)))\cdots))\\ &= y_1(y_2(\cdots(y_n(x((y_1(y_2(\cdots(y_n(xa))\cdots)))a)))\cdots))\\ &= (y_1(y_2(\cdots(y_n(xa))\cdots)))(y_1(y_2(\cdots(y_n(xa))\cdots)))\\ &= 1,\end{aligned}$$

and hence $((y_1(y_2(\cdots(y_n(xa))\cdots)))a)a \leq y_1(y_2(\cdots(y_n(xa))\cdots))$. This completes the proof. □

OBSERVATION 3.10. (Y. B. Jun et al. [3, Observation 2.8]) *Suppose $\mathcal{A}$ is a non-empty family of LI-ideals of a lattice implication algebra L. Then $A = \cap\mathcal{A}$ is also an LI-ideal of L.*

Let A be a subset of a lattice implication algebra L. Then the least LI-ideal containing A is called the *LI-ideal generated* by A, written $\langle A \rangle$.

Noticing that L is clearly an LI-ideal containing A, in view of Observation 3.10, we know that the above definition $\langle A \rangle$ of A is well-defined.

The next statement gives a description of elements of $\langle A \rangle$.

PROPOSITION 3.11. (Y. B. Jun et al. [3, Theorem 2.9]) *If A is a non-empty subset of a lattice implication algebra L, then*

$$\langle A \rangle = \{x \in L | a_n'(...(a_1'x')...) = 1 \text{ for some } a_1, ..., a_n \in A\}.$$

The following corollary is immediate from Proposition 3.11.

COROLLARY 3.12. *For any element a of a lattice implication algebra L, we have*

$$\langle a \rangle = \{x \in L | n(a')x' = 1 \text{ for some natural number } n\}.$$

Note that in a lattice H implication algebra the identity $x(xy) = xy$ holds (see [7, Corollary 1]). Hence using Corollary 3.12 we obtain

COROLLARY 3.13. *Let L be a lattice H implication algebra and $a \in L$. Then*

$$\langle a \rangle = \{x \in L | a'x' = 1\} = \{x \in L | xa = 1\}.$$

LEMMA 3.14. (Y. B. Jun et al. [3, Theorem 2.2]) *Let A be an LI-ideal of a lattice implication algebra L and let $x \in A$. If $y \leq x$ (or equivalently $x' \leq y'$), then $y \in A$ for all $y \in L$.*

The following theorem describes how to generate an LI-ideal by both an LI-ideal and an element.

THEOREM 3.15. *Let A be an LI-ideal of a lattice implication algebra L and let $a \in L$. Then*

$$\langle A \cup \{a\} \rangle = \{x \in L | (n(a')x')' \in A \text{ for some } n \in \mathbb{N}\}.$$

Proof. Denote $A_a = \{x \in L | (n(a')x')' \in A \text{ for some } n \in \mathbb{N}\}$. Clearly $0 \in A_a$. Let $(yx)' \in A_a$ and $x \in A_a$. Then there exist $m, n \in \mathbb{N}$ such that $(n(a')((yx)')')' \in A$ and $(m(a')x')' \in A$. It follows that $(n(a')(yx))' = u$ and $(m(a')x')' = v$ for some $u, v \in A$, so that $u' = n(a')(yx)$ and $v' = m(a')x'$. Then

$$1 = u'(n(a')(yx)) = u'(n(a')(x'y')) = u'(x'(n(a')y')) = x'(u'(n(a')y')),$$

i.e., $x' \leq u'(n(a')y')$. Using (6) we have

$$\begin{aligned} v' &= m(a')x' \\ &\leq m(a')(u'(n(a')y')) \\ &= u'(m(a')(n(a')y')) \\ &= u'((m+n)(a')y'), \end{aligned}$$

which implies that $v'(u'(((m+n)(a')y')')') = v'(u'((m+n)(a')y')) = 1$. Since $u, v \in A$, it follows from Proposition 3.11 that $((m+n)(a')y')' \in \langle A \rangle = A$. Hence $y \in A_a$ and A_a is an LI-ideal of L. Note that $(n(a')a')' = 1' = 0 \in A$ so that $a \in A_a$. Let $x \in A$. Since $x' \leq a'x' = ((a'x')')'$, it follows from Lemma 3.14 that $(a'x')' \in A$, i.e., $x \in A_a$. Thus $A \cup \{a\} \subseteq A_a$. Finally we should verify that A_a is the least LI-ideal containing A and a. Let B be any LI-ideal containing A and a, and let $x \in A_a$. Then $(n(a')x')' \in A \subseteq B$ for some $n \in \mathbb{N}$, and hence

$$(((n-1)(a')x')'a)' = (a'((n-1)(a')x'))' = (n(a')x')' \in B.$$

Since $a \in B$, it follows from (LI2) that $((n-1)(a')x')' \in B$. Repeating this process, we obtain $x = (x')' \in B$. Therefore A_a is the least LI-ideal containing A and a, i.e., $\langle A \cup \{a\} \rangle = A_a$. This completes the proof. □

LEMMA 3.16. (J. Liu et al. [4, Corollary 1]) *Let L be a lattice implication algebra and $a, b, x \in L$. If $n(a)x = 1$ and $m(b)x = 1$ for some $m, n \in \mathbb{N}$, then there exists $k \in \mathbb{N}$ such that $k(a \vee b)x = 1$.*

THEOREM 3.17. *Let A be an LI-ideal of a lattice implication algebra L. Then*

$$\langle A \cup \{a\} \rangle \cap \langle A \cup \{b\} \rangle = \langle A \cup \{a \wedge b\} \rangle$$

for all $a, b \in L$.

Proof. Let $x \in \langle A \cup \{a\}\rangle \cap \langle A \cup \{b\}\rangle$. By Theorem 3.15, there are $m, n \in \mathbb{N}$ such that $(n(a')x')' \in A$ and $(m(b')x')' \in A$. Hence $(n(a')x')' = u$ and $(m(b')x')' = v$ for some $u, v \in A$. It follows that $n(a')x' = u'$ and $m(b')x' = v'$ so that $1 = v'(u'(n(a')x')) = n(a')(v'(u'x'))$ and $1 = u'(v'(m(b')x')) = m(b')(v'(u'x'))$. Using Lemma 3.16, there exists $k \in \mathbb{N}$ such that $k(a' \vee b')(v'(u'x')) = 1$. Since $a' \vee b' = (a \wedge b)'$, we have

$$1 = k(a' \vee b')(v'(u'x')) = k((a \wedge b)')(v'(u'x')) = v'(u'(k((a \wedge b)')x')).$$

Applying Proposition 3.11 we get $(k((a \wedge b)')x')' \in \langle A\rangle = A$ and hence $x \in \langle A \cup \{a \wedge b\}\rangle$ by Theorem 3.15. Thus

$$\langle A \cup \{a\}\rangle \cap \langle A \cup \{b\}\rangle \subseteq \langle A \cup \{a \wedge b\}\rangle.$$

Conversely if $x \in \langle A \cup \{a \wedge b\}\rangle$, then $(n((a \wedge b)')x')' \in A$ for some $n \in \mathbb{N}$. Since $a \wedge b \leq a, b$, therefore $a' \leq (a \wedge b)'$ and $b' \leq (a \wedge b)'$. Using (6) repeatedly, we get $n((a \wedge b)')x' \leq n(a')x'$ and $n((a \wedge b)')x' \leq n(b')x'$, which imply that $(n(a')x')' \leq (n((a \wedge b)')x')'$ and $(n(b')x')' \leq (n((a \wedge b)')x')'$. Applying Lemma 3.14, we obtain $(n(a')x')' \in A$ and $(n(b')x')' \in A$, i.e., $x \in \langle A \cup \{a\}\rangle$ and $x \in \langle A \cup \{b\}\rangle$. Hence $x \in \langle A \cup \{a\}\rangle \cap \langle A \cup \{b\}\rangle$ and so $\langle A \cup \{a \wedge b\}\rangle \subseteq \langle A \cup \{a\}\rangle \cap \langle A \cup \{a\}\rangle$. This completes the proof. □

COROLLARY 3.18. *Let A be an LI-ideal of a lattice implication algebra L. For any $a_1, a_2, \cdots, a_n \in L$ we have*

$$\bigcap_{i=1}^{n} \langle A \cup \{a_i\}\rangle = \langle A \cup \{a_1 \wedge a_2 \wedge \cdots \wedge a_n\}\rangle.$$

Proof. Straightforward by induction on n. □

COROLLARY 3.19. *Let A be an LI-ideal of a lattice implication algebra L. If $a_1 \wedge a_2 \wedge \cdots \wedge a_n \in A$ for all $a_1, a_2, \cdots, a_n \in L$, then*

$$\bigcap_{i=1}^{n} \langle A \cup \{a_i\}\rangle = A.$$

Proof. Using Corollary 3.18, we have

$$\bigcap_{i=1}^{n} \langle A \cup \{a_i\}\rangle = \langle A \cup \{a_1 \wedge a_2 \wedge \cdots \wedge a_n\}\rangle = \langle A\rangle = A. \qquad \square$$

4. Prime LI-ideals

DEFINITION 4.1. A proper LI-ideal P of a lattice implication algebra L is said to be *prime* if whenever $x \wedge y \in P$ then $x \in P$ or $y \in P$ for all $x, y \in L$.

We provide an equivalent condition for a proper LI-ideal to be prime.

THEOREM 4.2. *Let P be a proper LI-ideal of a lattice implication algebra L. Then P is prime if and only if $(xy)' \in P$ or $(yx)' \in P$ for all $x, y \in L$.*

Proof. Assume that P is a prime LI-ideal of L and let $x, y \in L$. Then

$$\begin{aligned} 1 &= (x \vee y)(x \vee y) = ((x \vee y)x) \vee ((x \vee y)y) \\ &= (1 \wedge (yx)) \vee ((xy) \wedge 1) = (yx) \vee (xy), \end{aligned}$$

and so $(yx)' \wedge (xy)' = ((yx) \vee (xy))' = 1' = 0 \in P$. Since P is prime, it follows that $(xy)' \in P$ or $(yx)' \in P$. Conversely let P be a proper LI-ideal of L and suppose $(xy)' \in P$ or $(yx)' \in P$ for all $x, y \in L$. Let $x, y \in L$ be such that $x \wedge y \in P$. If $(yx)' \in P$, since

$$(y(yx)')' = ((yx)y')' = ((x'y')y')' = (x' \vee y')' = x \wedge y$$

we have $y \in P$ by (LI2). Similarly $x \in P$ whenever $(xy)' \in P$, ending the proof. □

Using Theorem 4.2 we have the extension property for prime LI-ideal. The proof is straightforward and is omitted.

THEOREM 4.3. (Extension property for prime LI-ideal) *Let P be a prime LI-ideal of a lattice implication algebra L. Then every proper LI-ideal containing P is also prime.*

THEOREM 4.4. (Prime LI-ideal theorem) *Let A be an LI-ideal of a lattice implication algebra L and S a $\wedge$-closed subset of L (i.e., $x \wedge y \in S$ whenever $x, y \in S$) such that $A \cap S = \emptyset$. Then there exists a prime LI-ideal P of L such that $A \subseteq P$ and $P \cap S = \emptyset$.*

Proof. The existence of an LI-ideal P being the maximal element of the family of all LI-ideals that contains A and have empty intersection with S follows from an application of Zorn's lemma. We now show that P is prime. Suppose P is not prime. Then there exist $x, y \in L \setminus P$ such that $x \wedge y \in P$. By means of the maximality of P, we know that $S \cap \langle P \cup \{x\} \rangle \neq \emptyset$ and $S \cap \langle P \cup \{y\} \rangle \neq \emptyset$. Taking u and v in $S \cap \langle P \cup \{x\} \rangle$ and $S \cap \langle P \cup \{y\} \rangle$, respectively, then $u \wedge v \in S$ because S is $\wedge$-closed. Since $u \wedge v \leq u, v$, it follows that $u \wedge v \in \langle P \cup \{x\} \rangle$ and $u \wedge v \in \langle P \cup \{y\} \rangle$. Using Theorem 3.17, one can know that

$$u \wedge v \in \langle P \cup \{x\} \rangle \cap \langle P \cup \{y\} \rangle = \langle P \cup \{x \wedge y\} \rangle = P.$$

Hence $u \wedge v \in S \cap P$, which is a contradiction. Therefore P is prime, ending the proof. □

COROLLARY 4.5. *Let A be an LI-ideal of a lattice implication algebra L. If $x \in L \setminus A$, then there is a prime LI-ideal P of L such that $A \subseteq P$ and $x \notin P$.*

Proof. Let $S = \{y \in L | xy = 1\}$. If $y_1, y_2 \in S$, then $xy_1 = 1$ and $xy_2 = 1$. It follows that $x(y_1 \wedge y_2) = (xy_1) \wedge (xy_2) = 1$ so that $y_1 \wedge y_2 \in S$, i.e., S is $\wedge$-closed. Let $y \in S$. Then $xy = 1$ and hence $(xy)' = 1' = 0 \in A$. Since $x \notin A$, we have $y \notin A$ by (LI2). Hence $A \cap S = \emptyset$. Using Theorem 4.4, there is a prime LI-ideal P of L such that $A \subseteq P$ and $P \cap S = \emptyset$. Since $x \in S$, the identity $P \cap S = \emptyset$ implies $x \notin P$. This completes the proof. □

THEOREM 4.6. *For a lattice implication algebra L the following are equivalent:*

(i) *LI-ideal $\{0\}$ is prime.*
(ii) *every proper LI-ideal of L is prime.*
(iii) *$(L, \leq)$ is a totally ordered set.*

Proof. (i) $\Rightarrow$ (ii) is by Theorem 4.3, and (ii) $\Rightarrow$ (i) is obvious. Assume that $(L, \leq)$ is a totally ordered set. Then $xy = 1$ or $yx = 1$, and hence $(xy)' = 1' = 0 \in \{0\}$ or $(yx)' = 1' = 0 \in \{0\}$ for all $x, y \in L$. It follows from Theorem 4.2 that $\{0\}$ is a prime LI-ideal of L. Conversely if $\{0\}$ is a prime LI-ideal of L, then $(xy)' \in \{0\}$ or $(yx)' \in \{0\}$ for all $x, y \in L$, that is, $(xy)' = 0$ or $(yx)' = 0$; hence $xy = 1$ or $yx = 1$

for all $x, y \in L$. This shows that $(L, \leq)$ is a totally ordered set. This completes the proof. □

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