J. Korean Math. Soc. **36** (1999), No. 2, pp. 355–367

MULTIPLICITY RESULTS OF ORDERED POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS ON $\mathbf{R}^n$

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ABSTRACT. We prove the existence of $2N-1$ distinct ordered positive solutions of a class of semilinear elliptic Dirichlet boundary value problems on $\mathbf{R}^n$ when the forcing term has N distinct positive stable zeros and the coefficient function decaying to the zero at infinity.

1. Introduction

In this paper, we are concerned with the existence of $2N-1$ distinct ordered positive solutions of the following semilinear elliptic boundary value problem;

$$(P_\lambda) \qquad \begin{cases} \triangle u + \lambda g(|x|) f(u) = 0 \quad \text{in} \quad \mathbf{R}^n \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $n \geq 3$. In what follows, we assume $f \in C(I, \mathbf{R})$, $I = [0, b] \subset \mathbf{R}$ is a bounded closed interval, and $g \in C(\mathbf{R}^+, \mathbf{R}^+)$, $g(r) > 0$ for some $r \geq 0$, denoting $\mathbf{R}^+ = [0, \infty)$. Furthermore, we assume the following conditions;

Received September 21, 1998.

1991 Mathematics Subject Classification: 35J25, 34B15.

Key words and phrases: existence, multiplicity, positive solution, semilinear elliptic problem, singular boundary value problem, upper and lower solution method, three solution theorem, Schauder fixed point.

† This work was supported in part by Cheju National University, Research Fund, 1997.

‡ The present studies were supported by the Matching Fund Programs of Research Institute for Basic Sciences, Pusan National University, Korea, 1998, Project No. RIBS-PNU-98-102.

(f_0) There exists a positive constant M such that

$$f(u) - f(v) \geq -M(u - v), \quad \text{if } u, v \in I, \; u \geq v.$$

(f_1) There exist exactly N positive numbers $0 < a_1 < a_2 < \cdots < a_N$ such that $f(a_i) = 0$, for all $i = 1, \cdots, N$.

(f_2) $[0, a_N] \subset I$ and there exists a positive constant K_f such that

$$f'(a_i) \leq -K_f, \quad \text{for all } i = 1, \cdots, N.$$

(f_3) $\int_s^{a_i} f(u)du > 0$, for all $s \in [0, a_i)$ and for all $i = 1, \cdots, N$.

(g) $g \in C^1(\mathbf{R}^+)$ satisfies $\int_0^\infty rg(r)dr < \infty$.

If the domain of the equation in problem (P_λ) is bounded and open with smooth boundary, then there have been several studies ([2],[3],[4],[5],[7],[8]) which prove generally that there exists $\lambda_o > 0$ such that (P_λ) has at least $2N - 1$ distinct ordered positive solutions for all $\lambda > \lambda_o$.

In those studies, the boundedness of the domain is crucial. If the domain is unbounded, the situation is quite different because the compactness of operator or functional induced from the problem is not guaranteed. Thus it is interesting to study the existence of multiple positive solutions when the problem is defined on $\mathbf{R}^n$, and as far as the authors know, related studies have not been made yet.

To obtain our desired result, we split $\mathbf{R}^n$ into a ball $B(0,1)$ and its compliment $\Omega = \mathbf{R}^n \setminus B(0,1)$ and solve problem (P_λ) defined on $B(0,1)$ and Ω respectively.

As mentioned above, we get the existence of N positive ordered solutions for (P_λ) defined on $B(0,1)$ if λ is sufficiently large.

To obtain the existence of N positive solutions for the problem defined on the exterior domain Ω, we transform (1_λ) into the singular boundary value problem (S_λ): More precisely, the problem on Ω, we concern, is as follows;

$$(1_\lambda) \qquad \begin{cases} \triangle u + \lambda g(|x|)f(u) = 0 & \text{in } \Omega, \\ u(x) = a_j \quad \text{if} \quad |x| = 1, \\ \lim\limits_{|x| \to \infty} u(x) = 0. \end{cases}$$

By transformations $r = |x|$ and $t = r^{2-n}$, (1_λ) can be transformed into

$$(S_\lambda) \qquad \begin{cases} u'' + \lambda q(t) f(u) = 0, & 0 < t < 1, \\ u(0) = 0, \ u(1) = a_j, \end{cases}$$

where q is given by

$$q(t) = \frac{1}{(n-2)^2} t^{\frac{2(1-n)}{n-2}} g(t^{\frac{1}{2-n}}).$$

It is known that a positive solution of (S_λ) corresponds to a positive radial solution of (1_λ).

In the problem (S_λ), the coefficient function q is of two types depending on the asymptotic growth condition on g as follows; if g in (P_λ) satisfies $\lim_{r\to\infty} r^{2(n-1)} g(r) < \infty$, then q can be extended continuously on $[0,1]$, so problem (S_λ) is regular. On the other hand, if g satisfies $\lim_{r\to\infty} r^{2(n-1)} g(r) = \infty$, then $q \in C(0,1]$ and singular at $t = 0$, see [9] for further detail.

For problem (S_λ), we show the existence of positive solutions for the singular case. We know the result for the regular case is followed by that of the singular case with less conditions.

Once positive solutions of two problems are provided, we glue them together to get N-pairs of lower and upper solutions of (P_λ), and then we use Three Solution Theorem ([1],[8]) and an argument of some convergence method ([10]) for proving our multiplicity.

In Section 2, we give a fundamental existence theorem of positive solutions on the method of G-upper and G-lower solutions which might have singular points in the interior of the interval $[0,1]$. In Section 3, we study the multiplicity of positive solutions for problem (P_λ) in the process as the previous paragraphs.

2. Existence of solutions for singular boundary value problems

In this section, we prove a fundamental existence theorem in terms of general upper and lower solutions for singular boundary value problems of the form $(SBVP)$;

$$(SBVP) \qquad \begin{cases} u'' + f(t,u) = 0, & 0 < t < 1, \\ u(0) = A, \ u(1) = B, \end{cases}$$

where $f : D \subset (0,1) \times \mathbf{R} \to \mathbf{R}$ is continuous. A solution $u(\cdot)$ means a function $u \in C[0,1] \cap C^2(0,1)$ such that $(t, u(t)) \in D$, for all $t \in (0,1)$ and u satisfies the ordinary differential equation pointwise on $(0,1)$ with $u(0) = A$ and $u(1) = B$.

DEFINITION 1. $\alpha \in C[0,1] \cap C^2(0,1)$ is called a *lower solution* of $(SBVP)$ if $(t, \alpha(t)) \in D$ for all $t \in (0,1)$ and

$$\alpha''(t) + f(t, \alpha(t)) \geq 0,$$
$$\alpha(0) \leq A, \ \alpha(1) \leq B.$$

Similarly, $\beta \in C[0,1] \cap C^2(0,1)$ is called an *upper solution* of $(SBVP)$ if the above inequalities are reversed.

The fundamental theorem on upper and lower solution method for the problem $(SBVP)$ is studied by Habets and Zanolin [6]. We give definitions of somewhat general type of upper and lower solutions and prove a fundamental theorem for this type.

DEFINITION 2. We say that a continuous function $\alpha(\cdot) : [0,1] \to \mathbf{R}$ is a *G-lower solution* of $(SBVP)$ if $\alpha \in C^2(0,1)$ except at finite points $\tau_1, \cdots, \tau_n$ with $0 < \tau_1 < \cdots < \tau_n < 1$ such that

(L_1) at each τ_i, there exist $\alpha'(\tau_i{}^-)$, $\alpha'(\tau_i{}^+)$ such that $\alpha'(\tau_i{}^-) < \alpha'(\tau_i{}^+)$ and

$$(L_2) \quad \begin{cases} \alpha''(t) + f(t, \alpha(t)) \geq 0, & \text{for all } t \in (0,1) \setminus \{\tau_1, \cdots, \tau_n\}, \\ \alpha(0) \leq A, \ \alpha(1) \leq B. \end{cases}$$

We also say that a continuous function $\beta(\cdot) : [0,1] \to \mathbf{R}$ is a *G-upper solution* of $(SBVP)$ if $\beta \in C^2(0,1)$ except at finite points $\sigma_1, \cdots, \sigma_m$ with $0 < \sigma_1 < \cdots < \sigma_m < 1$ such that

(U_1) at each σ_i, there exist $\beta'(\sigma_j{}^-)$, $\beta'(\sigma_j{}^+)$ and $\beta'(\sigma_j{}^-) > \beta'(\sigma_j{}^+)$ and

$$(U_2) \quad \begin{cases} \beta''(t) + f(t, \beta(t)) \leq 0, & \text{for all } t \in (0,1) \setminus \{\sigma_1, \cdots, \sigma_m\}, \\ \beta(0) \geq A, \ \beta(1) \geq B. \end{cases}$$

REMARK 1. We note that lower and upper solutions of $(SBVP)$ imply G-lower and G-upper solutions, respectively. But not vice versa. Hence, Theorem 1 below is a generalization of Theorem 1 in [6].

DEFINITION 3. If $\alpha,\ \beta \in C[0,1]$ are such that $\alpha(t) \leq \beta(t)$, for all $t \in [0,1]$, we define the set

$$D_\alpha^\beta = \{(t,u) \in (0,1) \times \mathbf{R} : \alpha(t) \leq u \leq \beta(t)\}.$$

THEOREM 1. *Let α and β be, respectively, a G-lower solution and a G-upper solution of $(SBVP)$ such that*
(a_1) $\alpha(t) \leq \beta(t)$, *for all* $t \in [0,1]$.
(a_2) $D_\alpha^\beta \subset D$.
Assume also that there is a function $h \in C\big((0,1), \mathbf{R}^+\big)$ such that
(a_3) $|f(t,u)| \leq h(t)$, *for all* $(t,u) \in D_\alpha^\beta$ *and*
(a_4) $\int_0^1 s(1-s)h(s)ds < \infty$.
Then $(SBVP)$ has at least one solution u such that

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \text{for all } t \in (0,1).$$

Proof. Define a modified function of f as follows;

$$F(t,u) = \begin{cases} f(t,\beta(t)) - \dfrac{u-\beta(t)}{1+u^2}, & \text{if } u > \beta(t), \\ f(t,u), \quad \text{if } \alpha(t) \leq u \leq \beta(t), & \\ f(t,\alpha(t)) - \dfrac{u-\alpha(t)}{1+u^2}, & \text{if } u < \alpha(t). \end{cases}$$

Then $F : (0,1) \times \mathbf{R} \to \mathbf{R}$ is continuous and

$$|F(t,u)| \leq m(\alpha,\beta) + h(t), \tag{1}$$

for all $(t,u) \in (0,1) \times \mathbf{R}$, where $m(\alpha,\beta) = \|\alpha\|_\infty + \|\beta\|_\infty + 1$.
Consider the problem

$$\begin{cases} u'' + F(t,u) = 0, \quad 0 < t < 1, \\ u(0) = A,\ u(1) = B. \end{cases} \tag{2}$$

We claim that any solution u of (2) satisfies $\alpha(t) \le u(t) \le \beta(t)$, for all $t \in [0,1]$. Without loss of generality, we only prove that $\alpha(t) \le u(t)$ for all $t \in [0,1]$. We can prove for the case that $u(t) \le \beta(t)$ for all $t \in [0,1]$ by a similar fashion.

Suppose that $\alpha \nleq u$. So let $(\alpha - u)(t_o) = \max_{t\in[0,1]}(\alpha - u)(t) > 0$. If $t_o \in (0,1) \setminus \{\tau_1, \cdots, \tau_n\}$, then $(\alpha - u)''(t_o) \le 0$. Since $u(t_o) < \alpha(t_o)$,

$$\begin{aligned} 0 \ge (\alpha - u)''(t_o) &= \alpha''(t_o) + F(t_o, u(t_o)) \\ &= \alpha''(t_o) + f(t_o, \alpha(t_o)) - \frac{u(t_o) - \alpha(t_o)}{1 + u^2(t_o)} \\ &\ge \frac{\alpha(t_o) - u(t_o)}{1 + u^2(t_o)} > 0, \end{aligned}$$

a contradiction.

Let $t_o = \tau_i$, for some $i = 1, \cdots, n$. Since $\alpha - u$ attains its positive maximum at τ_i,

$$(\alpha - u)'(\tau_i^-) \ge 0 \ \text{ and } \ (\alpha - u)'(\tau_i^+) \le 0.$$

Thus

$$\begin{aligned} 0 &\le (\alpha - u)'(\tau_i^-) - (\alpha - u)'(\tau_i^+) \\ &= \alpha'(\tau_i^-) - \alpha'(\tau_i^+). \end{aligned}$$

This leads to a contradiction to the definition of G-lower solution.

Let $t_o = 0$ or 1

$$\begin{aligned} 0 &< (\alpha - u)(0) = \alpha(0) - A \le 0, \\ 0 &< (\alpha - u)(1) = \alpha(0) - B \le 0, \end{aligned}$$

a contradiction.

Therefore, $\alpha(t) \le u(t) \le \beta(t)$, and so we can conclude u is a solution of $(SBVP)$.

We claim that (2) has at least one solution. It is well-known that problem (2) is equivalently written as

$$u = Tu \ \text{ on } X = C[0,1],$$

where

$$Tu(t) = A + (B - A)t + \int_0^1 G(t,s)F(s,u(s))ds$$

and $G(t,s)$ is the Green's function explicitly written as

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t, \\ t(1-s), & t \le s \le 1. \end{cases}$$

By (1) and (a_4), $T : X \to X$ is well-defined, continuous and TX is bounded. If T is a compact operator, then the proof of the existence of a solution is done by Schauder fixed point theorem.

To show T compact, making use of Arzela-Ascoli Theorem, it suffices to show that TX is equicontinuous.

Let $t \in (0,1)$. Then by (1) we get

$$\begin{aligned} &|\frac{d}{dt}Tu(t)| \\ &\le |B-A| + \int_0^t s|F(s,u(s))|ds + \int_t^1 (1-s)|F(s,u(s))|ds \\ &\le |B-A| + \frac{m(\alpha,\beta)}{2}(t^2 + (1-t)^2) + \int_0^t sh(s)ds + \int_t^1 (1-s)h(s)ds \\ &\triangleq |B-A| + \frac{m(\alpha,\beta)}{2}(t^2 + (1-t)^2) + \gamma(t). \end{aligned}$$

If $\gamma \in L^1(0,1)$, then the proof follows from that

$$\begin{aligned} \int_0^1 |\gamma(s)|ds \le & \lim_{t\to 1^-}(1-t)\int_0^t sh(s)ds + \lim_{t\to 0^+} t\int_t^1 (1-s)h(s)ds \\ & + 2\int_0^1 s(1-s)h(s)ds \\ \le & \ 4\int_0^1 s(1-s)h(s)ds < \infty. \end{aligned}$$

□

REMARK 2. It is not hard to see that if we replace (a_4) by the condition $\int_0^1 sh(s)ds < \infty$ $\left(\int_0^1 (1-s)h(s)ds < \infty\right)$, then the solution u which we have found belongs to $C^1(0,1]$ $\left(C^1[0,1)\right)$.

EXAMPLE. As an application of Theorem 1, we solve the following singular boundary value problem:

$$(3_\lambda) \qquad \begin{cases} u'' + \lambda u \left(u - \dfrac{1}{3t}\right)(1-u) = 0, & 0 < t < 1, \\ u(0) = 0, \ u(1) = 0. \end{cases}$$

We can easily check that problem (3_λ) satisfies (a_3) and (a_4) in Theorem 1. It seems to be quite difficult to find a pair of lower and upper solution for (3_λ). Because, the function $u(t) = \frac{1}{3t}$ intersects the function $u(t) = 1$ in the open interval $(0,1)$. However, by Theorem 1, it is enough to construct a pair of G-lower solution and G-upper solution to get a positive solution $u(t;\lambda)$ of (3_λ). The following theorem is useful to construct a pair of G-lower solution and G-upper solution.

THEOREM. (Theorem 4 in [8]) *Let $f \in C^1([a,b] \times I, \mathbf{R})$. Suppose that there is a positive real number u_1 satisfying the following conditions:*

(i) *$f(t,u_1) = 0$ for all $t \in [a,b]$,*
(ii) *$f_u(t,u_1) \le -k^2 < 0$ for all $t \in [a,b]$, and*
(iii) *$\int_s^{u_1} f(t,u)du > 0$ for $t = a$ and $t = b$ and $s \in [0,u_1)$.*

Then there is a positive number λ_o such that for all $\lambda > \lambda_o$, the problem

$$\begin{cases} u'' + \lambda f(t,u) = 0, & a < t < b, \\ u(a) = 0 = u(b) \end{cases}$$

has a positive lower solution $\bar{v}$ satisfying $\bar{v}(t) \le u_1$ for all $t \in [a,b]$.

Let $f(t,u) = u\left(u - \frac{1}{3t}\right)(1-u)$. At $t = 1$, we note that

$$\int_s^1 u\left(u - \frac{1}{3}\right)(1-u)du > 0$$

for all $s \in [0,1)$. Hence, there exists a positive number δ so that

$$\int_s^1 u\left(u - \frac{1}{3t}\right)(1-u)du > 0$$

uniformly on $t \in [1-\delta, 1]$ and for all $s \in [0,1)$, and we can apply the above Theorem with $u_1 = 1$ on the interval $[1-\delta, 1]$, and obtain a positive λ_o so that for all $\lambda > \lambda_o$, the following problem:

$$(3_\lambda^\delta) \qquad \begin{cases} u'' + \lambda u \left(u - \dfrac{1}{3t}\right)(1-u) = 0, & 1-\delta < t < 1, \\ u(1-\delta) = 0, \ u(1) = 0. \end{cases}$$

has a positive lower solution $v(t;\lambda)$ satisfying $v(t;\lambda) \leq 1$ for all $t \in [1-\delta, 1]$. Obviously $u_1 = 1$ is an upper solution of (3_λ^δ).

We define a G-lower solution $\alpha(t;\lambda)$ by

$$\alpha(t;\lambda) = \begin{cases} 0, & 0 \leq t \leq 1-\delta, \\ v(t;\lambda), & 1-\delta \leq t \leq 1. \end{cases}$$

Then $\alpha \leq \beta = 1$ are a G-lower solution and a G-upper solution of (3_λ), respectively. Consequently, by Theorem 1, there is a positive solution $u(t;\lambda)$ of (3_λ) which lies between $\alpha(t;\lambda)$ and 1.

3. Ordered positive solutions

We prove the existence of $2N-1$ distinct ordered positive solutions of the following problem;

$$(P_\lambda) \qquad \begin{cases} \triangle u + \lambda g(|x|) f(u) = 0 \quad \text{in} \quad \mathbf{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

If the domain Ω of the equation in (P_λ) is contained in the large ball $B(0,r)$ with $r > r_o$ and has the smooth boundary, if the function $g \in C^1(\bar{\Omega})$ satisfies $g(r_o) > 0$, and if we also assume the conditions $(f_0) \sim (f_3)$ for f, then the multiplicity result of positive solutions for large enough λ can be obtained ([4],[8]).

REMARK 3. Necessary and sufficient condition that g satisfies the integral condition in (g) is that q fulfills the following condition;

$$(C) \qquad \int_0^1 sq(s)ds < \infty$$

which corresponds condition (a_4) in Theorem 1. We also note that g satisfies condition in (g) if $\int_{\mathbf{R}^n\setminus B(0,r)} g(|x|)|x|^{2-n}dx < \infty$ for some $r > 0$. Furthermore, if we let $h(x) = g(|x|)$, we get the following statement: if for some p with $1 < p < \frac{n}{2}$, $h \in L^p(\mathbf{R}^n \setminus B(0,r))$, then q satisfies the integral condition (C). But the converse may not be true. This also implies that the nonlinear operator induced from (P_λ) might not be compact.

The following is the main result in this paper.

THEOREM 2. *Let $n \geq 3$ and assume $(f_0) \sim (f_3)$. If g satisfies condition (g), then there exists $\lambda_o > 0$ such that (P_λ) has $2N - 1$ distinct ordered positive solutions for all $\lambda > \lambda_o$.*

Proof. Without loss of generality, we assume that there exists a point x_0 such that $|x_0| < 1$ and $g(|x_0|) > 0$. Let

$$\Omega = \{x \in \mathbf{R}^n : 1 < |x| < \infty\}.$$

Then the problem

$$(1_\lambda) \qquad \begin{cases} \triangle u + \lambda g(|x|)f(u) = 0 & \text{in } \Omega, \\ u(x) = a_j \quad \text{if} \quad |x| = 1, \\ \lim_{|x|\to\infty} u(x) = 0, \end{cases}$$

can be transformed into

$$(S_\lambda) \qquad \begin{cases} u'' + \lambda q(t)f(u) = 0, & 0 < t < 1, \\ u(0) = 0, \ u(1) = a_j. \end{cases}$$

We notice by condition (g) that Theorem 1 is valid for (S_λ). It is not hard to check that 0 and a_j are a lower solution and an upper solution of (1_λ), respectively, for all λ, thus (S_λ) has solutions $y_j = y_j(t;\lambda)$, $j = 1, \cdots, N$ with $0 \leq y_j(t;\lambda) \leq a_j$, for all $t \in [0,1]$, and thus (1_λ) has positive radial solutions $\tilde{u}_j^\lambda(x) = y_j(|x|^{2-n};\lambda)$, $j = 1, \cdots, N$, for all λ such that

$$0 \leq \tilde{u}_j^\lambda(x) \leq a_j \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad \lim_{|x|\to\infty} \tilde{u}_j^\lambda(x) = 0.$$

On the other hand, since $g(|x_0|) > 0$, from Theorem 5 in [8], there exists $\lambda_o > 0$ such that for all $\lambda > \lambda_o$, the following problem

$$\begin{cases} \triangle u + \lambda g(|x|)f(u) = 0 & \text{in} \quad 0 \leq |x| < 1, \\ u(x) = 0 \quad \text{if} \quad |x| = 1 \end{cases}$$

has N distinct positive ordered solutions $\bar{v}_j^\lambda$ such that $0 \leq \bar{v}_j^\lambda(x) \leq a_j$ for all $0 \leq |x| \leq 1$ and $\bar{v}_j^\lambda$ converges to a_j as $\lambda \to \infty$ uniformly on every compact subsets of the unit open ball. For $\lambda \geq \lambda_o$ and $R > 0$, the function $\tilde{w}_j^\lambda$ defined by

$$\tilde{w}_j^\lambda(x) = \begin{cases} a_j & \text{if} \quad 0 \leq |x| \leq 1, \\ \tilde{u}_j^\lambda(x) & \text{if} \quad |x| \geq 1 \end{cases}$$

and the function $\bar{w}_j^\lambda$ defined by

$$\bar{w}_j^\lambda(x) = \begin{cases} \bar{v}_j^\lambda(x) & \text{if} \quad 0 \leq |x| \leq 1, \\ 0 \quad \text{if} \quad |x| \geq 1 \end{cases}$$

are a quasi-supersolution and a quasi-subsolution ([8]) respectively of the following problem;

$$(4_R) \qquad \begin{cases} \triangle u + \lambda g(|x|)f(u) = 0 & \text{in } |x| < R, \\ u(x) = 0 \quad \text{if} \quad |x| = R. \end{cases}$$

Since $\bar{w}_j^\lambda \prec \tilde{w}_j^\lambda$, $\bar{w}_{j+1}^\lambda \prec \tilde{w}_{j+1}^\lambda$, $\bar{w}_j^\lambda \prec \tilde{w}_{j+1}^\lambda$, and $\bar{w}_{j+1}^\lambda \not\preceq \tilde{w}_j^\lambda$, Theorem 3 in [9](or Three Solution Theorem in [1]) implies the existence of three distinct solutions $u_j^R \prec u_{j+\frac{1}{2}}^R \prec u_{j+1}^R$ of (4_R) such that

$$\bar{w}_j^\lambda \leq u_j^R \prec \tilde{w}_j^\lambda, \quad \bar{w}_{j+1}^\lambda \prec u_{j+1}^R \prec \tilde{w}_{j+1}^\lambda,$$
$$\tilde{w}_j^\lambda(x_R) < u_{j+\frac{1}{2}}^R(x_R) < \bar{w}_{j+1}^\lambda(x_R),$$

for some x_R with $|x_R| < 1$. Fix j and λ, letting $R \to \infty$, Theorem 2.10 in [10] implies that, without loss of generality (otherwise using diagonal process),

$$u_j = \lim_{R\to\infty} u_j^R,$$
$$u_{j+1} = \lim_{R\to\infty} u_{j+1}^R,$$
$$u_{j+\frac{1}{2}} = \lim_{R\to\infty} u_{j+\frac{1}{2}}^R$$

are solutions of problem (P_λ), and those are distinct since a_j, $\bar{v}_{j+1}^\lambda$ are independent of R and $\liminf_{R\to\infty} |x_R| \le 1$. Trivially, they are positive. Since $\lim_{|x|\to\infty} \tilde{u}_j^\lambda(x) = 0$, we get $u_j(x) \to 0$ as $|x| \to \infty$ and this completes the proof. □

From the simple scaling technique, we have the following result.

COROLLARY. *With assumptions in* Theorem 2, *there are* $2N-1$ *distinct ordered positive solutions of the following problem for all sufficiently large* λ*:*

$$\begin{cases} \triangle u + g\left(\left|\dfrac{x}{\lambda}\right|\right) f(u) = 0 & \text{in } \mathbf{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

REMARK 4. Let $g(0) = 1$. With the above corollary and some convergence method, we hope to prove the same multiplicity of positive ordered solutions for the problem $\Delta u + f(u) = 0$ in $\mathbf{R}^n$ with the same limiting behavior at the infinity. But the problem is still open.

ACKNOWLEDGEMENT. The authors return thanks to the referee for several comments on this paper.

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