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A MATRIX THEOREM AND FURTHER IMPROVEMENT OF THE STILES' SUBSERIES THEOREM

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ABSTRACT. In this paper, we obtain a matrix theorem which will be widely used in several theory and give it's applications. In particular, we present a further improvement of the improved Stiles' theorem.

1. Introduction

As stated in Kalton [4], the Stiles' subseries theorem [11] was a very important and significant departure from earlier Orlicz-Pettis type theorems and, therefore, the original Stiles theorem was generalized by Kalton [5] (1971), Basit [2] (1986) and Swartz [12] (1988), but all of these results established only for complete metric linear spaces. In 1995, Li and Cho [7] gave a substantial improvement for Stiles type result by dropping both conditions of the metrizability of spaces and the continuity of coordinate functionals in the past results. However, this result could not drop the completeness condition.

The main object of this paper is to obtain a matrix theorem which will be widely used in several theory and, by using this result, to present a further improvement of the improved Stiles' theorem which is given in [7].

In section 2 we introduce the very useful result concerning uniform convergence of the series whose terms are entries in each row of an infinite matrix over an abelian topological group and then give some applications of that result. In section 3, we improve the recent result in [7] so that the conclusion of Stiles theorem holds without any restriction

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for the concerned space. Thus, in the view of extension of the concerned space, we have obtained the best and, hence, the last result.

2. A matrix theorem in abelian topological groups

Let G be an abelian topological group. If $\{x_j\}$ is a sequence in G, we say that the series $\sum_{j=1}^{\infty} x_j$ is *subseries convergent* in G if for each subsequence $\{x_{j_k}\}$ of $\{x_j\}$, the series $\sum_{k=1}^{\infty} x_{j_k}$ converges in G. If Δ is an infinite subset of $\mathbb{N}$ and $\sum_{j=1}^{\infty} x_j$ is subseries convergent in G, we write $\sum_{j\in\Delta} x_j$ for the sum of the series $\sum_{k=1}^{\infty} x_{j_k}$, where the elements of Δ have been arranged in a subsequence $j_1 < j_2 < \cdots$. If Δ is a finite subset of $\mathbb{N}$, the meaning of $\sum_{j\in\Delta} x_j$ is clear.

A series $\sum_{j=1}^{\infty} x_{ij}$ in G is said to *converge uniformly* for $i \in \mathbb{N}$ if $\lim_{n\to\infty} \sum_{j=n}^{\infty} x_{ij} = 0$ uniformly for $i \in \mathbb{N}$, that is, for every neighborhood U of $0 \in G$ there is an $n_0 \in \mathbb{N}$ such that $\sum_{j=n}^{\infty} x_{ij} \in U$ whenever $n > n_0$ and $i \in \mathbb{N}$.

We begin with the following lemma.

LEMMA 1 (Antosik-Mikusinski Theorem ([1],[9])). *Let G be an abelian topological group and $x_{ij} \in G$ for $i, j \in \mathbb{N}$. Suppose*

(I) $\lim_{i\to\infty} x_{ij} = x_j$ *exists for each j and*

(II) *for each increasing sequence $\{m_j\}$ of positive integers there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that $\{\sum_{j=1}^{\infty} x_{in_j}\}_{i=1}^{\infty}$ is a Cauchy sequence.*

Then $\lim_{i\to\infty} x_{ij} = x_j$ uniformly for $j \in \mathbb{N}$. In particular,

$$\lim_{i\to\infty}\lim_{j\to\infty} x_{ij} = \lim_{j\to\infty}\lim_{i\to\infty} x_{ij} = 0 \quad \textit{and} \quad \lim_{i\to\infty} x_{ii} = 0.$$

By using the above lemma, we can obtain the following matrix theorem which is a generalization of the Antosik-Swartz theorem ([1], Theorem 8.1).

THEOREM 2. *Let G be an abelian topological group and $x_{\alpha j} \in G$ for all $j \in \mathbb{N}$ and $\alpha \in I$, where I is an index set. Suppose that for each $\alpha \in I$ the series $\sum_{j=1}^{\infty} x_{\alpha j}$ is subseries convergent and for each sequence $\{\alpha_n\}$ in I there exists a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that $\lim_{i\to\infty} \sum_{j\in\Delta} x_{\alpha_{n_i} j} = x_\Delta$ exists for each non-empty subset Δ of $\mathbb{N}$. Then*

$\lim_{\min \Delta \to \infty} \sum_{j \in \Delta} x_{\alpha j} = 0$ *uniformly for* $\alpha \in I$ *and, in particular, for every non-empty subset* Δ *of* $\mathbb{N}$ *the series* $\sum_{j \in \Delta} x_{\alpha j}$ *converges uniformly for* $\alpha \in I$.

Proof. Suppose that $\lim_{\min \Delta \to \infty} \sum_{j \in \Delta} x_{\alpha j} = 0$ is not uniform with respect to $\alpha \in I$. Then there is a neighborhood U of $0 \in G$ such that:

(*) for any $n_0 \in \mathbb{N}$ there exist a subset Δ_0 of $\mathbb{N}$ and an $\alpha \in I$ such that $\min \Delta_0 > n_0$ but $\sum_{j \in \Delta_0} x_{\alpha j} \notin U$.

Pick a neighborhood V of 0 for which $V + V \subseteq U$. By (*), there exist a subset Δ_1 of $\mathbb{N}$ and an $\alpha_1 \in I$ such that $\min \Delta_1 > 1$ but $\sum_{j \in \Delta_1} x_{\alpha_1 j} \notin U$. But, if m is a sufficiently large integer, then $\sum_{j \in \Delta_1, j > m} x_{\alpha_1 j} \in V$ and, therefore, $\sum_{j \in \Delta_1, j \leq m} x_{\alpha_1 j} \notin V$, i.e., there exist a finite subset Δ_1 of $\mathbb{N}$ and an $\alpha_1 \in I$ such that $\min \Delta_1 > 1$ but $\sum_{j \in \Delta_1} x_{\alpha_1 j} \notin V$. Similarly, there exist a finite subset Δ_2 of $\mathbb{N}$ and an $\alpha_2 \in I$ satisfying $\min \Delta_2 > \max \Delta_1$ and $\sum_{j \in \Delta_2} x_{\alpha_2 j} \notin V$. Continuing this construction we have a sequence $\{\Delta_n\}$ of finite subsets of $\mathbb{N}$ and a sequence $\{\alpha_n\}$ in I such that $\min \Delta_{n+1} > \max \Delta_n$ and

$$(**) \qquad \sum_{j \in \Delta_n} x_{\alpha_n j} \notin V \quad \text{for all } n \in \mathbb{N}.$$

There exists a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ such that $\lim_{i \to \infty} \sum_{j \in \Delta} x_{\alpha_{n_i} j} = x_\Delta$ exists for each non-empty subset Δ of $\mathbb{N}$. Now consider the matrix

$$\left[\sum_{j \in \Delta_{n_k}} x_{\alpha_{n_i} j} \right]_{i,k}.$$

For each k, $\lim_{i \to \infty} \sum_{j \in \Delta_{n_k}} x_{\alpha_{n_i} j} = x_{\Delta_{n_k}}$ exists. Let $\{k_l\}$ be an increasing sequence of positive integers. Then letting $\Delta = \bigcup_{l=1}^{\infty} \Delta_{n_{k_l}}$, we have

$$\sum_{l=1}^{\infty} \sum_{j\in\Delta_{n_{k_l}}} x_{\alpha_{n_i} j} = \sum_{j\in\Delta} x_{\alpha_{n_i} j}$$

for each i, and $\lim_{i\to\infty} \sum_{l=1}^{\infty} \sum_{j\in\Delta_{n_{k_l}}} x_{\alpha_{n_i} j} = \lim_{i\to\infty} \sum_{j\in\Delta} x_{\alpha_{n_i} j} = x_\Delta$ exists.

Thus, by Lemma 1, $\lim_{i\to\infty} \sum_{j\in\Delta_{n_i}} x_{\alpha_{n_i} j} = 0$ and, therefore, $\sum_{j\in\Delta_{n_i}} x_{\alpha_{n_i} j} \in V$ for sufficiently large i. This contradicts $(**)$ and the proof is complete. □

As an immediate consequence of Theorem 2, we have the following useful result.

COROLLARY 3. *Let G be an abelian topological group and $x_{ij} \in G$ for all $i, j \in \mathbb{N}$. Suppose that for each $i \in \mathbb{N}$ the series $\sum_{j=1}^{\infty} x_{ij}$ is subseries convergent and for each non-empty subset Δ of $\mathbb{N}$, $\lim_{i\to\infty} \sum_{j\in\Delta} x_{ij} = x_\Delta$ exists. Then $\lim_{\min \Delta\to\infty} \sum_{j\in\Delta} x_{ij} = 0$ uniformly for $i \in \mathbb{N}$ and, in particular, for every non-empty subset Δ of $\mathbb{N}$ the series $\sum_{j\in\Delta} x_{ij}$ converges uniformly for $i \in \mathbb{N}$.*

There is a famous version of the classical Schur lemma asserts that a sequence in l^1 converges weakly if and only if it converges strongly, i.e., converges in norm. This result and some of its more general forms have found many applications in functional analysis; for example, many of the proofs of the Orlicz-Pettis theorem, including the original proof of Pettis, use the Schur lemma in some form. Similarly, Phillips lemma has many applications in both measure theory and functional analysis.

Both of Schur lemma and Phillips lemma have been generalized to various abstract settings ([1],[8],[11],[15]). The most general and powerful result was obtained by Li Ronglu and C. Swartz recently ([8]) but this is a very abstract result and, in view of applications, the Antosik-Swartz theorem ([1], Theorem 8.1) is an excellent generalization of both Schur lemma and Phillips lemma. Fortunately, we would like to show that the Antosik-Swartz theorem is also a special case of our Theorem 2.

COROLLARY 4 ([1], Theorem 8.1). *Let G be an abelian topological group and $x_{ij} \in G$ for all $i, j \in \mathbb{N}$. Assume that the rows of the matrix $(x_{ij})_{i,j}$ are subseries convergent and $\lim_{i\to\infty} x_{ij} = x_j$ exists for each $j \in \mathbb{N}$. If $\lim_{i\to\infty} \sum_{j\in\Delta} x_{ij} = x_\Delta$ exists for each non-empty subset Δ of $\mathbb{N}$, then the following hold.*

(1) *The series $\sum_{j=1}^{\infty} x_j$ is subseries convergent.*

(2) *$\lim_{i\to\infty} \sum_{j\in\Delta} x_{ij} = \sum_{j\in\Delta} x_j$ uniformly for each non-empty subset Δ of $\mathbb{N}$.*

Proof. (1) By Corollary 3, the series $\sum_{j=1}^{\infty} x_{ij}$ converges uniformly for $i \in \mathbb{N}$. Therefore,

$$\sum_{j=1}^{\infty} x_j = \lim_{n\to\infty} \lim_{i\to\infty} \sum_{j=1}^{n} x_{ij} = \lim_{i\to\infty} \lim_{n\to\infty} \sum_{j=1}^{n} x_{ij} = \lim_{i\to\infty} \sum_{j\in\mathbb{N}} x_{ij} = x_{\mathbb{N}}.$$

The same argument is valid for every subseries of the series $\sum_{j=1}^{\infty} x_j$, i.e., the series $\sum_{j=1}^{\infty} x_j$ is subseries convergent.

(2) Let U be a neighborhood of 0 in G. Take any symmetric and closed neighborhood V of $0 \in G$ for which $V + V + V \subseteq U$. Since $\lim_{\min \Delta\to\infty} \sum_{j\in\Delta} x_{ij} = 0$ uniformly for $i \in \mathbb{N}$ by Corollary 3, there exists an $n_0 \in \mathbb{N}$ such that

$$\sum_{j\in\Delta, j>n_0} x_{ij} \in V$$

for all $\Delta \subseteq \mathbb{N}$ and $i \in \mathbb{N}$, and so

$$\sum_{j\in\Delta, j>n_0} x_j \in V$$

for all $\Delta \subseteq \mathbb{N}$. Choose again a neighborhood W of $0 \in G$ for which $\underbrace{W + W + \cdots + W}_{n_0 \text{ times}} \subseteq V$. Since $\lim_{i\to\infty} x_{ij} = x_j$ exists for each j, there is an $i_0 \in \mathbb{N}$ such that

$$i \geq i_0 \Rightarrow x_{ij} - x_j \in W, \ \ 1 \leq j \leq n_0.$$

Thus, if $i \geq i_0$, then

$$\sum_{j\in\Delta, j\leq n_0} x_{ij} - \sum_{j\in\Delta, j\leq n_0} x_j \in \underbrace{W + W + \cdots + W}_{n_0 \text{ times}} \subseteq V$$

for all $\Delta \subseteq \mathbb{N}$ so that

$$\begin{aligned}
&\sum_{j\in\Delta} x_{ij} - \sum_{j\in\Delta} x_j \\
&= \left\{ \sum_{j\in\Delta, j\leq n_0} x_{ij} - \sum_{j\in\Delta, j\leq n_0} x_j \right\} + \left\{ \sum_{j\in\Delta, j>n_0} x_{ij} - \sum_{j\in\Delta, j>n_0} x_j \right\} \\
&\in V + (V + V) \subseteq U
\end{aligned}$$

for all $\Delta \subseteq \mathbb{N}$. □

We would like to emphasize that our Theorem 2 gave a powerful result for a general matrix $(x_{\alpha j})_{\alpha\in I, j\in\mathbb{N}}$, where I is an index set which contains $\mathbb{N}$ as a special case, i.e., the usual matrix $(x_{ij})_{i,j\in\mathbb{N}}$ is a special case of our general matrix $(x_{\alpha j})_{\alpha\in I, j\in\mathbb{N}}$. Note that our general result Theorem 2 contains some important results which can not be obtained by the similar results about usual matrix $(x_{ij})_{i,j\in\mathbb{N}}$. To see this, recall the following Thomas theorem ([13], [14]):

Let Ω be a compact space and $\{f_j\}$ is a sequence of continuous scalar functions on Ω. If the series $\sum_{j=1}^{\infty} f_j$ is subseries convergent in the topology of pointwise convergence on Ω, then $\sum_{j=1}^{\infty} f_j$ is also subseries convergent in the topology of uniform convergence on Ω.

It is well known that a sequentially compact space need not be compact and of course, a compact space need not be sequentially compact. Our Theorem 2 just contains a Thomas type result as a special case. A function f from a topological space Ω into a topological space G is said to be *sequentially continuous* if $\{\omega_n\}$ is a sequence in Ω such that $\omega_n \longrightarrow \omega \in \Omega$, then $f(\omega_n) \longrightarrow f(\omega)$ in G. Clearly, a continuous function must be sequentially continuous but, in general, a sequentially continuous function need not be continuous, e.g., if a locally convex space X is not Mazur, then there exist sequentially continuous linear functionals on X which are not continuous. Fortunately, our Thomas type theorem just give a result for the family of sequentially continuous functions, though the Thomas theorem is a result only about continuous functions.

THEOREM 5. *Let Ω be a sequentially compact space and $SC(\Omega, G)$ the family of sequentially continuous functions from Ω into an abelian topological group G. If a series $\sum_{j=1}^{\infty} f_j$ on $SC(\Omega, G)$ is subseries convergent in the topology of pointwise convergence on Ω, then $\sum_{j=1}^{\infty} f_j$ is also subseries convergent in the topology of uniform convergence on Ω.*

Proof. Denote that $x_{\omega j} = f_j(\omega)$ for $\omega \in \Omega$ and $j \in \mathbb{N}$. Since the series $\sum_{j=1}^{\infty} f_j$ is subseries convergent in the topology of pointwise convergence on Ω, if Δ is a non-empty subset of $\mathbb{N}$, then there exists an $f_\Delta \in SC(\Omega, G)$ such that $\sum_{j \in \Delta} f_j(\omega) = f_\Delta(\omega)$ for all $\omega \in \Omega$, i.e., $\sum_{j \in \Delta} x_{\omega j} = f_\Delta(\omega)$ for all $\omega \in \Omega$. Let $\{\omega_n\}$ be a sequence in Ω. Then there exist a subsequence $\{\omega_{n_i}\}$ of $\{\omega_n\}$ and an $\omega \in \Omega$ such that $\lim_{i \to \infty} \omega_{n_i} = \omega$, since Ω is sequentially compact. Therefore, if Δ is a non-empty subset of $\mathbb{N}$, then

$$\lim_{i \to \infty} \sum_{j \in \Delta} x_{\omega_{n_i} j} = \lim_{i \to \infty} \sum_{j \in \Delta} f_j(\omega_{n_i}) = \lim_{i \to \infty} f_\Delta(\omega_{n_i}) = f_\Delta(\omega),$$

since $f_\Delta \in SC(\Omega, G)$ and $\omega_{n_i} \longrightarrow \omega$ in Ω. This shows that the generalized matrix $(x_{\omega j})_{\omega \in \Omega, j \in \mathbb{N}}$ satisfies all of hypothesis of Theorem 2 and the desired follows from Theorem 2 immediately. □

3. Improved Stiles' subseries theorem

Let X be a topological vector space and $\{f_k\}$ a sequence of linear functionals on X. A series $\sum_{j=1}^{\infty} x_j$ in X is said to be *subseries* $w\{f_k\}$-*convergent* if for every increasing sequence $\{j_m\}$ in $\mathbb{N}$ there exists an $x \in X$ such that $\sum_{m=1}^{\infty} f_k(x_{j_m}) = f_k(x)$ for all $k \in \mathbb{N}$.

A *basis* for a topological vector space X is a sequence $\{e_k\}$ in X such that each $x \in X$ has a unique representation $x = \sum_{k=1}^{\infty} t_k e_k$. A basis $\{e_k\}$ of X is called a *Schauder basis* if the coordinate functionals of $\{e_k\}$ are continuous ([15]). If X is a Fréchet space, i.e., complete metric linear space, then every basis for X is a Schauder basis; but there exists a normed space having a basis which is not a Schauder basis ([6]).

We now present an improvement of the main result in [7].

THEOREM 6. *Let X be a topological vector space with a basis $\{e_k\}$ and $\{f_k\}$ the sequence of coordinate functionals on X which are determined by $\{e_k\}$. If a series $\sum_{j=1}^{\infty} x_j$ in X is subseries $w\{f_k\}$-convergent, then $\sum_{j=1}^{\infty} x_j$ is also subseries convergent in the original topology of X.*

Proof. For each $x \in X$, $x = \sum_{k=1}^{\infty} f_k(x)e_k$. Letting $z_{ij} = \sum_{k=1}^{i} f_k(x_j)e_k$, consider the matrix $(z_{ij})_{i,j}$. For each j, $\lim_{i\to\infty} z_{ij} = \sum_{k=1}^{\infty} f_k(x_j)e_k = x_j$ and, hence, if Δ is a finite subset of $\mathbb{N}$, then $\lim_{i\to\infty}\sum_{j\in\Delta} z_{ij} = \sum_{j\in\Delta} x_j$. Suppose Δ is an infinite subset of $\mathbb{N}$. Since the series $\sum_{j=1}^{\infty} x_j$ is subseries $w\{f_k\}$-convergent, there exists an $x_\Delta \in X$ such that $\sum_{j\in\Delta} f_k(x_j) = f_k(x_\Delta)$ for all k. But $\sum_{k=1}^{\infty} f_k(x_\Delta)e_k = x_\Delta$, so

$$\lim_{i\to\infty}\sum_{j\in\Delta} z_{ij} = \lim_{i\to\infty}\sum_{j\in\Delta}\left[\sum_{k=1}^{i} f_k(x_j)e_k\right] = \lim_{i\to\infty}\sum_{k=1}^{i} f_k(x_\Delta)e_k = x_\Delta.$$

By Theorem 2, the series $\sum_{j=1}^{\infty} z_{ij} = \sum_{j=1}^{\infty}\left[\sum_{k=1}^{i} f_k(x_j)e_k\right]$ converges uniformly for $i \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty}\sum_{j=1}^{n} x_j = \lim_{n\to\infty}\sum_{j=1}^{n}\left[\sum_{k=1}^{\infty} f_k(x_j)e_k\right] = \lim_{n\to\infty}\lim_{i\to\infty}\sum_{j=1}^{n}\left[\sum_{k=1}^{i} f_k(x_j)e_k\right]$$

$$= \lim_{i\to\infty}\lim_{n\to\infty}\sum_{k=1}^{i}\left[\sum_{j=1}^{n} f_k(x_j)e_k\right] = \lim_{i\to\infty}\sum_{k=1}^{i}\left[\sum_{j=1}^{\infty} f_k(x_j)e_k\right]$$

$$= \lim_{i\to\infty}\sum_{k=1}^{i} f_k(x_{\mathbb{N}})e_k = \sum_{k=1}^{\infty} f_k(x_{\mathbb{N}})e_k = x_{\mathbb{N}},$$

i.e., $\sum_{j=1}^{\infty} x_j = x_{\mathbb{N}}$. The same argument is valid for any subseries $\sum_{k=1}^{\infty} x_{j_k}$ of the series $\sum_{j=1}^{\infty} x_j$. In fact, if the series $\sum_{j=1}^{\infty} x_j$ is subseries $w\{f_k\}$-convergent, then the subseries $\sum_{k=1}^{\infty} x_{j_k}$ is also subseries $w\{f_k\}$- convergent so $\sum_{k=1}^{\infty} x_{j_k}$ converges in the original topology of X. □

REMARK. As stated in section 1, Stiles' result was an expressive departure from earlier results. Especially, our improvement of Stiles' result in this paper shows that basis sequences are powerful even if the concerned spaces are neither metric nor locally convex.

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