J. Korean Math. Soc. **36** (1999), No. 2, pp. 243–256

COINCIDENCE OF MAPS BETWEEN SURFACES

DACIBERG L. GONÇALVES

ABSTRACT. We will consider $f, g : S_1 \to S_2$ a pair of maps between two orientable compact surfaces. The purpose of this paper is to decide when the pair can be deformed to a pair (f', g') such that $\#\text{coin}(f', g') = N(f, g)$, the Nielsen coincidence number of (f, g). We derive an equivalent algebraic condition and show that if we compose (f, g) with certain maps $h : S \to S_1$ then the answer is positive. Finally, we analyze the case of roots, i.e., g is the constant map. When S_2 is the torus we give a new proof of the converse of the Lefschetz theorem for coincidence.

0. Introduction

The purpose of this work is to study the Wecken problem for coincidence. Namely, given $f, g : S_1 \to S_2$, two continuous maps, when does it exist f' homotopic to f, g' homotopic to g such that $\#\text{coin}(f', g') = N(f, g)$ (where $\#$ stands for cardinality and $coin(f, g) = \{x \in S_1 | f(x) = g(x)\}$). This question, in the case of fixed points, is known to have a positive answer for some pairs (id, f) and negative for others. See, for example, [11] and [12]. Let us say that (f, g) has the Wecken property if the question above has a positive answer. In section 1, we derive algebraic conditions, in terms of the Braid groups, for a pair (f, g) to have the Wecken property. This is the Fundamental Lemma 1.2. We also prove Theorem 1.3 which says: *Given any pair of maps $(f, g) : S_l \to S_h$ between two surfaces, there is an integer n such that $(f \circ p_{l,n}, g \circ p_{l,n})$ has the Wecken property. Also $N(f \circ p_{l,n}, g \circ p_{l,n}) = N(f, g)$, for any*

Received April 8, 1997. Revised February 24, 1998.

1991 Mathematics Subject Classification: Primary 55M20; Secondary 55M25, 57N75.

Key words and phrases: Nielsen coincidence number, Braid group, Wecken property, compact surface.

This work has been supported by the international cooperation program GMD/Germany-CNPq/Brasil and Universidade de São Paulo during the author visit at the Mathematisches Institut der Universität Heidelberg.

integer n. Here $p_{l,n}$ looks like a projection. In section 2 we specialize to the case where the second space is the torus T. We reduce the problem to the case where one of the maps is the constant map. Then we write the algebraic equations which are equivalent to the geometric problem. These are Propositions 2.1 and 2.3, respectively. Finally we give a proof of the converse to the Lefschetz theorem for coincidence of maps on the torus. This is Theorem 2.8 which says: *Given* $f, g : S \to T$ *such that* $N(f, g) = 0$ *then* (f, g) *can be deformed to be a pair* (f', g') *which is coincidence free.* This theorem relies on the fact that a map $h : S \to T$, which has degree zero, can be deformed to a map which is not surjective. This result is known, but we give a new proof, that we expect to have its own interest.

We would like to thank Professor E. Fadell for many helpful conversations, for his interesting questions about the subject and for his encouragement. Finally, we would like to thank Professor R. F. Brown for pointing out the overlapping of this work with others and his suggestions, which improved the exposition of this work.

1. Algebraic version of the Wecken property and further results

We will start with a Fundamental Lemma, which is a natural generalization of Theorem 1.1 of [13] for coincidence. We would like to thank the referee for point out the reference [10]. The Fundamental Lemma here is similar to Lemma 1.2 together with Lemma 2.1 in [10].

Let S_l, S_h be the orientable compact surfaces of genus l, h respectively, and $e_1, ..., e_{2l}$ a fixed canonical basis of $\pi_1(S_l, x_0)$. Suppose we are given two maps $f, g : S_l \to S_h$, where we have $f_\#(e_i) = w_i$, $g_\#(e_i) = v_i$, $i = 1, 2, \ldots, 2l$, and w_i, v_i belong to $\pi_1(S_h, y_1), \pi_1(S_h, y_2)$, respectively, and $f(x_0) = y_1 \neq y_2 = g(x_0)$. Following [6] section 4, we have two subgroups of $\pi_1(S_h \times S_h - \Delta, (y_1, y_2))$ generated by $(\rho_{1,1}, \rho_{2,1}, \ldots, \rho_{2h-1,1}, \rho_{2h,1})$, $(\rho_{1,2}, \rho_{2,2}, \ldots, \rho_{2h-1,2}, \rho_{2h,2})$, which we denote by F_1, F_2 respectively. Let W_i, V_i be elements of F_1, F_2 which project over w_i, v_i, respectively, under the map $\pi_1(S_h \times S_h - \Delta, (y_1, y_2)) \to \pi_1(S_h \times S_h, (y_1, y_2))$ induced by the inclusion. Let $N(f, g) = r$ and $k_1, \ldots, k_r$ be the indices of the essential Nielsen classes. Also, let $\{\bar{1}, \bar{\alpha}_2, \ldots, \bar{\alpha}_r\}$ be a set of representatives of the Reidemeister classes of (f, g), which corresponds to the essential Nielsen classes. Let the class defined by α_i have index k_i and the class defined by 1 have index k_1. (For the definition of the local coincidence index, see e.g. [20]). We choose a base point in the class defined by 1.

DEFINITION 1.1. A pair (f, g) has the Wecken property if we can find f' homotopic to f, g' homotopic to g such that $\#\mathrm{coin}(f', g') = N(f, g)$.

Now let $F = \pi_1(S_h - y_1, y_2)$, $N = \mathrm{Ker}(\Phi : F \to \pi_1(S_h, y_2))$, $B = \prod_{i=1}^{h}[\rho_{2i-1,2}, \rho_{2i,2}^{-1}]$, $\Phi(\alpha) = \bar{\alpha}$ and $B_\alpha = \alpha B \alpha^{-1}$ as in [6].

FUNDAMENTAL LEMMA 1.2. *The pair (f, g) satisfies the Wecken property if and only if*

(a) *If $N(f, g) = 0$ then we can find a solution $\theta_i \in N$, $i = 1, 2, \dots, 2l$ of the equation*

$$\prod_{i=1}^{l}[\theta_{2i-1}W_{2i-1}V_{2i-1}, \theta_{2i}W_{2i}V_{2i}] = 1 .$$

(b) *If $N(f, g) = r \neq 0$ and $k_1, \dots, k_r$ are the indices of the Nielsen classes, then we can find a solution $\theta_i \in N$, $i = 1, \dots, 2l$ of the equation*

$$\prod_{i=1}^{l}[\theta_{2i-1}W_{2i-1}V_{2i-1}, \theta_{2i}W_{2i}V_{2i}] = B^{k_1}B_{\alpha_2}^{k_2}\cdots B_{\alpha_r}^{k_r}$$

for some set $\{\bar{1}, \bar{\alpha}_2, \dots, \bar{\alpha}_r\}$ of representatives of the correspondents essential Nielsen classes.

Proof. The case (a) can be proved in the same way as case (b) and it is simpler. So we will show only case (b). Suppose (f, g) satisfies the Wecken property. Let $f' \sim f$, $g' \sim g$, where $\sim$ means homotopic, such that $\#\mathrm{coin}(f', g') = N(f, g)$. Denote by $x_1, \dots, x_r$ the points of $\mathrm{coin}(f', g')$. Around each point we draw a small circle and connect them by a path. See figure I.

Using the notation of figure I the small circle around the point x_j is $\lambda_j * \beta_j^{-1}$. The base point is x_0 which belongs to the circle around the first point. Two consecutive circles are connected by a path γ_j. Call D_j the closed disk which boundary the circle around x_j and $\mathring{D}_j$ its interior. Since (f', g') have no coincidence in $S_l - \bigcup_{j=1}^{r} \mathring{D}_j$, this means that (f', g') is in fact a map $(f', g') : S_l - \bigcup_{j=1}^{r} \mathring{D}_j \to S_h \times S_h - \Delta$. But the loop

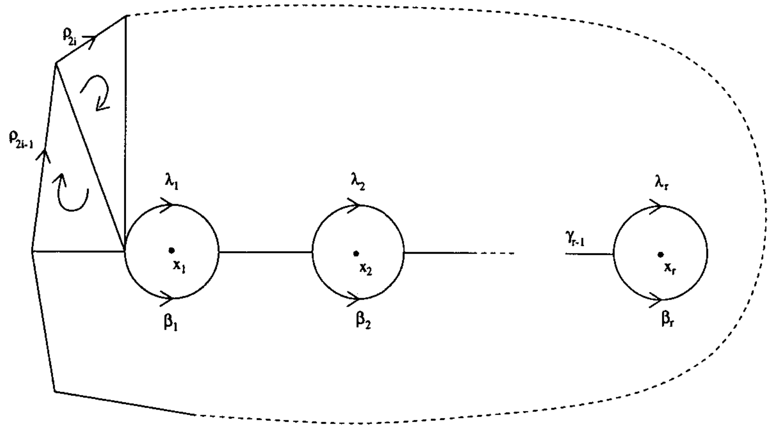

Figure I.

$\xi = \prod_{i=1}^{l}[\rho_{2i-1}, \rho_{2i}] \prod_{i=1}^{r} \xi_i$ is trivial in $S_l - \bigcup_{j=1}^{r} \mathring{D}_j$ where ξ_i is the class of the loop

$$\lambda_1 * \gamma_1 * \cdots * \lambda_{i-1} * \gamma_{i-1} * \beta_i * \lambda_i^{-1} * \gamma_{i-1}^{-1} * \lambda_{i-1}^{-1} * \cdots * \gamma_1^{-1} * \lambda_1^{-1} .$$

So $(f \times g)_{\#}(\xi)$ is 1 in $\pi_1(S_h \times S_h - \Delta)$. Let φ_i be the path $\lambda_1 * \gamma_1 * \cdots * \lambda_{i-1} * \gamma_{i-1}$ which goes from the base point x_0 to the base point of the circle around x_i. Since the coincidence index of x_i is k_i we have that $(f'(\varphi_i * \lambda_i * \beta_i^{-1} * \varphi_i^{-1}), g'(\varphi_i * \lambda_i * \beta_i^{-1} * \varphi_i^{-1}))$, as a braid, represents the class of $\alpha_i B^{k_i} \alpha_i^{-1} = B_{\alpha_i}^{k_i}$, where $\bar{\alpha}_i \approx g(\widetilde{\varphi}_i) f(\widetilde{\varphi}_i)^{-1}$ and $\widetilde{\varphi}_i$ is just φ_i followed by the radius from the end of φ_i to x_i. Also $(f'(\rho_i), g'(\rho_i))$ as a braid is $\theta_i W_i V_i$ for some $\theta_i \in N$. Since $(f \times g)_{\#}(\xi) = 1$, the equation given in part (b) follows.

Now suppose that we have a solution for the equation given in (b). Let us define two functions $f, g : S_l \to S_h$.

Using the surface and the notation in figure I, define $f(x) = y_0$ for $x \in D_1 \cup \gamma_1 \cup D_2 \cup \cdots \cup \gamma_{r-1} \cup D_r$, where y_0 is a chosen point of S_h. Define g on the boundary of D_i such that the image is a small circle around y_0 and g has degree k_i. Extend g to D_i radially and define g on the path γ_i in such way that $g(\widetilde{\varphi})$ represents the element α_i given by the equation. Finally on the edges γ, ρ_i define f and g such that $(f(e_i)$, $g(e_i))$ represent the words w_i, v_i respectively. There is no problem to define f, g in these edges with no coincidences. But the given equation

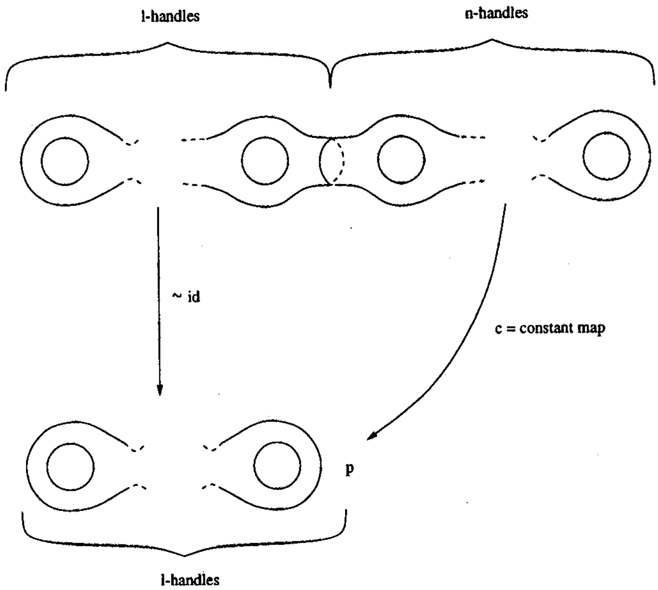

Figure II.

is precisely the algebraic condition to extend the map $\varphi = (f, g)$ over the 2-skeleton. See theorem 4.3.1 of [1].

If $\widetilde{\varphi}$ is the extension and $p_i : S_h \times S_h - \Delta \to S_h$ is the projection on the i-th coordinate, then the maps $f_1 = p_1 \circ \widetilde{\varphi}$, $g_1 = p_2 \circ \widetilde{\varphi}$ are certainly homotopic to f, g, respectively, and we get the result. □

This type of equation is basically the same one which appears in [13], Theorem 1.1, where the fixed point case is studied.

Let S_l be the orientable surface of genus l. Define $p_{l,n} : S_{l+n} \to S_l$ as the pinching map which takes the last n-handles to a point p, while the complement is mapped almost like the identity (see figure II).

Now we will state the main result of this section. For its proof we need Proposition 1.4.

THEOREM 1.3. *Given any pair of maps $(f, g) : S_l \to S_h$ among two surfaces, there is an integer n such that $(f \circ p_{l,n}, g \circ p_{l,n})$ has the Wecken property. Also $N(f \circ p_{l,n}, g \circ p_{l,n}) = N(f, g)$ for any integer n.*

Now let us consider the abelianized obstruction, (see [6]), to lift a map to the 2-skeleton. If we consider the diagram

$$\begin{array}{ccc} & & S_h \times S_h - \Delta \\ & & \downarrow \\ (f,g) : S_l & \longrightarrow & S_h \times S_h \end{array}$$

by [6] the abelianized obstruction to lift (f, g) from the 1-skeleton to the 2-skeleton is an element of $H^2(S_l, Z[\pi])$, where $\pi = \pi_1(S_h)$. This cohomology class can be represented by a 2-cocycle of the form $\sum n_i[\alpha_i]$ where n_i is the index of a Nielsen class and $[\alpha_i]$ is the corresponding Reidemeister class.

PROPOSITION 1.4. *The equations given in parts* (a) *and* (b) *of the Fundamental* Lemma 1.2, *when looked in* N_{ab}, *the abelianized of* N, *admits a solution.*

Proof. By classical obstruction theory, if a 2-cocycle represents an obstruction class which is a two dimensional cohomology class, then we can deform the function over the 1-skeleton such that the cocycle defined from this new function is precisely the given one. This new function together with the original one provide us with $\theta_1, \ldots, \theta_{2l}$ which is a solution for the equations in N_{ab}. □

REMARK. This can be done directly by working with the isomorphism $N_{ab} \approx Z[\pi]$.

Proof of Theorem 1.3. By proposition 1.4 if we are given a set of representatives $\{\bar{1}, \bar{\alpha}_2, \ldots, \bar{\alpha}_r\}$ of the Reidemeister classes which correspond to the essential Nielsen classes, we can find $\theta_1, \ldots, \theta_{2l}$ such that

$$\begin{aligned} W &= B_{\alpha_k}^{-n_k} \cdots B_{\alpha_2}^{-n_2} B^{-n_1} [\theta_1 W_1 V_1, \theta_2 W_2 V_2] \cdots [\theta_{2l-1} W_{2l-1} V_{2l-1}, \theta_{2l} W_{2l} V_{2l}] \\ &\in [N, N] \end{aligned}$$

or this element is zero in N_{ab}. So we have $\theta_{2l+1}, \ldots, \theta_{2l+2n} \in N$ such that

$$W = \prod_{j=1}^{n} [\theta_{2l+2n-2j+2}, \theta_{2l+2n-2j+1}] \qquad \text{or}$$

$$\prod_{i=1}^{l}[\theta_{2i-1}W_{2i-1}V_{2i-1}, \theta_{2i}W_{2i}V_{2i}] \prod_{j=l+1}^{l+n}[\theta_{2j-1}, \theta_{2j}] = B^{n_1}B_{\alpha_2}^{n_2}\cdots B_{\alpha_k}^{n_k}.$$

Now we have

$$f \circ p_{l,n}(e_i) = \begin{cases} f_\#(e_i), & 1 \le i \le 2l \\ 1, & 2l+1 \le i \le 2(l+n) \end{cases}$$

and

$$g \circ p_{l,n}(e_i) = \begin{cases} g_\#(e_i), & 1 \le i \le 2l \\ 1, & 2l+1 \le i \le 2(l+n) \end{cases}$$

So by the Fundamental Lemma 1.2, equation (1) above implies that $(f \circ p_{l,n}, g \circ p_{l,n})$ has the Wecken property.

Finally, we can assume that the point p which appears in the definition of the function $p_{l,n}$, does not belong to $\text{coin}(f,g)$. So $\text{coin}(f \circ p_{l,n}, g \circ p_{l,n}) \xrightarrow{p_{l,n}} \text{coin}(f,g)$ is an homeomorphism, which induces a map among the Nielsen classes. This induced map is certainly surjective and is also injective because $p_{l,n\#}(\pi_1(S_{l+n}) \to \pi_1(S_l))$ is surjective. So we have a bijection among the Nielsen classes which certainly preserves indexes, and the result follows. □

REMARK. The above result suggests the natural question of finding the minimum integer n for which $(f \circ p_{n,l}, g \circ p_{n,l})$ has the Wecken property. In case one of the maps is the identity, the answer is known in a reasonable number of cases. See e.g. [11].

To finish this section, we prove a proposition which tells us that, in order to minimize coin, it suffices to change one function, at least when the target space is a manifold. This geometric fact is certainly useful.

Let $f, g : M \to N$ be two continuous maps where M is a topological space, and N is a manifold. Let $\mu(f,g) = \min_{f' \in [5], g' \in [g]} \#\text{coin}(f', g')$ and $\mu_1(f,g) = \min_{g' \in [g]} \#\text{coin}(f, g')$.

PROPOSITION 1.5. *For any pair* $(f,g) : M \to N$ *we have* $\mu(f,g) = \mu_1(f,g)$.

Proof. Consider the fibered pair $(N \times N, N \times N - \Delta) \xrightarrow{p_1} N$ (see [5]), where p_1 is the projection on the first coordinate. Certainly we have $\mu(f,g) \leq \mu_1(f,g)$. So it suffices to show that $\mu_1(f,g) \leq \mu(f,g)$. For this let (f',g') such that $\#\mathrm{coin}(f',g') = k$. We have that $\overline{f}, \overline{f}' : M \times M \to N$ are homotopic where $\overline{f}(x,y) = f(x)$ and $\overline{f}'(x,y) = f'(x)$. Call H such homotopy. The map f' has a lift, namely (f',g') such that $(f',g')(M - \mathrm{coin}(f',g')) \subset N \times N - \Delta$. By the lifting property of fibered pairs it follows that there is a lift $\widetilde{H}$ of H such that $\widetilde{H}(M - \mathrm{coin}(f',g')) \subset N \times N - \Delta$. So $\widetilde{H}(\ ,1) = (f,g'')$ and $\mathrm{coin}(f,g'') \subset \mathrm{coin}(f',g')$. Therefore $\#\mathrm{coin}(f,g'') \leq \#\mathrm{coin}(f',g')$ and the result follows. □

Although the above result is known in a more general form, we decided to include it with this proof since it is simpler than the one in [2].

2. Root case and coincidence of maps into the torus

Let S be an orientable surface, T the torus and $f,g : S \to T$ two maps. Now, if we identify T with $S^1 \times S^1$, then T has a group structure given by the complex multiplication in each coordinate. So we consider the maps $h(x) = g(x)/f(x)$ and $c(x) = e$, where $e \in T$ is the identity element with respect to the above multiplication.

PROPOSITION 2.1. *We have* $\mathrm{coin}(f,g) = \mathrm{coin}(h,c)$ *and* $N(f,g) = N(h,c)$. *Further* (f,g) *has the Wecken property if and only if we can find a map* h_1 *homotopic to* h *such that* $\#h_1^{-1}(1) = N(f,g)$.

Proof. It is clear that $\#\mathrm{coin}(f,g) = \#\mathrm{coin}(h,c)$. Also, the Nielsen classes of (f,g) and (h,c) are the same. For let $x,y \in \mathrm{coin}(f,g)$, $\lambda : I \to S$, $\lambda(0) = x$, $\lambda(1) = y$ with $f(\lambda) \sim g(\lambda)$ relative to the end points and let H be a homotopy. Then $H_1(s,t) = H(s,t)/f(\lambda(s))$ gives a homotopy relative to the end points between $h(\lambda)$ and $c(\lambda)$. To see that a class has the same index with respect to both pair of maps, let us consider the map $\psi : (T \times T, T \times T - \Delta) \to (T, T - \{1\})$ given by $\psi(x,y) = y/x$. This is a well defined map of the pairs and $\psi^* : H^n(T, T - \{1\}) \to H^n(T \times T, T \times T - \Delta)$ takes the fundamental cohomology class to the Thom class. But $\psi \circ (f,g) = \psi \circ (h,c)$. So a Nielsen class has the same index either with respect to (f,g) or (h,c).

Finally, by Proposition 1.5, in order to minimize coincidence, it suffices to deform one of the maps, let us say h, so the result follows.

From now on let $h : S \to T$ be a map and let $h_\# : H_1(S) \to H_1(T)$ be given by $h_\# = \begin{pmatrix} a_1 & b_1 & \cdots & a_g & b_g \\ c_1 & d_1 & \cdots & c_g & d_g \end{pmatrix}$, where S is a compact surface of genus g. Let $\Lambda(h, c)$ be the Lefschetz coincidence number and $\deg(h)$ the degree of h.

PROPOSITION 2.2. *We have* $\Lambda(h, c) = \deg(h)$. *If* $\deg(h) = 0$ *then* $N(h, c) = 0$. *If* $\deg(h) \neq 0$ *then* $N(h, c) = \#\mathrm{coker}(h_\#)$ *and each Nielsen class has index equal to* $\deg(h)$ *divided by* $N(h, c)$.

Proof. The above result is true in general, i.e., whenever S and T are orientable manifolds of the same dimension. This follows from Proposition 5 of [17] and Corollary 7.3 of [14], or [18]. □

Now we will derive the algebraic condition for (h, c) to have the Wecken property. When $\deg(h)$ is not zero, then we certainly have that $h_\#(\pi_1(S)) \lhd \pi_1(T)$ has finite index. Denote by $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}$ a set of representatives of the elements of the Reidemeister classes $\pi_1(T)/h_\#(\pi_1(S))$. Let us pick a base point x_0 for S and y_0 for T, where y_0 is close to $1 \in T$. Denote by $j : T - \{1\} \hookrightarrow T$ the inclusion, by $F(x, y) = \pi_1(T - \{1\})$ the free group on two generators and $B_\alpha = \alpha B \alpha^{-1}$, $w_{2i-1} = x^{a_i} y^{b_i}$, $w_{2i} = x^{c_i} y^{d_i}$. As before, let $B = [x, y]$, $N = [F, F]$ and $h_\# = \begin{pmatrix} a_1 & b_1 & \cdots & a_g & b_g \\ c_1 & d_1 & \cdots & c_g & d_g \end{pmatrix}$.

PROPOSITION 2.3. *Let* $h : S \to T$. *Then there exists* $h_1 \sim h$ *such that* $\# h_1^{-1}(1) = N(h, c)$ *iff*

(a) *If* $\deg(h) = 0$ *then the equation*

$$\prod_{i=1}^{g} [\theta_{2i-1} w_{2i-1}, \theta_{2i} w_{2i}] = 1$$

has a solution $\theta_i \in N = [F, F]$, $i = 1, \ldots 2g$.

(b) *If* $\deg(h) = m \neq 0$, *let* $r = \#\mathrm{coker}(h_\#)$ *and* $k = m/r$. *Then the equation*

$$\prod_{i=1}^{g} [\theta_{2i-1} w_{2i-1}, \theta_{2i} w_{2i}] = B^k . B^k_{\alpha_2} \cdots B^k_{\alpha_r}$$

has a solution for some $\theta_j \in N$ *and for some set of representatives* $\{\bar{1}, \bar{\alpha}_2, \ldots, \bar{\alpha}_r\}$ *of the quotient* $\frac{\pi_1(T)}{h_\#(\pi_1(S))}$.

Proof. The proof follows from the Fundamental Lemma 1.2. It is enough to notice that since the second map is the constant map, the equation given in part b) of the Fundamental Lemma 1.2 is in fact an equation in the subgroup F_1, where F_1 is defined in Definition 1.1. □

In the special case where $N(f,g) = 0$, we will show, in a geometric way, that (f,g) satisfies the Wecken property.

LEMMA 2.4. *In order to show that a pair with $N(h,c) = 0$, has the Wecken property, it suffices to consider the case where $h_{\#}(\pi_1(S)) = \pi_1(T)$.*

Proof. If the $\mathrm{rank}(h_{\#}(\pi_1(S))) < 2$ then we can deform h to h_1 such that $h_1(S)$ lies inside a curve which does not contain $1 \in T$. If $h_{\#}(\pi_1(S))$ has rank two, take $\widetilde{T} \xrightarrow{p} T$ the finite cover which corresponds to the subgroup $h_{\#}(\pi_1(S))$ and let $\widetilde{h} : S \to \widetilde{T}$ be the lifting of h. So $\widetilde{h} : S \to \widetilde{T}$ has the property that $\widetilde{h}_{\#} : \pi_1(S) \to \pi_1(\widetilde{T})$ is surjective. If $\widetilde{h}$ can be deformed to a map which is not surjective, then it can be deformed to the one skeleton of $\widetilde{T}$ which we assume that does not intersect $p^{-1}(1)$, where $p : \widetilde{T} \to T$ is the cover map, and the proof is done. □

Let $\gamma \subset S_g$ be an embedded curve. We let $h_\gamma : S_g \to S^1$ be a map defined as follows: take a tubular neighbourhood of γ and let $\varphi : \gamma \times [-\varepsilon, \varepsilon] \to S_g$ be an homeomorphism onto that tubular neighborhood of γ. Now we define

$$h_\gamma(x) = \begin{cases} e^{\frac{\pi t i}{\varepsilon}} & \text{if } x = \varphi(x_1, t) \\ -1 & \text{otherwise.} \end{cases}$$

Roughly speaking, the homology class represented by γ is the Poincaré dual of the cohomology class $h_\gamma^{\#}(w_1) \in H^1(S_g)$ with proper orientations, where w_1 is a chosen generator of $H^1(S^1)$. See [8], Part II and [19] for more details.

Let $h : S_g \to T$ and

$$h_{\#} = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 & \cdots & a_g & b_g \\ c_1 & d_1 & c_2 & d_2 & \cdots & c_g & d_g \end{pmatrix}.$$

We will assume from now on that $\deg h = \Lambda(h,c) = 0$ and $h_{\#}(\pi_1(S)) = \pi_1(T)$.

PROPOSITION 2.5. *The elements* $(a_1, b_1, \dots, a_g, b_g)$ *and* $(c_1, d_1, \dots, c_g, d_g)$ *in* $\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2g}$ *are indivisible.*

Proof. This follows from the fact that $h_\#(\pi_1(S)) = \pi_1(T)$. □

Let γ_1 resp. γ_2 be two connected closed simple curves which represent the homology classes $(a_1, b_1, \dots, a_g, b_g)$ resp. $(c_1, d_1, \dots, c_g, d_g)$. Such curves do exist, see e.g. [8] or [21] section 3.6. We can assume that these curves intersect transversally.

PROPOSITION 2.6. *The intersection number of these curves is zero.*

Proof. See [3], chapter VIII, 13 or [21] section 3.6. □

Now we will modify one of the curves, γ_1, for example, so that the new curve γ_1' has the following properties: a) γ_1' is a simple curve, not necessarily connected; b) γ_1' represents the same homology class as γ_1; c) $\gamma_1' \cap \gamma_2 = \emptyset$.

PROPOSITION 2.7. *Given* γ_1 *and* γ_2 *we can construct* γ_1' *as above.*

Proof. The argument used to construct γ_1' can be found in [8], appendix. The figure below gives the idea of the construction. Namely, for each two consecutive points with intersection number $+1$ and -1 we get a new curve as shown below.

Since the number of points in the intersection is finite the process ends after a finite number of steps and we get the curve γ_1' with the required properties. □

THEOREM 2.8. *Given* $f, g : S \to T$ *such that* $N(f, g) = 0$ *then* (f, g) *can be deformed to a pair* (f', g') *which is coincidence free.*

Proof. It suffices to consider the map $h(x) = g(x)/f(x)$ and to show that h can be deformed to h' such that $1 \notin h'(S)$. We can also assume, by Lemma 2.4, that $h_\#(\pi_1(S)) = \pi_1(T)$. Let γ_1' and γ_2 be the two curves given by proposition 2.7. So we have two maps $h_{\gamma_1'}, h_{\gamma_2} : S \to S^1$. Therefore we have $h_1 = (h_{\gamma_1'}, h_{\gamma_2}) : S \to T$. Certainly

$$h_1^{-1}(1) = h_{\gamma_1'}^{-1}(1) \cap h_{\gamma_2}^{-1}(1) = \gamma_1' \cap \gamma_2 = \emptyset$$

So $h_1^{-1}(1) = \emptyset$ and h_1 is certainly homotopic to h because they induce the same homomorphism on π_1 or H_1. □

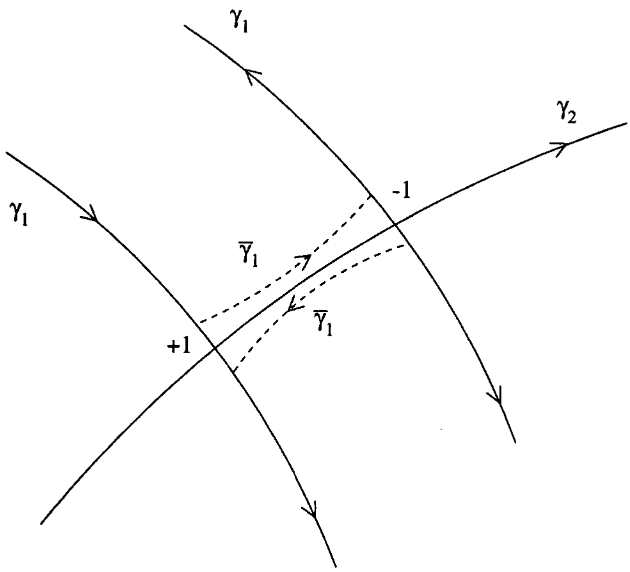

Figure III.

REMARK 1. Given a map $h : S_1 \to S_2$ between any two compact orientable surfaces, it is known that if $\deg(h) = 0$, then h can be deformed to a map which is not surjective. This is a consequence of deep results of Kneser, as it was pointed out in [4]. For this purpose you can also see [15], [16] and [21] section 3.3. This, in particular, shows, by means of the Fundamental Lemma 2.1, that certain quadratic equations on free group have solutions. It is not clear how to provide an explicit solution or even how to show algebraically, that such equations have a solution.

REMARK 2. Theorem 2.8 is certainly not new. In fact, it is weaker than the results pointed out in Remark 1. Nevertheless, we believe that the proof we presented here may have its own interest.

REMARK 3. In general, one can not expect that (h, c) has the Wecken property. The first example of a map $h : S_2 \to T$, S_2 being the surface of genus 2, where (h, c) does not satisfy the Wecken condition, was given in [9]. This naturally brings up the question of trying to classify the maps

h such that (h, c) satisfies the Wecken condition. A joint work with H. Zieschang is in preparation and discuss this question.

REMARK 4. We believe that the connection between geometry and algebra given in the Fundamental Lemma, looks worthwhile exploring.

References

[1] Baues, H. J., *Obstruction theory*, Lectures Notes in Mathematics, vol. 628, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[2] Brook, R. B. S., *On removing coincidences of two maps when only one, rather than both of them, may be deformed by a homotopy*, Pacific Journal of Mathematics **40** (1972) no. 1, 45–52.

[3] Dold, A., *Lectures on algebraic topology*, Die Grundlehren der Mathematischen Wissenschaften, vol. 200, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[4] Epstein, D. B. A., *The degree of a map*, Proceedings of the London Mathematical Society **16** (1966) no. 3, 369–383.

[5] Fadell, E., *Generalized normal bundles for locally-flat imbeddings*, Trans. Am. Math. Soc. **114** (1965), 488–513.

[6] Fadell, E. and Husseini, S., *The Nielsen number on surfaces*, Contemporary Mathematics **21** (1982), Topological methods in non linear functional analysis 59–99.

[7] ———, *Fixed point theory for non simply connected manifolds*, Topology **20** (1981), 53–93.

[8] Gonçalves, D. L., *Fixed points of S^1-fibrations*, Pacific Journal of Mathematics **129** (1987) no. 12, 297–306.

[9] Hopf, H., *Zur Topologie der Abbildungen von Mannigfalltigkeiten II*, Math. Annalen **102** (1930), 562-623.

[10] Jezierski, J., *The least number of coincidence points on surfaces*, J. Austral. Math. Soc. (Series A) **58** (1995), 27–38.

[11] Jiang, B., *The fixed point classes from the differential view point*, Lectures Notes in Math. **886** (1981), Fixed point theory, 163–170.

[12] ———, *Fixed points and braids II*, Math. Ann. **272** (1985), 249–256.

[13] ———, *Surface maps and braid equations, I*, Lecture Notes in Math. **1369**, 125–141.

[14] Kiang, T. H., *The theory of fixed point classes*, Springer-Verlag, 1989.

[15] Kneser, H., *Glättung von Flächenabbildungen*, Math. Ann. **100** (1928), 609–617.

[16] ———, *Die Kleinste Bedeckungszahlinnerhalb einer Klasse von Flächenabbildungen*, Math. Ann. **103** (1930), 347–358.

[17] Oliveira, E., *Teoria de Nielsen para coincidência e algumas aplicações*, Ph. D. Thesis., 1986.

[18] Thom, R., *Quelques propriétés globales des variétés differentiables*, Comm. Math. Helv. **28** (1954), 17–86.

[19] Vick, J., *Homology theory*, Academic Press, New York, 1976.

[20] Zieschang, H., Vogt, E., Coldewey, H. D., *Surfaces and planes discontinuous groups*, Lectures Notes in Mathematics, vol. 835, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

Departamento de Matemática - IME-USP
Caixa Postal 66.281 - Ag. Cidade de São Paulo
05315-970 - São Paulo - SP - Brasil
E-mail: dlgoncal@ime.usp.br