Bull. Korean Math. Soc. **36** (1999), No. 4, pp. 629–636

APPROXIMATION IN LIPSCHITZ ALGEBRAS OF INFINITELY DIFFERENTIABLE FUNCTIONS

T. G. HONARY AND H. MAHYAR

ABSTRACT. We introduce Lipschitz algebras of differentiable functions on a perfect compact plane set X and then extend the definition to Lipschitz algebras of infinitely differentiable functions on X. Then we define the subalgebras generated by polynomials, rational functions, and analytic functions in some neighbourhood of X, and determine the maximal ideal spaces of some of these algebras. We investigate the polynomial and rational approximation problems on certain compact sets X.

Let X be a perfect compact plane set, and let $0 < \alpha \leq 1$. The algebra of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in X, x \neq y\right\} < \infty,$$

is denoted by $Lip(X, \alpha)$ and the subalgebra of those functions f for which $|f(x) - f(y)|/|x - y|^\alpha \to 0$ as $|x - y| \to 0$, by $\ell ip(X, \alpha)$. These Lipschitz algebras were first studied by Sherbert [6]. The algebras $Lip(X, \alpha)$ for $\alpha \leq 1$ and $\ell ip(X, \alpha)$ for $\alpha < 1$ are Banach function algebras on X under the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup_{x\in X} |f(x)|$. It is interesting to note that $Lip(X, 1) \subseteq \ell ip(X, \alpha)$. In fact, $Lip(X, 1)$ is dense in $\ell ip(X, \alpha)$ [1].

A complex-valued function f on X is called differentiable on X if at each point $z_0 \in X$,

$$f'(z_0) = \lim\left\{\frac{f(z) - f(z_0)}{z - z_0} : z \in X, z \to z_0\right\}$$

Received May 22, 1997.

1991 Mathematics Subject Classification: 46J10, secondary 46J15, 46J20, 41A10, 41A20.

Key words and phrases: Banach function algebras, Lipschitz algebras of differentiable functions, maximal ideal space, polynomial and rational approximation.

exists.

DEFINITION 1. The algebra of functions f on X whose derivatives up to order n exist and for each k $(0 \leq k \leq n)$, $f^{(k)} \in Lip(X,\alpha)$ is denoted by $Lip^n(X,\alpha)$. The algebra $\ell ip^n(X,\alpha)$ is defined in a similar way. For f in $Lip^n(X,\alpha)$ or in $\ell ip^n(X,\alpha)$, let

$$\|f\| = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\alpha}{k!} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X + p_\alpha(f^{(k)})}{k!}.$$

The algebra of functions f with derivatives of all orders for which $f^{(k)} \in Lip(X,\alpha)$ $(f^{(k)} \in \ell ip(X,\alpha))$ for all k is denoted by $Lip^\infty(X,\alpha)$ $(\ell ip^\infty (X,\alpha))$.

We also introduce certain subalgebras of $Lip^\infty(X,\alpha)$ and $\ell ip^\infty(X,\alpha)$. Let $M = \{M_k\}_{k=0}^\infty$ be a sequence of positive numbers such that

$$M_0 = 1 \quad \text{and} \quad \frac{M_k}{M_r.M_{k-r}} \geq \binom{k}{r} \qquad r = 0,1,\dots,k.$$

Whenever we refer to $M = \{M_k\}$ we mean this sequence satisfies the above conditions.

DEFINITION 2. Let

$$Lip(X,M,\alpha) = \left\{ f \in Lip^\infty(X,\alpha) : \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_\alpha}{M_k} < \infty \right\},$$

$$\ell ip(X,M,\alpha) = \left\{ f \in \ell ip^\infty(X,\alpha) : \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_\alpha}{M_k} < \infty \right\},$$

and for f in $Lip(X,M,\alpha)$ or in $\ell ip(X,M,\alpha)$ let

$$\|f\| = \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_\alpha}{M_k}.$$

For convenience, we regard $Lip^n(X,\alpha)$ and $\ell ip^n(X,\alpha)$ as being algebras of the type $Lip(X,M,\alpha)$ and $\ell ip(X,M,\alpha)$, respectively, by setting $M_k = k!$ $(k = 0,1,\dots,n)$ and $1/M_k = 0$ $(k = n+1,\dots)$.

These algebras have similar properties to $D^n(X)$, the algebra of functions on X with continuous n^{th} derivatives, and $D(X,M)$, the algebra of infinitely differentiable functions f on X such that $\|f\| = \sum_{k=0}^\infty \|f^{(k)}\|_X/M_k$

$< \infty$, which were introduced by Dales and Davie [2]. It is also known that $D^1(X) \subseteq R(X)$, where $R(X)$ is the uniform closure of $R_0(X)$, the algebra of all rational functions with poles off X [2].

Now we introduce the type of compact sets which we shall consider.

DEFINITION 3. Let X be a compact plane set which is connected by rectifiable arcs, and suppose $\delta(z, w)$ is the geodesic metric on X, the infimum of the lengths of the arcs joining z and w.

(i) X is called regular if for each $z_0 \in X$ there exists a constant C such that for all $z \in X$, $\delta(z, z_0) \leq C|z - z_0|$.

(ii) X is called uniformly regular if there exists a constant C such that for all $z, w \in X$, $\delta(z, w) \leq C|z - w|$.

If X is a finite union of regular sets then for each $z_0 \in X$ there exists a constant C such that for every $z \in X$ and any $f \in D^1(X)$,

$$|f(z) - f(z_0)| \leq C|z - z_0|(\|f\|_X + \|f'\|_X).$$

This inequality implies that $D^1(X)$ is complete under the norm $\|f\|_1 = \|f\|_X + \|f'\|_X$ [2]. It is also interesting to note that the above condition is, in fact, a necessary and sufficient condition for the completeness of $D^1(X)$. To see this, let $D^1(X)$ be complete and define another norm on $D^1(X)$ by

$$\|f\| = \|f\|_X + \|f'\|_X + \sup_{\substack{z \in X \\ z \neq z_0}} \frac{|f(z) - f(z_0)|}{|z - z_0|} \qquad (f \in D^1(X)),$$

where z_0 is a fixed point in X. Then $D^1(X)$ is also a Banach function algebra on X under this new norm. Thus there exists a constant C such that for all $f \in D^1(X)$ and for every $z \in X$

$$|f(z) - f(z_0)| \leq C|z - z_0|(\|f\|_X + \|f'\|_X).$$

The completeness of $D^1(X)$ implies that $Lip(X, M, \alpha)$ and $\ell ip(X, M, \alpha)$ are Banach function algebras on X. From now on we assume that X is a perfect compact plane set such that $D^1(X)$ is complete, unless otherwise specified.

Now we introduce subalgebras of $Lip(X, M, \alpha)$ and $\ell ip(X, M, \alpha)$.

DEFINITION 4. The closed subalgebra of $Lip(X, M, \alpha)$ ($\ell ip(X, M, \alpha)$) which is generated by the polynomials, by the rational functions with poles off X that belong to $Lip(X, M, \alpha)$ $(\ell ip(X, M, \alpha))$, or by those functions of $Lip(X, M, \alpha)$ $(\ell ip(X, M, \alpha))$ which are analytic in some neighbourhood of X, is denoted by $Lip_P(X, M, \alpha)$ $(\ell ip_P(X, M, \alpha))$, $Lip_R(X, M, \alpha)$ $(\ell ip_R(X, M, \alpha))$ or $Lip_H(X, M, \alpha)$ $(\ell ip_H(X, M, \alpha))$, respectively.

Clearly $Lip_P(X, M, \alpha)$ is uniformly dense in $P(X)$, the uniform closure of polynomials. Moreover, when $\sqrt[k]{M_k/k!} \to \infty$ as $k \to \infty$, we have $R_0(X) \subseteq Lip(X, M, 1)$, so $Lip(X, M, 1)$ and hence $\ell ip(X, M, \alpha)$ and $Lip(X, M, \alpha)$ are uniformly dense in $R(X)$. In particular, $Lip_R(X, M, \alpha)$ is uniformly dense in $R(X)$. For convenience, we set $P_k = \sqrt[k]{M_k/k!}$. Note that, when $P_k \to \infty$ as $k \to \infty$, $Lip_R(X, M, \alpha)$ is generated by the all rational functions with poles off X, and hence it is a natural Banach function algebra on X. So by the Theorem in [4] we have $M_{Lip_P(X,M,\alpha)} \cong M_{P(X)} \cong \hat{X}$, where M_A is the maximal ideal space of the algebra A. Thus when $P_k \to \infty$ as $k \to \infty$, $Lip_P(X, M, \alpha) = Lip_R(X, M, \alpha)$ if and only if $\hat{X} = X$. Also when $P_k \to \infty$ as $k \to \infty$, by the Functional Calculus Theorem [3; 3.4.5], $Lip_R(X, M, \alpha)$ contains all analytic functions in a neighbourhood of X, and so $Lip_R(X, M, \alpha) = Lip_H(X, M, \alpha)$.

THEOREM 1. *For each $n \geq 0$, $\ell ip^n(X, \alpha)$ and $Lip^n(X, \alpha)$ are natural Banach function algebras on X.*

Proof. The algebras $\ell ip(X, \alpha)$ and $Lip(X, \alpha)$ are uniformly dense in $C(X)$ [6], and for $n \geq 1$, $\ell ip^n(X, \alpha)$ and $Lip^n(X, \alpha)$ are uniformly dense in $R(X)$. By the naturality of $C(X)$ and $R(X)$, and the Theorem in [4], it is sufficient to prove that for each $f \in Lip^n(X, \alpha)$ we have $\|\hat{f}\| \leq \|f\|_X$, where $\hat{f}$ is the Gelfand transform of f.

But straightforward calculations show that:

$$\|(f^m)^{(k)}\|_X \leq 2^{n^2} \delta^n m^n \|f\|_X^{m-n}, \qquad (0 \leq k \leq n)$$

$$p_\alpha[(f^m)^{(k)}] \leq 2^{n^2} m^{n+1} \lambda \delta^{n-1} \|f\|_X^{m-n}, \qquad (0 \leq k \leq n)$$

for all $m > n$, where δ and λ are constants independent of m and k. Hence

$$\|f^m\|^{1/m} \leq \|f\|_X^{1-n/m} [(n+1) 2^{n^2} \delta^{n-1} m^n]^{1/m} (\delta + m\lambda)^{1/m}.$$

Therefore $\|\hat{f}\| = lim_{m \to \infty} \|f^m\|^{1/m} \leq \|f\|_X$. This completes the proof of the theorem. □

We recall that $\ell ip_P(X,\alpha)$ ($\ell ip_R(X,\alpha)$) is the closed subalgebra of $\ell ip(X,\alpha)$ which is generated by polynomials (rational functions with poles off X) and $\ell ip_A(X,\alpha)$ is defined by $\ell ip_A(X,\alpha) = A(X) \cap \ell ip(X,\alpha)$, where $A(X)$ is the uniform algebra of functions which are continuous on X and analytic in the interior of X. It is interesting to note that $\ell ip_A(X,\alpha)$ is a natural Banach function algebra on X [5].

THEOREM 2. *If X is uniformly regular and $\ell ip_P(X,\alpha) = \ell ip_A(X,\alpha)$ then $\ell ip_P^n(X,\alpha) = \ell ip^n(X,\alpha)$ for all $n \geq 1$ and $\alpha < 1$.*

Proof. As we know if $f \in D^1(X)$ then $p_\alpha(f) \leq Cd^{1-\alpha}\|f'\|_X$, where $d = diam(X)$. Now let $n \geq 1$ and $f \in \ell ip^n(X,\alpha)$. Since $f^{(n)} \in \ell ip_A(X,\alpha) = \ell ip_P(X,\alpha)$, for every $\epsilon > 0$ there exists a polynomial P_0 such that

$$\|f^{(n)} - P_0\|_{\ell ip(X,\alpha)} = \|f^{(n)} - P_0\|_X + p_\alpha(f^{(n)} - P_0) < \epsilon.$$

Let z_0 be a fixed point in X and P_1 be the antiderivative of P_0 with the initial condition $P_1(z_0) = f^{(n-1)}(z_0)$. Since $f^{(n-1)} - P_1 \in \ell ip^1(X,\alpha) \subseteq D^1(X)$ we have

$$p_\alpha(f^{(n-1)} - P_1) \leq Cd^{1-\alpha}\|f^{(n)} - P_0\|_X < Cd^{1-\alpha}\epsilon.$$

Continuing in this manner, we obtain polynomials $P_2, P_3, \ldots, P_n$ such that $P_k' = P_{k-1}$, $P_k(z_0) = f^{(n-k)}(z_0)$, $\|f^{(n-k)} - P_k\|_X < C^k d^k \epsilon$, and $p_\alpha(f^{(n-k)} - P_k) \leq C^k d^{k-\alpha}\epsilon$, for $k = 1, 2, \ldots, n$. Clearly $P_n^{(k)} = P_{n-k}$ on X and

$$\|f - P_n\|_{\ell ip^n(X,\alpha)} \leq \sum_{k=0}^{n-1} \frac{C^{n-k}d^{n-k}\epsilon + C^{n-k}d^{n-k-\alpha}\epsilon}{k!} + \frac{\epsilon}{n!} = \lambda\epsilon,$$

for some constant λ. Hence $f \in \ell ip_P^n(X,\alpha)$. □

Now we investigate rational approximation on circles and annuli. We note that when X is uniformly regular and $\alpha < 1$, then $\ell ip(X,M,\alpha) = Lip(X,M,\alpha)$ if $1/M_k \neq 0$ for infinitely many k.

THEOREM 3. *If $T = \{z \in \mathbb{C} : |z - z_0| = R\}$ then $\ell ip_R(T,M,\alpha) = \ell ip(T,M,\alpha)$.*

Proof. We assume that $z_0 = 0$ and $R = 1$. Let $f \in \ell ip(T,M,\alpha)$ and $\sum_{-\infty}^{\infty} a_j z^j$ be the Fourier series generated by f, where $a_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})$

$e^{-ij\theta}d\theta$. The Cesaro means of this series are

$$\sigma_n(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})K_n(\theta - t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i(\theta-t)})K_n(t)dt,$$

where $K_n(t)$ is the Fejer kernel. It is known that σ_n is a rational function on T with the only pole $z = 0$, and $\|\sigma_n - f\|_T \to 0$ as $n \to \infty$. Since for each $k \geq 0$, $f^{(k)}$ is continuous on T we have

$$\sigma_n^{(k)}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt} f^{(k)}(ze^{-it})K_n(t)dt \qquad (z \in T),$$

and so $\|\sigma_n^{(k)}\|_T \leq \|f^{(k)}\|_T$.

On the other hand,

$$\frac{|\sigma_n^{(k)}(z) - \sigma_n^{(k)}(w)|}{|z-w|^\alpha} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|f^{(k)}(ze^{-it}) - f^{(k)}(we^{-it})|}{|ze^{-it} - we^{-it}|^\alpha} K_n(t)dt \leq p_\alpha(f^{(k)}).$$

Hence $p_\alpha(\sigma_n^{(k)}) \leq p_\alpha(f^{(k)})$ and so $\sigma_n \in \ell ip_R(T, M, \alpha)$.

Now we prove that $\|\sigma_n - f\|_{\ell ip(T,M,\alpha)} \to 0$ as $n \to \infty$. Since

$$\|\sigma_n - f\|_{\ell ip(T,M,\alpha)} \leq 2\|f\|_{\ell ip(T,M,\alpha)},$$

by the dominated convergence theorem, it is enough to show that for each $k \geq 0$, $\|\sigma_n^{(k)} - f^{(k)}\|_T + p_\alpha(\sigma_n^{(k)} - f^{(k)}) \to 0$ as $n \to \infty$.

By the uniform continuity of each $f^{(k)}$ on T we have $\|\sigma_n^{(k)} - f^{(k)}\|_T \to 0$ as $n \to \infty$.

Since $f^{(k)} \in \ell ip(T, \alpha)$, for $\epsilon > 0$ there exists $\delta > 0$ such that for all $z, w \in T$, if $0 < |z - w| < \delta$ then $|f^{(k)}(z) - f^{(k)}(w)|/|z-w|^\alpha < \epsilon/2$. Let $k \geq 0$ and $z, w \in T$, $(z \neq w)$. If $|z - w| < \delta$, then

$$\frac{|\sigma_n^{(k)}(z) - f^{(k)}(z) - \sigma_n^{(k)}(w) + f^{(k)}(w)|}{|z-w|^\alpha} < \epsilon.$$

If $|z - w| \geq \delta$ and n is large enough, then

$$\frac{|\sigma_n^{(k)}(z) - f^{(k)}(z) - \sigma_n^{(k)}(w) + f^{(k)}(w)|}{|z-w|^\alpha} \leq \frac{2\|\sigma_n^{(k)} - f^{(k)}\|_T}{\delta^\alpha} < \epsilon.$$

Hence $p_\alpha(\sigma_n^{(k)} - f^{(k)}) \to 0$ as $n \to \infty$. This completes the proof of the theorem. $\square$

REMARK. Note that the following results are not satisfied when $\ell ip(X, M, \alpha)$ reduces to $\ell ip(X, \alpha)$. But in these cases we have $\ell ip_R(X, \alpha) = \ell ip_A(X, \alpha)$.

THEOREM 4. *If $X = \{z \in \mathbb{C} : r \leq |z - z_0| \leq R\}$, where $0 < r < R$, then $\ell ip_R(X, M, \alpha) = \ell ip(X, M, \alpha)$.*

Proof. Without loss of generality we can assume that $z_0 = 0$. Let $f \in \ell ip(X, M, \alpha)$. Since f is analytic in $r < |z| < R$ it has a Laurent series of the form $f(z) = \sum_{-\infty}^{\infty} a_j z^j$ on $r < |z| < R$, where $a_j = (2\pi\rho^j)^{-1} \int_{-\pi}^{\pi} e^{-ijt} f(\rho e^{it}) dt$, for $r < \rho < R$. The Cesaro means of the Laurent series of f is

$$\sigma_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(ze^{-it}) K_n(t) dt \qquad (r < |z| < R),$$

where $K_n(t)$ is the Fejer kernel. Clearly for each $k \geq 0$ we have

$$\sigma_n^{(k)}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} f^{(k)}(ze^{-it}) K_n(t) dt \qquad (r < |z| < R),$$

and so $|\sigma_n^{(k)}(z)| \leq \|f^{(k)}\|_X$ for $r < |z| < R$. Since σ_n is a rational function with the only pole $z = 0$, $\sigma_n^{(k)}$ is analytic in $r \leq |z| \leq R$. Therefore the above inequality holds for all z in $r \leq |z| \leq R$. Hence $\|\sigma_n^{(k)}\|_X \leq \|f^{(k)}\|_X$ and $p_\alpha(\sigma_n^{(k)}) \leq p_\alpha(f^{(k)})$ for all $k \geq 0$ and for every positive integer n, and so $\sigma_n \in \ell ip_R(X, M, \alpha)$.

Now we can proceed exactly the same as in the proof of theorem 3 to show that $\|\sigma_n - f\|_{\ell ip(X,M,\alpha)} \to 0$ as $n \to \infty$. Therefore $f \in \ell ip_R(X, M, \alpha)$ and this completes the proof of the theorem. □

If $r \to 0$, the above theorem implies the following result.

COROLLARY 1. *If $X = \{z : |z| \leq R\}$ then $\ell ip_P(X, M, \alpha) = \ell ip(X, M, \alpha)$.*

THEOREM 5. *Let X be a regular set for which there exists $z_0 \in X$ such that for $0 \leq \beta < 1$, $\beta(z - z_0) + z_0 \in intX$ for all $z \in X$. Or, equivalently, the segment $[z_0, z)$ is contained in the interior of X for all $z \in X$. If $P_k \to \infty$ as $k \to \infty$, then $\ell ip_P(X, M, \alpha) = \ell ip(X, M, \alpha)$.*

Proof. Clearly X is star-shaped and so it is polynomially convex. Thus

$$\ell ip_P(X, M, \alpha) = \ell ip_R(X, M, \alpha) = \ell ip_H(X, M, \alpha).$$

Without loss of generality we can assume that $z_0 = 0$. By the hypothesis for each positive integer n and every $z \in X$, $r_n z \in intX$, where $r_n = n/(n+1)$. Let $f \in \ell ip(X, M, \alpha)$ and define the sequence $\{f_n\}$ on X by $f_n(z) = f(r_n z)$. Each f_n is analytic in a neighbourhood of X and so $f_n \in \ell ip_H(X, M, \alpha)$. Moreover for each $k \geq 0$, $f_n^{(k)}(z) = r_n^k f^{(k)}(r_n z)$ and so $\|f_n^{(k)}\|_X \leq \|f^{(k)}\|_X$, $p_\alpha(f_n^{(k)}) \leq p_\alpha(f^{(k)})$ for all $k \geq 0$ and every n. By the uniform continuity of each $f^{(k)}$ on X, $lim_{n\to\infty}\|f_n^{(k)} - f^{(k)}\|_X = 0$. Since $f^{(k)} \in \ell ip(X, \alpha)$ for each $k \geq 0$, $p_\alpha(f_n^{(k)} - f^{(k)}) \to 0$ as $n \to \infty$. Consequently by the dominated convergence theorem $\|f_n - f\|_{\ell ip(X,M,\alpha)} \to 0$ as $n \to \infty$, and so $f \in \ell ip_H(X, M, \alpha)$. □

COROLLARY 2. *If X is a compact convex set with non-empty interior and $P_k \to \infty$ as $k \to \infty$, then $\ell ip_P(X, M, \alpha) = \ell ip(X, M, \alpha)$.*

ACKNOWLEDGEMENTS. This work was carried out when the authors were visiting the Department of Mathematics of the Australian National University. We would like to thank the Department for its hospitality and particularly Dr R. J. Loy for several valuable conversations and comments.

The authors are also grateful to the University for Teacher Education in Tehran for granting us sabbatical leave.

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INSTITUTE OF MATHEMATICS, UNIVERSITY FOR TEACHER EDUCATION, TEHRAN 15614, IRAN

E-mail: Hmahyar@saba.tmu.ac.ir