Bull. Korean Math. Soc. **36** (1999), No. 1, pp. 147–159

# ELLIPTIC BIRKHOFF'S BILLIARDS WITH $C^2$-GENERIC GLOBAL PERTURBATIONS

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ABSTRACT. Tabanov investigated the *global symmetric* perturbation of the integrable billiard mapping in the ellipse [3]. He showed the nonintegrability of the Birkhoff billiard in the perturbed domain by proving that the principal separatrices splitting angle is not zero. In this paper, using the *exact separatrix map* of an one-degree-of-freedom Hamiltonian system with time periodic perturbation, we show the existence the stochastic layer including the uniformly hyperbolic invariant set which implies the nonintegrability near the separatrices of a Birkhoff's billiard in the domain bounded by a $C^2$ convex simple curve constructed by the *generic global* perturbation of the ellipse.

## 1. Introduction

A dynamical system defined by the free motion of a point in some domain $\Omega$ of the plane bounded by an analytic closed convex curve $\partial\Omega$, is called a Birkhoff's billiard [1,2]. The point follows a straight line path with unit velocity inside the domain and reflects from the boundary according to the law "angle of incidence is equal to the angle of reflection".

It is believed that the ellipse is the only example of the boundary of the domain on which the billiard mapping is integrable in the plane. So, a class of Birkhoff's billiards with the $C^2$ convex boundary close to the ellipse may be not integrable. Donnay (unpublished) has considered the *local analytic* perturbations of the ellipse and proved the analogous theorem on the separatrices splitting of the corresponding billiard mapping.

Received April 20, 1998.
1991 Mathematics Subject Classification: 34C35, 58F39.
Key words and phrases: Birkhoff's billiard, twist map, Hamiltonian system, separatrix, separatrix map, uniform hyperbolicity.

Tabanov has proved a theorem on the nonintegrability of the Birkhoff's billiard in a special domain with the *global symmetric* perturbation for the ellipse and established the asymptotic formulas for the separatrices splitting angle for the Birkhoff's billiard [3].

In this paper we investigate the Birkhoff's billiard in a special domain, bounded by a $C^2$ closed convex curve constructed by the perturbation of the ellipse. But our work is devoted to the *generic global* perturbations of the ellipse without the restriction of symmetry to the perturbations.

In this work, applying the twist map theory to the billiard map and using Moser's theorem [4], we will prove that the billiard map is identified to the time-one-map of the one-degree-of-freedom time periodic Hamiltonian flow, particularly, the perturbed elliptic billiard map (the unperturbed elliptic billiard map) is identified to the time-one-map of the time periodic (integrable) Hamiltonian flow with hyperbolic periodic orbit respectively. Furthermore, using the theory of exact separatrix map of the one-degree-of-freedom Hamiltonian system with time-periodic perturbation [5], we will prove a more general theorem on the separatrices splitting of the billiard map than that of Tabanov's.

We now outline this paper. In section 2, we show that the Birkhoff's billiard map is an exact symplectic monotone twist (ESMT) map. In section 3, we introduce Moser's theorem for an ESMT map and show the perturbed elliptic billiard map is identified to the time-one-map of the Hamiltonian flow with time-periodic perturbation. Using the theory of an exact separatrix map, we prove that the separatrices splitting angle is not zero. In section 4, we conclude our results.

## 2. Billiard map and its properties

Let $S = \frac{\mathbb{R}}{\mathbb{Z}}$ be the circle with unit circumference, i.e., $S$ is the interval $[0,1]$ with 1 and 0 identified. Let $T = S\times[0,1]$ be the standard annulus. We will be studying a diffeomorphism of $T$ to itself, so called *billiard map*, however it will be easier to state the results if we have a global coordinate system. So let $A = \mathbb{R}\times[0,1]$. Then $A$ is the universal cover of $T$ with natural projection $\psi : A \to T$ which sends $(x, y)$ and $(x + r, y) \in A$ to the same point in $T$ whenever $r \in \mathbb{Z}$.

Now, we consider an area preserving, orientation preserving and boundary components preserving $C^1$-diffeomorphism $f : A \to A$ satisfying

$$f(x+1,y) = f(x,y) + (1,0) \quad \text{for any} \quad (x,y) \in A \tag{2.1}$$

$$\pi_1(f(0,0)) \in [0,1), \quad \text{where} \quad \pi_1(x,y) = x \tag{2.2}$$

We say $f : A \to A$ is an *exact symplectic* map if $f$ is symplectic with respect to the usual symplectic structure and for any embedding $\gamma : \mathbb{R} \to A$ satisfying $\forall x, \gamma(x+1) = \gamma(x) + (1,0)$ and second argument coordinate function $\pi_2 : (x,y) \to y$, we have

$$\int_0^1 \pi_2(\gamma(s))\frac{d}{ds}(\pi_1(\gamma(s)))\,ds = \int_0^1 \pi_2(f\circ\gamma(s))\frac{d}{ds}(\pi_1(f\circ\gamma(s)))\,ds.$$

In two dimensional case, the condition that $f$ be symplectic is the same as requiring that $f$ be area preserving, i.e., $|Df| \equiv 1$. Furthermore the condition that $f$ be exact symplectic adds to the area preservation a condition saying that the net area between a nontrivial loop on $T$ and its image under $f$ is zero. In 2-d case for an area preserving map $f : A \to A$, this condition is satisfied automatically.

DEFINITION 2.1. A map $f : A \to A$ is called a monotone twist map if there exists $\epsilon > 0$ such that for all $(x,y) \in A$

$$|\frac{\partial}{\partial y}(\pi_1(f(x,y))| > \epsilon.$$

In twist map theory, we say that a twist map $f : A \to A$ has a generating function $G : B \to \mathbb{R}$ if for $B = \{(x,x') \in \mathbb{R}^2 : \{f(x,y) : y \in [0,1]\} \bigcap \{(x',y) : y \in [0,1]\} \neq \phi\}$,

$$y = -\frac{\partial G(x,x')}{\partial x}, y' = \frac{\partial G(x,x')}{\partial x'} \quad \text{implies} \quad (x',y') = f(x,y).$$

In studying dynamics of Birkhoff's billiard, we usually use the *Birkhoff coordinate* $(s,\theta)$. the bounce position is measured by the arc-length

$s$ along the boundary from a given point. The direction of motion is measured by the angle $\theta$ between a tangent to the boundary and the trajectory. It is easy to see that $s'(s,\theta)$ is a monotone increasing function of $\theta$ because of the convexity of the boundary. Thus the Birkhoff's billiard in Birkhoff coordinate has twist. In fact, $s$ is an anglelike coordinate since the map is periodic with period equal to the perimeter of the boundary.

In our work, for convenience we use the *canonical Birkhoff coordinate* $(s, \cos\theta)$. With this coordinate we prove the next lemma:

LEMMA 2.1. *For a given Birkhoff's billiard, the generating function of the billiard map $f$ is the function that gives the length between boundary points at which the point particle bounces successively.*

*Proof.* Let $(X, Y)$ represents rectangular coordinates in the plane of the billiard. Using the canonical Birkhoff coordinates, we have $(s', \cos\theta') = f(s, \cos\theta)$.

Consider

$$G(s,s') = \left\{[X(s) - X(s')]^2 + [Y(s) - Y(s')]^2\right\}^{\frac{1}{2}},$$

where $(X(s), Y(s))$ represents the billiard boundary. Since

$$\begin{aligned} -G_1(s,s') &= -\left(\frac{X(s)-X(s')}{G(s,s')}, \frac{Y(s)-Y(s')}{G(s,s')}\right) \circ (\dot{X(s)}, \dot{Y(s)}) \\ &= \cos\theta, \quad \text{similarly} \\ G_2(s,s') &= \cos\theta', \end{aligned}$$

the function $G(s,s')$ is the generating function of the billiard map $f$.□

PROPOSITION 2.1. *The billiard map $f$ of a Birkhoff's billiard is an exact symplectic monotone twist (ESMT) map.*

*Proof.* Let$(x,y) = (s, \cos\theta)$ then, by lemma (2.1), the billiard map $f$ has a generating function $G$. So we have

$$dy' = G_{12}dx + G_{22}dx' \quad \text{and} \quad dy = -G_{11}dx + -G_{12}dx'.$$

This implies that

$$dy \wedge dx = -G_{12}dx' \wedge dx = G_{12}dx \wedge dx' = dy' \wedge dx'.$$

That is, $f$ is symplectic. Moreover, $f$ is an exact symplectic map for it preserves area. Since an exact symplectic map is monotone twist if and only if it has a generating function; the billiard map $f$ is an exact symplectic monotone twist map. □

PROPOSITION 2.2. *Let $f$ be a Birkhoff's billiard map. Then $f \circ f$ is also an ESMT map on $A$.*

*Proof.* For the billiard map $f : (s_1, \cos\theta_1) \mapsto (s_2, \cos\theta_2)$, let $G$ be the generating function of $f$. With $G$ we can represent the Jacobian of $f$ as follows;

$$Df = \begin{pmatrix} \frac{-G_{11}}{G_{12}} & \frac{-1}{G_{12}} \\ \frac{\triangle}{G_{12}} & \frac{-G_{22}}{G_{12}} \end{pmatrix},$$

$$\text{where} \quad \triangle = {G_{12}}^2 - G_{11}G_{22} \quad \text{and} \quad G_{ij} = \frac{\partial G}{\partial s_i \partial s_j}.$$

Furthermore $G_{12}$, $G_{11}$ and $G_{22}$ are explicitly given by as follows;

$$\begin{aligned} G_{12} &= \frac{\sin\theta_1 \sin\theta_2}{G}, \\ G_{11} &= \frac{\sin^2\theta_1}{G} - \kappa(s_1)\sin\theta_1, \qquad (2.3) \\ G_{22} &= \frac{\sin^2\theta_2}{G} - \kappa(s_2)\sin\theta_2 \end{aligned}$$

where $\kappa(s)$ is the curvature function at $s$ on the boundary.

Since $\frac{\sin\theta_i}{G} < \kappa(s_i)$, for $i = 1, 2$, we have

$$G_{11} \quad \text{and} \quad G_{22} < 0 \quad \text{for} \quad 0 < \theta_i < \pi.$$

The Jacobian of $f \circ f$ is given by

$$\begin{aligned} D(f \circ f) &= Df \circ Df \\ &= \begin{pmatrix} -\frac{\bar{G}_{11}}{\bar{G}_{12}} & -\frac{1}{\bar{G}_{12}} \\ \frac{\bar{\triangle}}{\bar{G}_{12}} & -\frac{\bar{G}_{22}}{\bar{G}_{12}} \end{pmatrix} \begin{pmatrix} -\frac{G_{11}}{G_{12}} & -\frac{1}{G_{12}} \\ \frac{\triangle}{G_{12}} & -\frac{G_{22}}{G_{12}} \end{pmatrix} \\ &= \begin{pmatrix} * & \frac{1}{\bar{G}_{12}G_{12}}(\bar{G}_{11} + G_{22}) \\ * & * \end{pmatrix}, \end{aligned}$$

where $\bar{G_{ij}} = G_{ij}(s_2, s_3)$.

From equation (2.3) and the fact that $\lim_{\theta_i \to 0} \frac{G_{11}}{G_{12}} = -\frac{1}{2}$ for $i = 1, 2$ monotonically, there must exist $\epsilon > 0$ such that

$$|D_2(\pi_1(f \circ f))| = |\frac{1}{\bar{G}_{12} G_{12}}(\bar{G}_{11} + G_{22})| \geq \epsilon > 0 \quad \text{in} \quad A.$$

So, $f \circ f$ is an exact symplectic monotone twist map on $A$. □

## 3. Analysis near the separatrix of the billiard map

In this section, using a theorem of Moser [4], we reduce dynamics near separatrices of a Birkhoff's billiard to dynamics near separatrices in Hamiltonian system. First, we recall that theorem;

THEOREM 3.1. *Given an exact symplectic monotone twist (ESMT) map $f : A \to A$, there exists a Hamiltonian $H : A \times \mathbb{R} \to \mathbb{R}$ which satisfies*

$$\forall (x, y, t); H(x+1, y, t) = H(x, y, t) = H(x, y, t+1) \tag{3.1}$$

$$\forall (x, y, t); \frac{\partial^2}{\partial y^2} H(x, y, t) > 0 \tag{3.2}$$

*such that $f$ is the time one map of the Hamiltonian system given by $H$.*

Since a one-degree-of-freedom Hamiltonian system with time periodic perturbation generally shows chaotic dynamics near separatrices, it is extremely difficult to obtain a precise quantitative description of the properties of dynamics in the stochastic layer near separatrices by simple numerical integrations of equations of motion due to the complexity of orbit structures. Hence, in the study of separatrix motion, we have used the *separatrix map* [6] which is an approximate map of the energy and the phase describing dynamics near separatrices. But in this study, rather than working with an approximate separatrix map, we use a new map more efficient in describing dynamics near separatrices in Hamiltonian system, so called *exact separatrix map* introduced by T. Ahn, G. I. Kim and S. Kim [5].

Now we explain the construction of the exact separatrix map briefly. For details of this construction, we refer readers to [5]. We consider one-degree-of-freedom Hamiltonian systems with time-periodic perturbations given by

$$\begin{aligned} \dot{x} &= JDH_0(X) + JDH_1(x, \phi), \\ \dot{\phi} &= 1, \end{aligned} \qquad (x, \phi) \in \mathbb{R}^2 \times S^1, \tag{3.3}$$

where $H_0(x) + H_1$ is an analytic Hamiltonian function, $H_0$ the unperturbed Hamiltonian function, $H_1$ the small time periodic perturbation, $S^1 = \mathbb{R}/(T\mathbb{Z})$ the circle of length $T$, $T \in \mathbb{R}^+$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with two assumptions;

(A.1) For $H_1 = 0$, the $x$ component of (3.3) has a heteroclinic orbit $x^0(t)$ to a hyperbolic periodic point $x_0$ of period 2 for a Poincare section map of $H$ whose trajectory is called the *separatrix.*

(A.2) The interior of $x^0(t)$ is filled with a family of periodic orbits $x^h(t), h < 0$ of period $T(h)$ such that $H_0(x^h(t)) = h$ for all $t$ and $T'(h) > 0$, and that

$$\lim_{h \to 0} x^h(t) = x^0(t), \quad \text{and} \quad \lim_{h \to 0} T(h) = \infty \quad \text{(See Figure 1).}$$

In the full phase space, $x_0$ corresponds to the periodic orbit $\gamma_0(t) = (x_0, t + \phi)$, which is a normally hyperbolic invariant manifold. By the invariant manifold theorem $\gamma_0(t)$ persists under the perturbation as $\gamma_1(t)$, which has the stable and unstable manifolds.

In order to study the dynamics near the separatrix, let us consider a section $\Sigma \subset \mathbb{R}^2$ for the $x$ component of (3.3) which transversally intersects the separatrix when $H_1 = 0$. Then $\Sigma \times S^1$ is transversal to the flow of (3.3) for small enough $H_1$. The full phase space for $H_1 = 0$ is shown in Figure 2. The transversality of $\Sigma \times S^1$ to the system (3.3) and an application of the implicit function theorem yield that $(\phi, p)$ is a sympletic coordinate system of $\Sigma \times S^1$ since the energy $p$ and the time $\phi$ are canonical variables. Let $q(t; \phi, p) = (x(t), \phi(t))$ be a solution of (3.3) with an initial condition $(\phi, p) \in \Sigma \times S^1$. Now we define the return map $\hat{S}$ called the *exact separatrix map* by

$$\hat{S}(\phi, p) = q(\tau; \phi, p) \quad \tau \text{ is the first return time,}$$

for the domain $D$ given by

$$D = \{(\phi, p) \in \Sigma \times S^1 : \exists \tau > 0, \qquad \text{s.t.} \qquad q(\tau; \phi, p) \in \Sigma \times S^1\}.$$

In [5], it is proved that $\hat{S}$ is a symplectic twist map. By lifting $S^1$ to $\mathbb{R}$, we extend $\hat{S}$ on the whole plane since the extended map is more convenient for use in the twist map theory. Also in [5], the explicit functional form of the lifted return map $S : \mathbb{R}^2 \to \mathbb{R}^2$ generated by the flow of (3.3) is given by:

$$S : \left\{ \begin{array}{l} \phi' = \phi + N(\phi, p) \\ p' = p + M(\phi, p) \end{array} \right\}. \quad S^{-1} : \left\{ \begin{array}{l} \bar{\phi} = \phi - \hat{N}(\phi, p) \\ \bar{p} = p + \hat{M}(\phi, p), \end{array} \right\}.$$

$N(\phi, p)$, $\hat{N}(\phi, p)$, $M$, and $\hat{M}$ satisfy

$$N(Y(\phi) - \hat{Y}(\phi), \phi) = \infty, \quad \hat{N}(0, \phi) = \infty, |M| << 1 \quad \text{and} \quad |\hat{M}| << 1,$$

where $Y(\phi)$ and $\hat{Y}(\phi)$ are graphs of differentiable functions given by the intersections of the stable and unstable manifolds of $\gamma_1(t)$ and $\Sigma \times S^1$ respectively (See Figure 2).

Alan Weinstein has pointed out that the stable and unstable manifolds of the perturbed Hamiltonian system must intersect [7]. By Lagrangian intersection theory [7,8], the perturbed manifolds transversally intersect for *generic* perturbation $H_1$ [9, 10]. Hence we have $Y(\phi) - \hat{Y}(\phi)$ has simple zeros. We denote the set of simple zeros of $\hat{Y}(\phi) - Y(\phi)$ by $\bar{\mathcal{A}}$. Let $\mathcal{A}$ be the lifted set of $\bar{\mathcal{A}}$ on the whole real line. To prove that the separatrix map has an chaotic invariant set near the separatrices, we define connected open sets near $(a_k, 0), a_k \in \mathcal{A}$ as follows (see Figure 3):

DEFINITION 3.1. For sufficiently large $K$ we choose a sequence $\{a_k\}_{-\infty}^{\infty}$ such that $a_k - a_{k-1} > K$ and $a_k \in \mathcal{A}$ for all $k \in N$. We define connected open sets near $(a_k, 0)$ for each $k$, $D(a_k; K)$, by

$$D(a_k; K) = \{(\phi, p) : N(\phi, p) > K \quad \text{and} \quad \hat{N}(\phi, p) > K\}.$$

THEOREM 3.2. *For sufficiently large $K$ and any sequence $\{a_k\}$ such that $a_k - a_{k-1} > K$ and $a_k \in \mathcal{A}$ for all $k$, there exists a unique orbit $\{(\phi_k, p_k)\}$ of $S$ such that $(\phi_k, p_k) \in D(a_k; K)$ for all $k$.*

*Proof.* See the reference [5]. □

Note that if we take the sequence $\{a_k\}$ arbitrarily random, the corresponding orbit $\{(\phi_k, p_k)\}$ should be also arbitrarily random. That is, theorem (3.2) implies that dynamics near separatrices is chaotic.

DEFINITION 3.2. The projection of the orbits in theorem 3.2 to $R \times S^1$ forms an invariant set. We denote this invariant set by $\Lambda_K$.

REMARK 3.1. Let $\Lambda_{K,K'}$ be a subset of $\Lambda_K$ obtained by the set of projections of the orbits corresponding to the sequences $\{a_k\}$ in theorem (3.2) with $K < |a_k - a_{k-1}| \leq K'$. Then $\Lambda_{K,K'}$ is a compact invariant set. In [5], it is proved that $\Lambda_{K,K'}$ is uniformly hyperbolic. By the definition of uniform hyperbolicity [11] and theorem (3.2), the separatrices splitting angle is not zero.

Now, we are ready to prove our main theorem.

THEOREM 3.3. *For the generic $C^2$ convex analytic perturbation of an ellipse the Birkhoff's billiard has non zero separatrices splitting.*

*Proof.* Let $f_u$ and $f_p$ be the unperturbed and perturbed elliptic Birkhoff's billiard maps respectively. Then, by proposition (2.2) $f_u \circ f_u$ and $f_p \circ f_p$ are ESMT maps. So, By theorem (3.1), there exist an integrable Hamiltonian $H_{f_u \circ f_u}$ and a perturbed Hamiltonian $H_{f_p \circ f_p}$ of which $f_u \circ f_u$ and $f_p \circ f_p$ are the time one maps of them respectively. Since the entire orbit structure of $f_u \circ f_u$ in the phase space consists of the rotational invariant curves (RICs), the hyperbolic fixed point $p_h$ connected by the heteroclinic loop and the elliptic fixed point $p_e$

surrounded by a family of invariant curves, $H_{f_u \circ f_u}$ satisfies $(A.1)$ and $(A.2)$.

Let $H_{f_p \circ f_p} = H_{f_u \circ f_u} + H_1 = H_{f_u \circ f_u} + (H_{f_p \circ f_p} - H_{f_u \circ f_u})$. As the boundary perturbation is sufficiently small, we have $|H_1| << 1$. Hence we can apply the *exact separatrix map* theory to this perturbed Hamiltonian system $H_{f_p \circ f_p}$. By theorem (3.2) and remark (3.1), there exists a uniform hyperbolic invariant set $\Lambda_{K,K'}$ near the separatrices of $H_{f_p \circ f_p}$, which includes chaotic orbits. As mentioned in remark (3.1), the existence of the uniformly hyperbolic invariant set with chaotic orbits near separatrices implies that the separatrices splitting angle is not zero. □

## 4. Conclusion

Using the theory of *exact separatrix map* of an one-degree-of-freedom Hamiltonian system with time periodic perturbation, we have proved that the Birkhoff's billiard in a special domain bounded by the $C^2$ convex simple curve constructed by the *generic global* perturbation for the ellipse, has the stochastic layer near the separatrices. Furthermore in the stochastic layer a uniformly hyperbolic invariant set including chaotic orbits near separatrices exists. This uniformly hyperbolic invariant set has the Smale's horse shoe-like structure, implying the separatrices splitting angle of the perturbed elliptic billiard is not zero. As author knows, from the definition of uniform hyperbolicity [11], we can explicitly compute the least bound of the principal separatrices splitting angle. This explicit computation of the splitting angle will be discussed in detail elsewhere.

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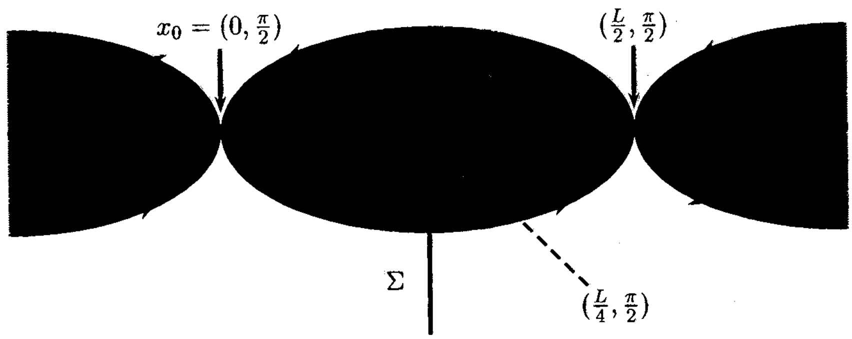


FIGURE 1. The unperturbed phase space of the $x$ component of (2.1) with the homoclinic orbit $x^0(t)$ to a hyperbolic equilibrium $x_0$ and the transversal section $\Sigma$ are shown.

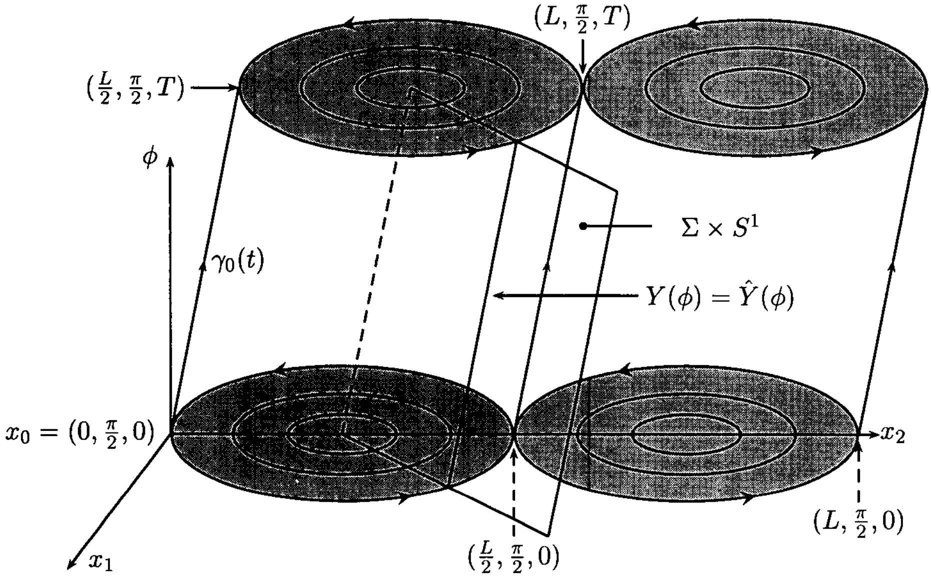


FIGURE 2. The full phase space of (2.1) with a periodic orbit $\gamma_0(t)$ and the section $\Sigma \times S^1$ are shown when $\epsilon = 0$.

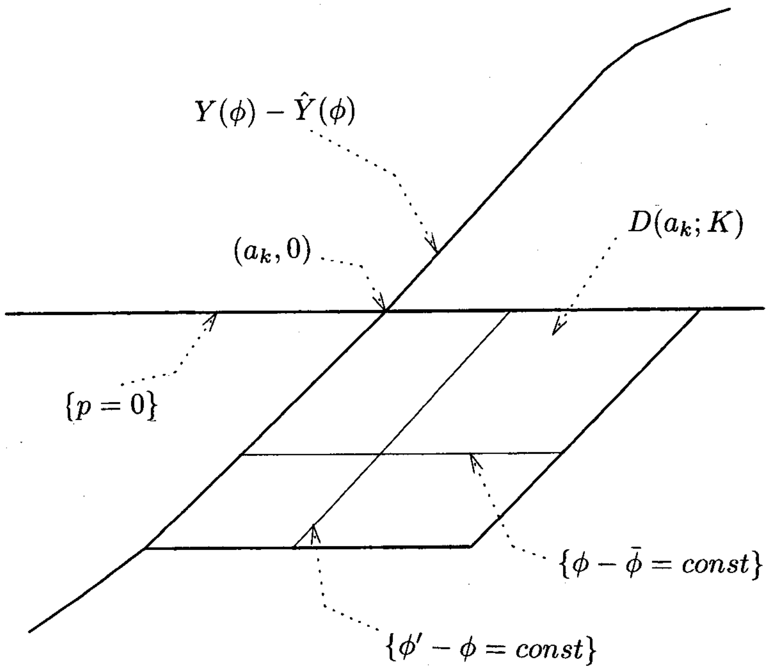


FIGURE 3. The connected open set $D(a_k; K)$ near the simple zero $(a_k, 0)$ of $Y(\phi) - \hat{Y}(\phi)$ in the coordinate system $(\phi - \bar{\phi}, \phi' - \phi)$ is shown.

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