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AN EXISTENCE OF THE FULLY DISCRETE SOLUTION FOR THE NAVIER STOKES EQUATION

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ABSTRACT. In this paper, we construct a fully discrete solution of the incompressible Navier Stokes equations using implicit Runge Kutta method. We prove the existence of the fully discrete solution.

1. Introduction

In this paper we consider the following Navier Stokes equations:

$$
\begin{cases}
u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f & \text{in } \Omega \times (0,T] \\
u = 0 & \text{on } \partial\Omega \times [0,T] \\
\text{div} u = 0 & \text{in } \Omega \times [0,T] \\
u(x,0) = u^0(x) & \text{in } \overline{\Omega}
\end{cases}
\tag{1.1}
$$

where u is a R^N-valued function, $\Omega \subset R^N$, $N = 2, 3$, and $0 < T < \infty$. We seek a R^N-valued velocity function $u = (u_1, \ldots, u_N)$ and a real-valued pressure function p, defined on $\bar{\Omega} \times [0,T]$ when Ω has a sufficiently smooth boundary $\partial\Omega$.

We assume that u^0 is a given R^N-valued function defined on $\bar{\Omega}$ with $u^0 = 0$ on $\partial\Omega$ and $\text{div} u^0 = 0$ in Ω. In (1.1), $\nu > 0$ is kinematic viscosity constant.

The results of existence, uniqueness and regularity of a pair of solutions (u,p) are proved. We refer the reader to the book by Temam [8].

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In this paper we apply the implicit Runge-Kutta method to construct the fully discrete solution. We prove the existence of the Galerkin fully discrete solution.

In [1], Baker proved an unique existence of the semidiscrete solution of the Navier Stokes equation. In [5], Crouzeix and Raviart used the conforming and nonconforming finite element method to solve the stationary Stokes equation. In [2,3,6,7], several results on the unique existence and convergence of the fully discrete solution of Navier Stokes equation are obtained.

2. Preliminaries and Notations

We introduce the appropriate spaces of functions. For an integer $s \geq 0$ and a real number $1 \leq p \leq \infty$, we define H^s as the usual Sobolev space, with the associated norm

$$|f|_s = \left(\sum_{|\alpha| \leq s} \int |D^\alpha f|^2 \right)^{\frac{1}{2}}.$$

As usual, we let $\overset{\circ}{H}{}^1$ be the space of those functions in H^1 which vanish on $\partial\Omega$ in the sense of trace. We let $\mathbb{H}^s = (H^s)^N = H^s \times \cdots \times H^s$ and $\overset{\circ}{\mathbb{H}}{}^1 = (\overset{\circ}{H}{}^1)^N$. We equip $\mathbb{H}^s$ with the inner product $(u, v)_s = \sum_{i=1}^{N} \langle u_i, v_i \rangle_s$, generating the product norm $\| \cdot \|_s = (\cdot, \cdot)_s^{\frac{1}{2}}$. We construct a finite dimensional subspace $\mathbb{S}_h^r$ of $\overset{\circ}{\mathbb{H}}{}^1$ consisting of ordered N-tuples of piecewise polynomials of degree $\leq r - 1$ defined on a quasi uniform partition of $\bar{\Omega}$ and satisfying the approximation property:

$$\inf_{\chi \in \mathbb{S}_h^r} (\|u - \chi\| + h\|u - \chi\|_1) \leq Ch^s \|u\|_s, \quad \forall u \in \mathbb{H}^s \cap \overset{\circ}{\mathbb{H}}{}^1, \quad 1 \leq s \leq r,$$

and

$$\|\chi\|_1 \leq Ch^{-1} \|\chi\|, \quad \forall \chi \in \mathbb{S}_h^r$$

where C is independent of h.

We let $P^r_{\hat{h}}$ be the family of finite dimensional subspaces of H^1, which consist of piecewise polynomials of degree $r \leq 1$ if $r = 2$ and $\leq r-2$ if $r > 2$. We assume that $P^r_{\hat{h}}$ satisfies the approximation properties:

$$\inf_{\phi \in P^r_{\hat{h}}} \left(|p-\phi| + \hat{h}|p-\phi|_1 \right) \leq c\hat{h}^s |p|_s, \quad \forall p \in H^s, \quad 1 \leq s \leq r-1,$$

and

$$|\phi|_1 \leq c\hat{h}^{-1}|\phi|, \quad \forall \phi \in P^r_{\hat{h}}.$$

Now we define the bilinear form

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} dx, \quad \forall u, v \in \mathbb{H}^1.$$

It is well known that there exists a constant $C_a = C_a(\Omega)$ such that

$$a(u,v) \leq \|u\|_1 \|v\|_1, \quad \forall u, v \in \mathbb{H}^1,$$

$$a(u,u) \geq C_a \|u\|_1^2, \quad \forall u \in \overset{\circ}{\mathbb{H}}{}^1.$$

We also consider the following trilinear form

$$b_1(u,v,w) = \sum_{i,j=1}^{N} \int_\Omega u_i \frac{\partial v_i}{\partial x_j} w_j dx, \quad \forall u, v, w \in \overset{\circ}{\mathbb{H}}{}^1.$$

In this paper, we shall use the following trilinear form

$$b(u,v,w) = \frac{1}{2}(b_1(u,v,w) - b_1(u,w,v)).$$

It is well known that, for $u \in \overset{\circ}{\mathbb{H}}{}^1$ with $\mathrm{div} u = 0$ in Ω,

$$b(u,v,w) = b_1(u,v,w) = -b_1(u,w,v), \quad \forall v, w \in \overset{\circ}{\mathbb{H}}{}^1.$$

We also remark here that it is well known that there exists a constant $C_b = C_b(\Omega)$ such that

$$|b(u,v,w)| \leq C_b \|u\|_1 \|v\|_1 \|w\|_1, \quad \forall u, v, w \in \overset{\circ}{\mathbb{H}}{}^1.$$

Since, for a fixed $u \in H^1$, $a(u,v)$ is a bounded linear functional on $(\mathbb{S}_h^r, \|\cdot\|)$, by Riesz-representation theorem, for any fixed $u \in H^1$ there exists an unique $a(u) \in (\mathbb{S}_h^r, \|\cdot\|)$ such that

$$(a(u), v) = a(u,v), \quad \forall v \in \mathbb{S}_h^r,$$

and also the following inequality holds,

$$\begin{aligned}\|a(u)\| &= \sup_{\|v\|=1} |(a(u),v)| = \sup_{\|v\|=1} |a(u,v)| \\ &\le \|u\|_1 \|v\|_1 \le Ch^{-1}\|u\|_1\|v\| = Ch^{-1}\|u\|_1.\end{aligned}$$

Since, for a fixed $u \in W_{1,\infty}$, $b(u,u,v)$ is a bounded linear functional on $(\mathbb{S}_h^r, \|\cdot\|)$, by Riesz-representation theorem, there exists an unique $b(u) \in (\mathbb{S}_h^r, \|\cdot\|)$ such that

$$(b(u), v) = b(u,u,v), \quad \forall v \in \mathbb{S}_h^r,$$

and also the following inequality holds,

$$\begin{aligned}\|b(u)\| &= \sup_{\substack{\|v\|=1 \\ v\in\mathbb{S}_h^r}} |b(u,u,v)| \\ &\le C \sup_{\|v\|=1} (\|u\|\|u\|_{1,\infty}\|v\| + \|u\|\|u\|_\infty\|v\|_1) \\ &\le \sup_{\|v\|=1} C(\|u\|\|u\|_{1,\infty}\|v\|_1) \le Ch^{-1}\|u\|\|u\|_{1,\infty}.\end{aligned}$$

And also define $f_{ij}^* \in (\mathbb{S}_h^r, \|\cdot\|)$ by $(f_{ij}^*, v) = (f_{ij}, v)$. Then $\|f_{ij}^*\| \le \|f_{ij}\|$.

We let $\widetilde{\mathbb{S}}_h^r = \{v \in \mathbb{S}_h^r : (v, \nabla q) = 0, \quad \forall q \in P_h^r(\Omega)/R\}$.

3. Implicit Runge-Kutta Method

For the temporal approximation of the solution to (1.1), the implicit Runge-Kutta (IRK) methods are now introduced. For an integer $q \ge 1$, q-stage IRK method is characterized by a set of constants arranged in the following tableau form

$$\begin{array}{ccc|c} a_{11} & \cdots & a_{1q} & \tau_1 \\ \vdots & & \vdots & \vdots \\ a_{q1} & \cdots & a_{qq} & \tau_q \\ \hline b_1 & \cdots & b_q & \end{array} \qquad \text{or} \qquad \begin{array}{c|c} A & T \\ \hline b^T & \end{array}$$

In this paper, we assume that there exists for each q, a diagonal $q \times q$ matrix D with positive diagonal elements such that DAD^{-1} is positive definite on $\mathbb{R}^q$.

Given the initial value problem

$$\begin{aligned} y' &= f(t,y), \quad 0 \le t \le T \\ y(0) &= y^0. \end{aligned} \tag{3.1}$$

IRK methods can be applied to generate approximations $\{y^n\}_{n=0}^J$ to $\{y(t^n)\}_{n=0}^J$ as follows. Let

$$y^{n+1} = y^n + k \sum_{j=1}^{q} b_j f(t^{nj}, y^{nj}) \tag{3.2}$$

where $t^{nj} = t^n + \tau_j k$, $k = T/J$ and the intermediate stages y^{nj} are given by the coupled system of equations

$$y^{nj} = y^n + k \sum_{m=1}^{q} a_{jm} f(t^{nm}, y^{nm}), \quad j = 1, 2, \ldots, q. \tag{3.3}$$

For more details about the application of Runge-Kutta method, refer to [4]. We apply IRK method to (1.1) to get

$$\begin{aligned} &(U^{ni}, v) + k \sum_{j=1}^{q} a_{ij} \{ va(U^{nj}, v) + b(U^{nj}, U^{nj}, v) \} \\ &= (U^n, v) + k \sum_{j=1}^{q} a_{ij} (f_{nj}, v), \quad \forall v \in \widetilde{\mathbb{S}}_h^r, \quad 1 \le i \le q, \end{aligned} \tag{3.4}$$

$$\begin{aligned} (U^{n+1}, v) = (U^n, v) + k \sum_{i=1}^{q} b_i [&-va(U^{ni}, v) - b(U^{ni}, U^{ni}, v) \\ &+ (f_{ni}, v)], \qquad \forall v \in \widetilde{\mathbb{S}}_h^r. \end{aligned} \tag{3.5}$$

4. Existence of the Fully Discrete Solution

THEOREM 4.1 **(Brouwer's fixed point theorem)**. *Let H be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. Let $g : H \to H$ be continuous function. If there exists $\alpha > 0$ such that $(g(z), z)_H > 0$ for all z with $\|z\|_H = \alpha$, then there exists $z^* \in H$ such that $\|z^*\|_H \leq \alpha$ such that $g(z^*) = 0$.*

THEOREM 4.2. *The fully discrete solution of (3.4) exists. And trivially (3.5) has a unique solution.*

PROOF. From (3.4) we have

$$
\begin{aligned}
&(U^{ni}, v) + k\sum_{j=1}^{q} a_{ij}\{\nu a(U^{nj}, v) + b(U^{nj}, U^{nj}, v)\} \\
&= (U^n, v) + k\sum_{j=1}^{q} a_{ij}(f_{nj}, v), \quad \forall v \in \widetilde{\mathbb{S}}_h^r.
\end{aligned} \tag{4.1}
$$

Set

$$
\begin{aligned}
F_i(v) =&(U^{ni}, v) + k\sum_{j=1}^{q} a_{ij}\{\nu a(U^{nj}, v) + b(U^{nj}, U^{nj}, v) - (f_{nj}, v)\} \\
&- (U^n, v), \quad \forall v \in \widetilde{\mathbb{S}}_h^r.
\end{aligned}
$$

For fixed $U = \{U^{ni}\}_{i=1}^q \in (\widetilde{\mathbb{S}}_h^r)^q = \mathbb{H}$, F_i is bounded linear functional on $(\widetilde{\mathbb{S}}_h^r, \|\cdot\|)$ for $i = 1, 2, \ldots, q$. By the Riesz representation theorem, there exists unique $u_i \in \mathbb{S}_h^r$ such that

$$
\begin{aligned}
&(U^{ni}, v) + k\sum_{j=1}^{q} a_{ij}\{\nu a(U^{nj}, v) + b(U^{nj}, U^{nj}, v) - (f_{nj}, v)\} \\
&- (U^n, v) = (u_i, v).
\end{aligned} \tag{4.2}
$$

Define $\mathbb{F} : (\widetilde{\mathbb{S}}_h^r)^q \longrightarrow (\widetilde{\mathbb{S}}_h^r)^q$ such that $\mathbb{F}(U) = (F_i(U_{ni}))_{1\leq i\leq q} = (u_1, u_2, \ldots, u_q)^t$. We need to show that there exists $U \in (\widetilde{\mathbb{S}}_h^r)^q$ such that $\mathbb{F}(U) = 0$.

If we rewrite (4.2), we have

$$\mathbb{F}(U) = U + kA\{\nu a(U) + b(U) - f^*\} - \bar{U}_n$$

in $((\widetilde{\mathbb{S}}^r_h)^q, \|\cdot\|_{\mathbb{H}})$ where $a(U) = (a(U^{n1}), a(U^{n2}), \cdots, a(U^{nq}))^t$, $b(U) = (b(U^{n1}), b(U^{n2}), \cdots, b(U^{nq}))^t$, $f^* = (f^*_{n1}, f^*_{n2}, \ldots, f^*_{nq})^t$ and $\overline{U}^n = (U^n, U^n, \ldots, U^n)^t$.

To prove the continuity of $\mathbb{F}$, it is sufficient to prove the continuity of $a(U)$ and $b(U)$. Now we first prove the continuity of $a(U)$.

$$\begin{aligned}
\|a(U) - a(V)\|_{\mathbb{H}} &= \left(\sum_{i=1}^{q} \|a(U_i) - a(V_i)\|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{q} \|a(U_i - V_i)\|^2\right)^{\frac{1}{2}} \\
&\leq Ch^{-2}\|U - V\|_{\mathbb{H}}
\end{aligned}$$

which implies the continuity of $a(U)$.

Continuity of $b(U)$ can be proved as follows.

$$\begin{aligned}
&\|b(U) - b(V)\|^2_{\mathbb{H}} \\
&= \sum_{i=1}^{q} \|b(U_i) - b(V_i)\|^2 \\
&= \sum_{i=1}^{q} (b(U_i) - b(V_i), b(U_i) - b(V_i)) \\
&= \sum_{i=1}^{q} [(b(U_i), b(U_i)) - (b(U_i), b(V_i)) - (b(V_i), b(U_i)) + (b(V_i), b(V_i))] \\
&= \sum_{i=1}^{q} [b(U_i, U_i, b(U_i)) - b(U_i, U_i, b(V_i)) - b(V_i, V_i, b(U_i)) \\
&\quad + b(V_i, V_i, b(V_i))]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{q} [b(U_i - V_i, U_i, b(U_i)) - b(U_i - V_i, U_i, b(V_i)) - b(V_i, V_i - U_i, b(U_i)) \\
&\quad + b(V_i, V_i - U_i, b(V_i))] \\
&\leq \sum_{i=1}^{q} [C_b \|U_i - V_i\| \|U_i\|_{1,\infty} \|b(U_i)\|_1 + C_b \|U_i - V_i\| \|U_i\|_{1,\infty} \|b(V_i)\|_1 \\
&\quad + C_b \|V_i\| \|V_i - U_i\|_{1,\infty} \|b(U_i)\|_1 + C_b \|V_i\| \|V_i - U_i\|_{1,\infty} \|b(V_i)\|_1] \\
&\leq \sum_{i=1}^{q} C_b' [\|U_i - V_i\| h^{-(1+\frac{N}{2})} \|U_i\| h^{-(3+\frac{N}{2})} \|U_i\|^2 \\
&\quad + \|U_i - V_i\| h^{-(1+\frac{N}{2})} \|U_i\| h^{-(3+\frac{N}{2})} \|V_i\|^2 \\
&\quad + \|V_i\| h^{-(1+\frac{N}{2})} \|V_i - U_i\| h^{-(3+\frac{N}{2})} \|U_i\|^2 \\
&\quad + \|V_i\| h^{-(1+\frac{N}{2})} \|V_i - U_i\| h^{-(3+\frac{N}{2})} \|V_i\|^2] \\
&= \sum_{i=1}^{q} C_b' h^{-(4+N)} \{\|U_i\|^3 + \|U_i\| \|V_i\|^2 + \|V_i\| \|U_i\|^2 + \|V_i\|^3\} \|V_i - U_i\|.
\end{aligned}
$$

If $\|U - V\|_{\mathbb{H}} < 1$, then $\|U_i - V_i\| < 1$, i.e., $\|V_i\| < 1 + \|U_i\| = M_i$. For any given $\epsilon > 0$, choose δ such that

$$
\delta < \min \left\{ 1, \frac{\epsilon^2 h^{(4+N)}}{C_b' 4 q M^3} \right\}
$$

where $M = \max_{1 \leq i \leq q} M_i$.

If $\|U - V\|_{\mathbb{H}} < \delta$, then

$$
\begin{aligned}
\|b(U) - b(V)\|_{\mathbb{H}}^2 &\leq \sum_{i=1}^{q} C_b' h^{-(4+N)} 4 M^3 \|V_i - U_i\| \\
&\leq C_b' h^{-(4+N)} 4 q M^3 \delta \\
&\leq \epsilon^2,
\end{aligned}
$$

which implies the continuity of $b(U)$ on $\mathbb{H}$.

Now we obtain the following equalities,

$$\begin{aligned}(\mathbb{F}(U),U)_{\mathbb{H}} =&(U+kA\{\nu a(U)+b(U)-f^*\}-\bar{U}_n,U)_{\mathbb{H}},\\ (D^2A^{-1}\mathbb{F}(U),U)_{\mathbb{H}} =&(D^2A^{-1}U+kD^2\{\nu a(U)+b(U)-f^*\}\\ &-D^2A^{-1}\bar{U}_n,U)_{\mathbb{H}},\\ (D^2A^{-1}U,U)_{\mathbb{H}} =&(DA^{-1}U,DU)_{\mathbb{H}}=(DA^{-1}D^{-1}DU,DU)_{\mathbb{H}}\\ &\equiv(\mathbb{C}DU,DU)_{\mathbb{H}},\end{aligned}$$

where $\mathbb{C}=DA^{-1}D^{-1}=(DAD^{-1})^{-1}$ is positive definite on $\mathbb{R}^q$. And also we have $(D^2A^{-1}U,U)_{\mathbb{H}}\geq C'\|DU\|_{\mathbb{H}}^2\geq C_1\|U\|_{\mathbb{H}}^2$ for some positive constant C_1.

$$\begin{aligned}&k(D^2[\nu a(U)+b(U)],U)_{\mathbb{H}}\\ &\geq k\underline{d}^2\left\{\nu\sum_{i=1}^q(a(U_i),U_i)+\sum_{i=1}^q(b(U_i),U_i)\right\}\\ &=k\underline{d}^2\left\{\nu\sum_{i=1}^q[a(U_i,U_i)+b(U_i,U_i,U_i)]\right\}\\ &\geq k\underline{d}^2\{\nu C_0\sum_{i=1}^q\|U_i\|_1^2\}\geq kC_2\|U\|_{\mathbb{H}}^2\end{aligned}$$

for some positive constant C_2 which depends on the matrix A, ν and Ω, where $\underline{d}=\min_{1\leq i\leq q}d_i$.

$$|(kD^2f^*+D^2A^{-1}\bar{U}^n,U)_{\mathbb{H}}|\leq C_3(k\|f^*\|_{\mathbb{H}}\|U\|_{\mathbb{H}}+\|\bar{U}^n\|_{\mathbb{H}}\|U\|_{\mathbb{H}}),$$

for some positive constant C_3.

$$\begin{aligned}&|(D^2A^{-1}\mathbb{F}(U),U)_{\mathbb{H}}|\\ &\geq\ C_1\|U\|_{\mathbb{H}}^2+kC_2\|U\|_{\mathbb{H}}^2-C_3(k\|f^*\|_{\mathbb{H}}\|U\|_{\mathbb{H}}+\|\bar{U}^n\|_{\mathbb{H}}\|U\|_{\mathbb{H}})\\ &=\ \|U\|_{\mathbb{H}}\{\|U\|_{\mathbb{H}}(C_1+C_2k)-C_3(k\|f^*\|_{\mathbb{H}}+\|\bar{U}^n\|_{\mathbb{H}})\}\end{aligned}$$

for some positive constant C_3. Choose $U\in(\widetilde{\mathbb{S}}_h^r)^q=\mathbb{H}$ such that

$$\|U\|_{\mathbb{H}}=\frac{2C_3\{k\|f^*\|_{\mathbb{H}}+\|\bar{U}^n\|_{\mathbb{H}}\}}{C_1+C_2k}=\alpha.$$

If either $f^* \neq 0$ or $\bar{U}^n \neq 0$, then $\alpha > 0$, which implies that there exists $U^* \in \mathbb{H}$ such that $D^2A^{-1}\mathbb{F}(U^*) = 0$ and $\|U^*\| \leq \alpha$ by fixed point theorem 4.1. We proved that there exists $U^* \in \mathbb{H}$ satisfying $\mathbb{F}(U^*) = 0$, which implies the existence of $\{U^{ni}\}_{i=1,2,\ldots,q}$ satisfying (4.1).

If $f^* = 0$ and $\bar{U}^n = 0$, then (4.1) has a trivial solution $0 \in \mathbb{H}$.

This ends the proof of the existence of $\{U_{ni}\}$ which satisfies (4.1). □

We leave the stability and convergence of the fully discrete solution introduced in this paper for the future work.

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