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HIGH–ORDER WEIGHTED DIFFERENCE SCHEMES FOR THE CONVECTION-DIFFUSION PROBLEMS

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ABSTRACT. High-order weighted difference schemes with uniform meshes are considered for the convection-diffusion problem depending on Reynolds numbers. For small Reynolds numbers, a weighted central difference scheme is suggested since there is no boundary layer. For large Reynolds numbers, we propose a modified upwind method with an artificial diffusion in order to overcome nonphysical oscillation of central schemes and obtain good accuracy in the boundary layer. Existence and corresponding error estimates of the solution for the difference schemes have been shown. Numerical experiments are provided to back up the analysis.

1. Introduction

Consider the one dimensional steady linear convection-diffusion problem

$$-\epsilon\frac{d^2u}{dx^2}+b\frac{du}{dx}=f, \qquad x\in(0,1), \tag{1.1a}$$

with boundary conditions

$$u(0)=u(1)=0. \tag{1.1b}$$

Here $u(x)$ is the transport quantity, b is a positive transport velocity and $f(x)$ is a sufficiently smooth function. The diffusion constant $\epsilon>0$ is positive, which is proportional to the reciprocal of Reynolds number. The solution u has a boundary layer for small $\epsilon>0$.

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The convection-diffusion problem (1.1) arises in diverse areas such as the ground water pollution problem, the reservoir displacement problem, a steady state of the linearized Navier-Stokes equations and the drift-diffusion equation of semiconductor device modeling.

For small ϵ, many numerical methods have been proposed for the solution of (1.1). It is well known that the standard central difference scheme has order $O(h^2)$, but it gives a nonphysical oscillation in the numerical solution. In order to remove this nonphysical oscillatory phenomena, Kellogg and Tsan [5] have developed an upwind method with uniform meshes. The uniform mesh upwind method is of lower order of accuracy $O(h)$.

In general, we may expect that the error becomes small as $h \to 0$. But the numerical error of the upwind method with a uniform mesh does not decrease even though the mesh h decreases because the error depends on small ϵ. That is, the uniform mesh upwind method is not ϵ-uniform(a numerical method is said to be ϵ-uniform if numerical solution converges independently of ϵ). In order to obtain ϵ-uniform convergence, Miller, O'Riordan and Shiskin [8] and Stynes and Roos [9] adopted nonuniform mesh upwind- and central- difference schemes. Gartland [4] has used a uniform mesh difference scheme with perturbation technique.

Several finite element methods such as adaptive finite element methods and finite volume methods have been also applied to the problem (1.1)(see, [1], [3], and [6]). In particular, Liang and Zhao[6] have proposed a high-order upwind finite element method for problem (1.1) with accuracy $O(h^3\|u^{(4)}\|)$ using the dual partition. They showed that their scheme is more accurate than the standard central difference scheme for various ϵ numbers. But the scheme in [6] is neither simple nor ϵ-uniform.

In this paper, we propose two finite difference schemes with uniform meshes for the problem (1.1) depending on ϵ. One is a weighted central difference scheme which gives a good accuracy for moderately large ϵ. The other is a modified upwind difference scheme with artificial diffusion, which satisfies the maximum principle and produces monotonic solutions to a boundary layer type problems for large Reynolds numbers. Thus, this scheme overcomes numerical oscillation and gives astonishingly high order accuracy.

In section 2, we introduce a weighted central difference scheme for

(1.1) and show existence and convergence of the numerical solutions when ϵ is not small. In section 3, we introduce a modified upwind method with artificial diffusion in order to make the scheme satisfy the maximum principle. In section 4, we compare numerical results of the proposed schemes with those of the standard upwind method, the standard central scheme and the scheme in [6].

2. A Weighted Central Difference Scheme

Let Ω be the unit interval (0,1) and $h = \frac{1}{M}$ for a positive integer M. Let $\Omega_h = \{x_i = ih | i = 1, \cdots, M-1\}$ and $\partial\Omega_h = \{x_0, x_M\}$. For a function $v = (v_0, \cdots, v_M)$ defined on $\Omega_h \cup \partial\Omega_h$ with $v = 0$ on $\partial\Omega_h$, denote $v_i = v(x_i)$ for $0 \leq i \leq M$.

We now introduce the discrete L^2-space with an inner product

$$(v, w)_h = h \sum_{i=1}^{M-1} v_i w_i$$

for functions v and w defined on Ω_h. Define the difference operators as

$$D_+ v_i = \frac{v_{i+1} - v_i}{h}, \qquad D_- v_i = \frac{v_i - v_{i-1}}{h},$$

$$D^2 v = D_+(D_- v).$$

The discrete L^2 norm is defined by

$$\|v\|_{0,h} = (v, v)_h^{\frac{1}{2}},$$

and the supremum norm and the discrete H^1-seminorm are defined, respectively, by

$$\|v\|_\infty = \max_{1 \leq i \leq M-1} |v_i|, \qquad |v|_1^2 = h \sum_{i=1}^{M} (D_- v)_i^2.$$

Whenever there is no confusion, we use $(\cdot, \cdot)$ and $\| \cdot \|$ instead of $(\cdot, \cdot)_h$ and $\| \cdot \|_{0,h}$, respectively.

Then the approximate solution U_i for (1.1) is defined as a solution of

$$(2.1a) \quad -\epsilon D^2 U_i + b\{\alpha D_- U_i + (1-\alpha) D_+ U_i\} = F_i, \quad 1 \le i \le M-1,$$

$$(2.1b) \qquad U_0 = U_M = 0,$$

where the weighting factor α is given by

$$(2.2) \qquad \alpha = \frac{1}{2} + \frac{\epsilon h}{\frac{12\epsilon^2}{b} + h^2 b}$$

and $F_i = f_i + \tilde{f}_i$ with

$$\tilde{f}_i = (1-2\alpha)\frac{h}{2}\left\{\frac{\epsilon}{b} f_i'' + f_i'\right\} + \frac{h^2}{6} f_i'' + \alpha \frac{1}{24h}\int_{x_{i-1}}^{x_i} (x_{i-1}-s)^4 f^{(4)}(s)ds$$
$$+ (1-\alpha)\frac{1}{24h}\int_{x_i}^{x_{i+1}} (x_{i+1}-s)^4 f^{(4)}(s)ds.$$

REMARK 2.1. The weighting factor α satisfies an inequality

$$\frac{1}{2} < \alpha < \frac{1}{2} + \min\left\{\frac{1}{12}\text{Re}, \ \frac{1}{\text{Re}}\right\},$$

where Re$=\frac{bh}{\epsilon}$ is the Reynolds number.

Now, we prove the existence of the numerical solution of (2.1).

THEOREM 2.1. *There exists a unique solution of the scheme* (2.1).

PROOF. The equation (2.1) can be expressed in the form

$$(2.3) \qquad A_i U_{i-1} + B_i U_i + C_i U_{i+1} = h^2 F_i, \qquad 1 \le i \le M-1,$$

where $A_i + B_i + C_i = 0$ and

$$A_i = -\epsilon - hb\alpha, \quad B_i = 2\epsilon + hb(2\alpha - 1) > 0, \quad C_i = -\epsilon + hb(1-\alpha).$$

Since the matrix having B_i as the diagonal element and A_i, C_i as off-diagonal elements is nonsingular, the solution U exists uniquely. □

REMARK 2.2. If Reynolds number $Re < 2$, then the matrix in Theorem 2.1 becomes an M-matrix.

The identities in Lemma 2.1 are obtained by the differentiation of each side of the equation (1.1), which will be used for the error analysis.

LEMMA 2.1. *For the solution u of* (1.1), *the following identities hold.*

(1) $u^{(5)} = \frac{\epsilon}{b}u^{(6)} + \frac{1}{b}f^{(4)}$.
(2) $u''' = \frac{\epsilon}{b}u^{(4)} + \frac{1}{b}f''$.
(3) $u'' = \frac{\epsilon}{b}u''' + \frac{1}{b}f' = (\frac{\epsilon}{b})^2u^{(4)} + \frac{\epsilon}{b^2}f'' + \frac{1}{b}f'$.

LEMMA 2.2. *For the solution u of* (1.1), *we obtain*

$$\begin{aligned}
&\epsilon(u_i'' - D^2u_i) + b[\{\alpha D_-u_i + (1-\alpha)D_+u_i\} - u_i'] \\
&= \alpha\frac{\epsilon}{24h}\int_{x_{i-1}}^{x_i}(x_{i-1}-s)^4u^{(6)}(s)ds \\
&\quad + (1-\alpha)\frac{\epsilon}{24h}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^4u^{(6)}(s)ds \\
&\quad + \epsilon O(h^4\|u^{(6)}\|) + \tilde{f}_i, \qquad\qquad 1 \le i \le M-1.
\end{aligned}$$

PROOF. From the definition of D^2, we obtain

$$\epsilon(u_i'' - D^2u_i) = \epsilon\left\{-\frac{h^2}{12}u^{(4)}\right\} + \epsilon O(h^4\|u^{(6)}\|). \tag{2.4}$$

It follows from Lemma 2.1 that

$$\begin{aligned}
&\alpha D_-u_i + (1-\alpha)D_+u_i \\
&= u_i' + (1-2\alpha)\frac{h}{2}u_i'' + \frac{h^2}{6}u_i''' + (1-2\alpha)\frac{h^3}{24}u^{(4)} \\
&\quad + \alpha\frac{1}{24h}\int_{x_{i-1}}^{x_i}(x_{i-1}-s)^4\left\{\frac{\epsilon}{b}u^{(6)}(s) + \frac{1}{b}f^{(4)}(s)\right\}ds \\
&\quad + (1-\alpha)\frac{1}{24h}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^4\left\{\frac{\epsilon}{b}u^{(6)}(s) + \frac{1}{b}f^{(4)}(s)\right\}ds.
\end{aligned} \tag{2.5}$$

Thus, from (2.4)–(2.5) and Lemma 2.1, we have

$$
\begin{aligned}
&\epsilon(u_i'' - D^2u_i) + b[\{\alpha D_-u_i + (1-\alpha)D_+u_i\} - u_i'] \\
&= \left[-\epsilon\frac{h^2}{12} + b\left\{(1-2\alpha)\frac{h}{2}\left(\frac{\epsilon}{b}\right)^2 + \frac{h^2}{6}\frac{\epsilon}{b} + (1-2\alpha)\frac{h^3}{24}\right\}\right]u^{(4)} \\
&+ \epsilon O(h^4\|u^{(6)}\|) + (1-2\alpha)\frac{h}{2}\left\{\frac{\epsilon}{b}f_i'' + f_i'\right\} + \frac{h^2}{6}f_i'' \\
&+ \alpha\frac{\epsilon}{24h}\int_{x_{i-1}}^{x_i}(x_{i-1}-s)^4u^{(6)}(s)ds \\
&+ (1-\alpha)\frac{\epsilon}{24h}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^4u^{(6)}(s)ds \\
&+ \alpha\frac{1}{24h}\int_{x_{i-1}}^{x_i}(x_{i-1}-s)^4f^{(4)}(s)ds \\
&+ (1-\alpha)\frac{1}{24h}\int_{x_i}^{x_{i+1}}(x_{i+1}-s)^4f^{(4)}(s)ds.
\end{aligned}
$$

Since the first term of the right hand side vanishes for α given in (2.2), the result follows from the definition of $\tilde{f}_i$. □

The following lemma can be verified using summation by parts and minimum eigenvalue of a symmetric matrix. The proof can be found in Agarwal [2].

LEMMA 2.3. *For a function v defined on $\Omega_h \cup \partial\Omega_h$ with $v = 0$ on $\partial\Omega_h$, the following inequalities hold.*

(1) *the discrete Poincaré inequality:* $\frac{2\sin(\frac{\pi h}{2})}{h}\|v\| \le |v|_1$.
(2) $\|v\|_\infty^2 \le 2(\|v\|^2 + |v|_1^2)$.

THEOREM 2.2. *Let u be the exact solution of (1.1) and U the numerical solution of the scheme (2.1). Let $e = u - U$. Then there exists a constant C such that*

$$\|e\|_\infty \le Ch^4\|u^{(6)}\|.$$

PROOF. Forming the inner product between (2.1a) and e, we obtain

$$\begin{aligned} &-\epsilon(D^2e,e)+b(\{\alpha D_-e+(1-\alpha)D_+e\},e)\\ &=-\epsilon(D^2u,e)+b(\{\alpha D_-u+(1-\alpha)D_+u\},e)-(F,e). \end{aligned} \tag{2.6}$$

Since $-(D^2v,v)=|v|_1^2$ holds for a function v defined on $\Omega_h\cup\partial\Omega_h$ with $v_0=v_M=0$, the left hand side of (2.6) becomes

$$-\epsilon(D^2e,e)+b(\{\alpha D_-e+(1-\alpha)D_+e\},e)=\epsilon|e|_1^2+b\frac{2\alpha-1}{2}\sum_{i=1}^{M}(e_i-e_{i-1})^2.$$

Hence, it follows from (2.6) that

$$\begin{aligned} &\epsilon|e|_1^2+b\frac{2\alpha-1}{2}\sum_{i=1}^{M}(e_i-e_{i-1})^2\\ &=(-\epsilon D^2u+b\{\alpha D_-u+(1-\alpha)D_+u\}-F,e). \end{aligned} \tag{2.7}$$

It follows from Lemma 2.2 and Young's inequality that we obtain

(2.8)

$$\begin{aligned} &\left|(-\epsilon D^2u+b\{\alpha D_-u+(1-\alpha)D_+u\}-F,e)\right|\\ &=\left|h\sum_{i=1}^{M-1}[-\epsilon D^2u_i+b\{\alpha D_-u_i+(1-\alpha)D_+u_i\}-F_i]e_i\right|\\ &=\left|h\sum_{i=1}^{M-1}[-\epsilon D^2u_i+b\{\alpha D_-u_i+(1-\alpha)D_+u_i\}-f_i-\tilde{f}_i]e_i\right|\\ &=\left|h\sum_{i=1}^{M-1}[-\epsilon D^2u_i+b\{\alpha D_-u_i+(1-\alpha)D_+u_i\}-(-\epsilon u_i''+bu_i')-\tilde{f}_i]e_i\right|\\ &=\left|h\sum_{i=1}^{M-1}\{\epsilon(u_i''-D^2u_i)+b[\{\alpha D_-u_i+(1-\alpha)D_+u_i\}-u_i']-\tilde{f}_i\}e_i\right| \end{aligned}$$

$$
\begin{aligned}
&= \left| h \sum_{i=1}^{M-1} \left\{ \alpha \frac{\epsilon}{24h} \int_{x_{i-1}}^{x_i} (x_{i-1} - s)^4 u^{(6)}(s) ds \right.\right. \\
&\qquad \left.\left. + (1-\alpha) \frac{\epsilon}{24h} \int_{x_i}^{x_{i+1}} (x_{i+1} - s)^4 u^{(6)}(s) ds + \epsilon O(h^4) \right\} e_i \right| \\
&\le \epsilon (C h^8 \|u^{(6)}\|^2 + \frac{1}{2} \|e\|^2).
\end{aligned}
$$

Using relations (2.6)–(2.8) and Lemma 2.3, we obtain

$$
\begin{aligned}
\epsilon |e|_1^2 + b \frac{2\alpha - 1}{2} \sum_{i=1}^{M} (e_i - e_{i-1})^2 &\le \epsilon \left(C h^8 \|u^{(6)}\|^2 + \frac{1}{2} \|e\|^2 \right) \\
&\le \epsilon \left(C h^8 \|u^{(6)}\|^2 + \frac{1}{2} |e|_1^2 \right).
\end{aligned}
$$

Since $\alpha > \frac{1}{2}$, the above inequality becomes

$$
|e|_1 \le C h^4 \|u^{(6)}\|.
$$

Lemma 2.3 completes the proof. □

REMARK 2.3. Since $\|u^{(n)}\| = O(\epsilon^{-n})$ for the exact solution u of (1.1), the weighted central difference scheme (2.1) is not ϵ-uniform.

3. A Modified Upwind Method with Artificial Diffusion

Since the scheme (2.1) is not ϵ-uniform, it may not give good numerical results for small ϵ. To overcome this shortage, we will consider a new finite difference scheme which has more meaningful error bound for large Reynolds number Re$=\frac{bh}{\epsilon}$. Let $Q(\alpha)$ be a cubic polynomial in α defined by

$$
\begin{aligned}
Q(\alpha) =& \alpha(\delta - 1)^2 + (1-\alpha)\delta^2 + \frac{bh}{3\epsilon} \{ \alpha(\delta-1)^3 + (1-\alpha)\delta^3 \} \\
& - \beta \frac{\epsilon}{bh} (2\delta - 1) - \frac{\beta}{3} (1 - 3\delta + 3\delta^2) \\
& + \frac{2\epsilon}{(bh)^2} \{ \epsilon(1-\beta) + bh(\delta - \alpha) \},
\end{aligned}
\tag{3.1}
$$

where $\delta = \alpha - \beta$ with $\beta = (\epsilon/bh)^2$.

LEMMA 3.1. *For Re$\geq$ 2, there exists a unique solution α of $Q(\alpha) = 0$ such that $1 < \alpha < 1 + \beta$.*

PROOF. It follows from (3.1) that

$$\begin{aligned}
Q(1) &= \frac{1}{3}\frac{1}{\epsilon^2(hb)^6}(5\mathrm{Re}^4 - 9\mathrm{Re}^3 + 5\mathrm{Re} - 3) > 0,\\
Q(1+\beta) &= -\frac{1}{3}\frac{1}{\epsilon^2(hb)^4}(\mathrm{Re}^3 - 2\mathrm{Re}^2 + 9\mathrm{Re} + 6) < 0,\\
Q'(0) &= \frac{\mathrm{Re}}{6}\{1 - 12(\frac{1}{\mathrm{Re}})^4 + 12(\frac{1}{\mathrm{Re}})^5\} > 0,\\
Q'(1) &= -1 - \frac{h}{3\epsilon}\{1 - 3(\frac{1}{\mathrm{Re}})^2\} - 2(\frac{1}{\mathrm{Re}})^3 b - (\frac{1}{\mathrm{Re}})^3\{1 - 2(\frac{1}{\mathrm{Re}})\} < 0,\\
Q'(1+\beta) &= -1 - 3(\frac{1}{\mathrm{Re}})^2 - \frac{h}{3\epsilon}\{1 + 3(\frac{1}{\mathrm{Re}})^2\} - 2(\frac{1}{\mathrm{Re}})^3 b < 0.
\end{aligned}$$

Since $Q(\alpha)$ is a cubic polynomial and Re$\geq$ 2, we obtain the desired result. □

REMARK 3.1. The root α of $Q(\alpha) = 0$ can be calculated easily from (3.2) by using the Newton's method.

We now consider a modified upwind method with an artificial diffusion. Let α be the root of $Q(\alpha) = 0$ with $1 < \alpha < 1 + \beta$ and $\Omega_h^* = \{x_{i-1/2} | i = 1, 2, \cdots, M\}$ the dual partition of (0,1) such that $x_{i-1/2} = x_i - \delta h, i = 1, 2, \cdots, M$. Denote $u_{i-1/2} = u(x_{i-1/2})$ and $f_{i-1/2} = f(x_{i-1/2})$ for $i = 1, 2, \cdots, M$. Then $U = (U_0, U_1, \cdots, U_M)$ is defined as a solution of the modified upwind method with artificial diffusion

$$\begin{aligned}
\mathcal{L}_h U_i := &- \epsilon\beta(D_+U_i - D_-U_i)\\
&+ b[\{\alpha U_i + (1-\alpha)U_{i+1}\} - \{\alpha U_{i-1} + (1-\alpha)U_i\}]\\
= &\int_{x_{i-1/2}}^{x_{i+1/2}} f(x)dx + \tilde{f}_i, \qquad 1 \leq i \leq M-1,
\end{aligned} \tag{3.3a}$$

$$U_0 = U_M = 0, \tag{3.3b}$$

where

$$\begin{aligned}\tilde{f}_i =& \Big[-\frac{bh^3}{6\epsilon}\{\alpha(\delta-1)^3+(1-\alpha)\delta^3\} + \frac{h^2\beta}{6}\{\delta^3-(\delta-1)^3\}\Big] \\ &\cdot \{f'(x_{i+1/2}) - f'(x_{i-1/2})\} \\ &+ \frac{1}{b}(-bh\beta - \epsilon\beta + \epsilon)(f_{i+1/2} - f_{i-1/2}).\end{aligned}$$

We prove that the scheme (3.3) satisfies the maximum principle which implies that the scheme (3.3) does not give nonphysical oscillation even for large Reynolds numbers.

THEOREM 3.1. *The modified upwind scheme* (3.3) *satisfies the maximum principle.*

PROOF. The equation (3.3) can be expressed as

$$(3.4)\quad A_iU_{i-1}+B_iU_i+C_iU_{i+1} = \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)dx+\tilde{f}_i, \qquad 1 \le i \le M-1,$$

where

$$A_i = -\frac{\epsilon\beta}{h} - b\alpha, \quad B_i = \frac{2\epsilon\beta}{h} + b(2\alpha-1), \quad C_i = -\frac{\epsilon\beta}{h} + b(1-\alpha).$$

Since $\alpha > 1$, we obtain

$$A_i < 0, \qquad B_i > 0, \qquad C_i < 0, \qquad \text{and} \quad A_i + B_i + C_i = 0.$$

Hence the matrix derived in (3.4) is an $M-$matrix. Thus we obtain the maximum principle. □

Integrating each side of (1.1a) from $x_{i-1/2}$ to $x_{i+1/2}$, we obtain the following lemma.

LEMMA 3.3. *For the solution u of* (1.1), *we have, for $1 \le i \le M-1$,*

$$-\epsilon(u'_{i+1/2} - u'_{i-1/2}) + b(u_{i+1/2} - u_{i-1/2}) = \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)dx.$$

LEMMA 3.4. *For the solution u of (1.1), we have, for $1 \le i \le M-1$,*

(1)
$$u_{i+1} = u_{i+1/2} + \delta h u'_{i+1/2} + \frac{1}{2}\delta^2 h^2 (1 + \frac{bh\delta}{3\epsilon}) u''_{i+1/2} - \frac{\delta^3 h^3}{6\epsilon} f'_{i+1/2} + \frac{1}{6}\int_{x_{i+1/2}}^{x_{i+1}} u^4(t)(x_{i+1}-t)^3 dt.$$

(2)
$$u_i = u_{i+1/2} + (\delta-1) h u'_{i+1/2} + \frac{1}{2}(\delta-1)^2 h^2 (1 + \frac{bh(\delta-1)}{3\epsilon}) u''_{i+1/2} - \frac{(\delta-1)^3 h^3}{6\epsilon} f'_{i+1/2} + \frac{1}{6}\int_{x_{i+1/2}}^{x_i} u^4(t)(x_i-t)^3 dt.$$

PROOF. Since the proof of (2) is similar to that of (1), we will only prove (1). Applying Taylor expansion and differentiating (1.1a), we obtain

$$u_{i+1} = u_{i+1/2} + \delta h u'_{i+1/2} + \frac{1}{2}\delta^2 h^2 u''_{i+1/2} + \frac{1}{6}\delta^3 h^3 u'''_{i+1/2} + \frac{1}{6}\int_{x_{i+1/2}}^{x_{i+1}} u^4(t)(x_{i+1}-t)^3 dt, \tag{3.5}$$

$$u'''_{i+1/2} = \frac{b}{\epsilon} u''_{i+1/2} - \frac{1}{\epsilon} f'_{i+1/2}. \tag{3.6}$$

Replacing $u'''_{i+1/2}$ in (3.5) by (3.6), we complete the proof. □

LEMMA 3.5. *For the solution u of (1.1), we have, for $1 \le i \le M-1$,*

$$-\epsilon\beta(D_+u_i - D_-u_i) + b[\{\alpha u_i + (1-\alpha)u_{i+1}\} - \{\alpha u_{i-1} + (1-\alpha)u_i\}] = I_1 + I_2 + I_3 + I_4 + \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)dx + \tilde{f}_i,$$

where

$$I_1 = \frac{1}{6}\left\{-\frac{\epsilon\beta}{h} + b(1-\alpha)\right\}\int_{x_{i+1/2}}^{x_{i+1}} u^{(4)}(t)(x_{i+1}-t)^3dt,$$

$$I_2 = \frac{1}{6}\left(\frac{\epsilon\beta}{h} + b\alpha\right)\int_{x_{i+1/2}}^{x_i} u^{(4)}(t)(x_i-t)^3dt,$$

$$I_3 = \frac{1}{6}\left\{\frac{\epsilon\beta}{h} - b(1-\alpha)\right\}\int_{x_{i-1/2}}^{x_i} u^{(4)}(t)(x_i-t)^3dt,$$

$$I_4 = \frac{1}{6}\left(-\frac{\epsilon\beta}{h} - b\alpha\right)\int_{x_{i-1/2}}^{x_{i-1}} u^{(4)}(t)(x_{i-1}-t)^3dt.$$

PROOF. From definitions of D_+ and D_-, we obtain

(3.7)

$$\begin{aligned}&-\epsilon\beta(D_+u_i - D_-u_i) + b[\{\alpha u_i + (1-\alpha)u_{i+1}\} - \{\alpha u_{i-1} + (1-\alpha)u_i\}]\\ &= \left\{-\frac{\epsilon\beta}{h} + b(1-\alpha)\right\}u_{i+1} + \left(\frac{\epsilon\beta}{h} + b\alpha\right)u_i\\ &\quad + \left\{\frac{\epsilon\beta}{h} - b(1-\alpha)\right\}u_i - \left(\frac{\epsilon\beta}{h} + b\alpha\right)u_{i-1}.\end{aligned}$$

Using Lemma 3.4, (3.7) can be expressed as

(3.8)

$$\begin{aligned}&-\epsilon\beta(D_+u_i - D_-u_i) + b[\{\alpha u_i + (1-\alpha)u_{i+1}\} - \{\alpha u_{i-1} + (1-\alpha)u_i\}]\\ &= b(u_{i+1/2} - u_{i-1/2}) + (-b\beta h - \epsilon\beta)(u'_{i+1/2} - u'_{i-1/2})\\ &\quad + \frac{b}{2}h^2\Big[\alpha(\delta-1)^2 + (1-\alpha)\delta^2 + \frac{bh}{3\epsilon}\{\alpha(\delta-1)^3 + (1-\alpha)\delta^3\}\\ &\qquad - \frac{\beta\epsilon}{bh}(2\delta-1) - \frac{\beta}{3}(3\delta^2 - 3\delta + 1)\Big](u''_{i+1/2} - u''_{i-1/2})\\ &\quad + \left[-\frac{h^3b}{6\epsilon}\{(1-\alpha)\delta^3 + \alpha(\delta-1)^3\} - \frac{h^2\beta}{6}(-3\delta^2 + 3\delta - 1)\right](f'(x_{i+1/2})\\ &\quad - f'(x_{i-1/2})) + I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Since

$$u'_{i+1/2} - u'_{i-1/2} = \frac{1}{b}(f_{i+1/2} - f_{i-1/2}) + \frac{\epsilon}{b}(u''_{i+1/2} - u''_{i-1/2}) \tag{3.9}$$

and the first two terms of the right hand side of (3.8) become

$$\int_{x_{i-1/2}}^{x_{i+1/2}} f(x)dx + (-b\beta h - \epsilon\beta + \epsilon)(u'_{i+1/2} - u'_{i-1/2}),$$

the equation (3.8) is expressed as

(3.10)

$$\begin{aligned}
&-\epsilon\beta(D_+u_i - D_-u_i) + b[\{\alpha u_i + (1-\alpha)u_{i+1}\} - \{\alpha u_{i-1} + (1-\alpha)u_i\}]\\
&= \frac{b}{2}h^2 Q(\alpha)(u''_{i+1/2} - u''_{i-1/2}) + \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)dx\\
&+ \left[-\frac{h^3 b}{6\epsilon}\{(1-\alpha)\delta^3 + \alpha(\delta-1)^3\}\right.\\
&\qquad \left. + \frac{h^2\beta}{6}(3\delta^2 - 3\delta + 1)\right](f'(x_{i+1/2}) - f'(x_{i-1/2}))\\
&+ \frac{1}{b}\{-b\beta h + \epsilon(1-\beta)\}(f_{i+1/2} - f_{i-1/2}) + I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Since $Q(\alpha) = 0$, we obtain the required result from the definition of $\tilde{f}_i$.□

The following lemma is obtained by simple calculation.

LEMMA 3.6. *There exists a positive constant C such that*

$$\mathcal{L}_h x_i \geq Ch, \quad 1 \leq i \leq M-1.$$

THEOREM 3.2. *Let u be the exact solution of* (1.1) *and U the numerical solution of* (3.3). *Let $e = u - U$. Then there exists a constant C such that*

$$\|e\|_\infty \leq C\left(\frac{1}{Re}\right)^2 h^{\frac{5}{2}}\|u^{(4)}\|.$$

PROOF. It follows from (3.3a) and Lemma 3.5 that

$$\begin{aligned}
\mathcal{L}_h e_i &= -\epsilon\beta(D_+u_i - D_-u_i)\\
&\quad + b[\{\alpha u_i + (1-\alpha)u_{i+1}\} - \{\alpha u_{i-1} + (1-\alpha)u_i\}]\\
&\quad - \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)dx - \tilde{f}_i\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Using Hölder inequality, we obtain

$$|I_1| \le \frac{1}{6}\left(\frac{\epsilon\beta}{h} - b + b\alpha\right)\frac{h^{7/2}}{\sqrt{7}}\delta^{7/2}\|u^{(4)}\|_{L^2(x_{i+1/2},x_{i+1})},$$
$$|I_2| \le \frac{1}{6}\left(\frac{\epsilon\beta}{h} + b\alpha\right)\frac{h^{7/2}}{\sqrt{7}}(1-\delta)^{7/2}\|u^{(4)}\|_{L^2(x_i,x_{i+1/2})},$$
$$|I_3| \le \frac{1}{6}\left(\frac{\epsilon\beta}{h} - b + b\alpha\right)\frac{h^{7/2}}{\sqrt{7}}\delta^{7/2}\|u^{(4)}\|_{L^2(x_{i-1/2},x_i)},$$
$$|I_4| \le \frac{1}{6}\left(\frac{\epsilon\beta}{h} + b\alpha\right)\frac{h^{7/2}}{\sqrt{7}}(1-\delta)^{7/2}\|u^{(4)}\|_{L^2(x_{i-1},x_{i-1/2})}.$$

Since $1 < \alpha < 1+\beta$, $\delta = \alpha - \beta$ and $\beta = (\frac{\epsilon}{bh})^2$, we have

$$\begin{aligned} &|I_1 + I_2 + I_3 + I_4| \\ &\le \frac{bh^{\frac{7}{2}}}{6\sqrt{7}}\left\{\beta\left(\frac{\epsilon}{bh}+1\right) + (2\beta)^{\frac{7}{2}}\left(\frac{\epsilon\beta}{bh}+\alpha\right)\right\}\|u^{(4)}\| \\ &\le \frac{bh^{\frac{7}{2}}}{6\sqrt{7}}\{\beta(\beta^{\frac{1}{2}}+1) + (2\beta)^{\frac{7}{2}}(\beta^{\frac{3}{2}}+1+\beta)\}\|u^{(4)}\| \\ &\le C\beta h^{\frac{7}{2}}\|u^{(4)}\|. \end{aligned} \tag{3.11}$$

Thus we have from (3.11) and Lemma 3.7 that

$$|\mathcal{L}_h e_i| \le C\beta h^{\frac{7}{2}}\|u^{(4)}\| \le \mathcal{L}_h(C\beta h^{\frac{5}{2}}\|u^{(4)}\|x_i).$$

We obtain the desired result by the maximum principle of the operator $\mathcal{L}_h$. □

4. Numerical Experiments

In this section, numerical results of the standard upwind scheme (SUS), the standard centered scheme (SCS), the high-order upwind scheme (HOUS) in [6], the weighted central scheme (WCS) in section 2 and the modified upwind scheme (MUS) in section 3 are given. Numerical computations are performed using MAPLE with digits=40.

EXAMPLE 4.1. Consider a problem

$$-\epsilon\frac{d^2u}{dx^2}+\frac{du}{dx}=f, \qquad u(0)=u(1)=0. \tag{4.1}$$

If we take $f(x)=-1$ in (4.1), then the exact solution u of (4.1) is

$$u(x)=-x+\frac{e^{\frac{x}{\epsilon}}-1}{e^{\frac{1}{\epsilon}}-1}.$$

Table 4.1 shows the estimates of error $\|e\|_\infty$ for various grid sizes with large ϵ. We observe that the weighted central scheme (2.1) is $O(h^4)$ accurate and more accurate than the standard upwind scheme and the standard centered scheme. The WCS is not better than HOUS, but it is comparable to HOUS when $f(x)$ is constant.

ϵ	h	SUS	SCS	HOUS	WCS	Re
1000	$\frac{1}{10}$	$.62\times10^{-8}$	$.10\times10^{-12}$	$.17\times10^{-22}$	$.70\times10^{-22}$	.0001
	$\frac{1}{20}$	$.31\times10^{-8}$	$.26\times10^{-13}$	$.11\times10^{-23}$	$.43\times10^{-23}$	.00005
	$\frac{1}{40}$	$.16\times10^{-8}$	$.65\times10^{-14}$	$.68\times10^{-25}$	$.27\times10^{-24}$	.000025
100	$\frac{1}{10}$	$.62\times10^{-6}$	$.10\times10^{-9}$	$.17\times10^{-17}$	$.70\times10^{-17}$	.001
	$\frac{1}{20}$	$.31\times10^{-6}$	$.26\times10^{-10}$	$.11\times10^{-18}$	$.43\times10^{-18}$	.0005
	$\frac{1}{40}$	$.16\times10^{-6}$	$.65\times10^{-11}$	$.68\times10^{-20}$	$.27\times10^{-19}$	.00025
10	$\frac{1}{10}$	$.62\times10^{-4}$	$.10\times10^{-6}$	$.17\times10^{-12}$	$.70\times10^{-12}$	.01
	$\frac{1}{20}$	$.31\times10^{-4}$	$.26\times10^{-7}$	$.11\times10^{-13}$	$.43\times10^{-13}$	.005
	$\frac{1}{40}$	$.16\times10^{-4}$	$.65\times10^{-8}$	$.68\times10^{-15}$	$.27\times10^{-14}$	.0025
1	$\frac{1}{10}$	$.70\times10^{-2}$	$.10\times10^{-3}$	$.17\times10^{-7}$	$.67\times10^{-7}$	.1
	$\frac{1}{20}$	$.30\times10^{-2}$	$.25\times10^{-4}$	$.10\times10^{-8}$	$.42\times10^{-8}$	.05
	$\frac{1}{40}$	$.15\times10^{-2}$	$.63\times10^{-5}$	$.66\times10^{-10}$	$.26\times10^{-9}$	.025
0.1	$\frac{1}{10}$	.13	$.34\times10^{-1}$	$.47\times10^{-3}$	$.20\times10^{-2}$	1
	$\frac{1}{20}$	$.76\times10^{-1}$	$.79\times10^{-2}$	$.31\times10^{-4}$	$.13\times10^{-3}$	.5
	$\frac{1}{40}$	$.42\times10^{-1}$	$.19\times10^{-2}$	$.20\times10^{-5}$	$.80\times10^{-5}$	.25

Table 4.1. $\|e\|_\infty$ with large ϵ.

Table 4.2 shows the estimates of error $\|e\|_\infty$ for various grid sizes with small ϵ. We observe that the modified upwind method with an artificial diffusion (MUS) is comparable to HOUS in [6].

ϵ	h	SUS	SCS	HOUS	WCS	MUS	Re
0.01	$\frac{1}{10}$	.90	.70	$.20\times10^{-2}$	.45	$.41\times10^{-2}$	10
	$\frac{1}{20}$	.16	.44	$.83\times10^{-2}$	.20	$.17\times10^{-1}$	5
	$\frac{1}{40}$	.20	.20	$.68\times10^{-2}$	$.46\times10^{-1}$	$.19\times10^{-1}$	2.5
0.001	$\frac{1}{10}$	$.10\times10^{-1}$	5.0	$.20\times10^{-5}$	2.5	$.58\times10^{-5}$	100
	$\frac{1}{20}$	$.20\times10^{-1}$	1.4	$.16\times10^{-4}$	.93	$.44\times10^{-4}$	50
	$\frac{1}{40}$	$.38\times10^{-1}$	.85	$.13\times10^{-3}$	.73	$.33\times10^{-3}$	25

Table 4.2. $\|e\|_\infty$ with small ϵ.

EXAMPLE 4.2. We now consider the test problem (4.1) when $f(x) = 20\epsilon x^3 - 5x^4$. Then the exact solution u of (4.1) is

$$u(x) = -x^5 + \frac{e^{\frac{x}{\epsilon}} - 1}{e^{\frac{1}{\epsilon}} - 1}.$$

Table 4.3 and Table 4.4 show error $\|e\|_\infty$ for various grid sizes and ϵ. We observe that WCS is more accurate than HOUS when ϵ is large and MUS is comparable to HOUS for small ϵ when $f(x)$ is not constant.

ϵ	h	SUS	SCS	HOUS	WCS	Re
1000	$\frac{1}{10}$	$.63\times10^{-2}$	$.64\times10^{-2}$	$.13\times10^{-8}$	$.70\times10^{-22}$	.0001
	$\frac{1}{20}$	$.15\times10^{-2}$	$.16\times10^{-2}$	$.81\times10^{-10}$	$.43\times10^{-23}$	.00005
	$\frac{1}{40}$	$.40\times10^{-3}$	$.40\times10^{-3}$	$.51\times10^{-11}$	$.27\times10^{-24}$	.000025
100	$\frac{1}{10}$	$.61\times10^{-2}$	$.64\times10^{-2}$	$.13\times10^{-7}$	$.70\times10^{-17}$	.001
	$\frac{1}{20}$	$.15\times10^{-2}$	$.16\times10^{-2}$	$.81\times10^{-9}$	$.43\times10^{-18}$	.0005
	$\frac{1}{40}$	$.33\times10^{-3}$	$.40\times10^{-3}$	$.51\times10^{-10}$	$.27\times10^{-19}$	.00025
10	$\frac{1}{10}$	$.36\times10^{-2}$	$.60\times10^{-2}$	$.13\times10^{-6}$	$.70\times10^{-12}$	.01
	$\frac{1}{20}$	$.38\times10^{-3}$	$.15\times10^{-2}$	$.81\times10^{-8}$	$.43\times10^{-13}$	.005
	$\frac{1}{40}$	$.31\times10^{-3}$	$.38\times10^{-3}$	$.51\times10^{-9}$	$.27\times10^{-14}$	.0025
1	$\frac{1}{10}$	$.18\times10^{-1}$	$.27\times10^{-2}$	$.13\times10^{-5}$	$.67\times10^{-7}$	.1
	$\frac{1}{20}$	$.98\times10^{-2}$	$.67\times10^{-3}$	$.80\times10^{-7}$	$.42\times10^{-8}$	.05
	$\frac{1}{40}$	$.50\times10^{-2}$	$.17\times10^{-3}$	$.50\times10^{-8}$	$.26\times10^{-9}$	.025
0.1	$\frac{1}{10}$	$.18\times10^{-1}$	$.27\times10^{-2}$	$.13\times10^{-5}$	$.67\times10^{-7}$	.1
	$\frac{1}{20}$	$.98\times10^{-2}$	$.67\times10^{-3}$	$.80\times10^{-7}$	$.42\times10^{-8}$	.05
	$\frac{1}{40}$	$.50\times10^{-2}$	$.17\times10^{-3}$	$.50\times10^{-8}$	$.26\times10^{-9}$	.025

Table 4.3. $\|e\|_\infty$ with large ϵ.

ϵ	h	SUS	SCS	HOUS	WCS	MUS	Re
	$\frac{1}{10}$	.11	.65	$.19\times10^{-2}$	.45	$.41\times10^{-2}$	10
0.01	$\frac{1}{20}$	$.72\times10^{-1}$	.42	$.83\times10^{-2}$	.20	$.17\times10^{-1}$	5
	$\frac{1}{40}$	.16	.20	$.68\times10^{-2}$	$.46\times10^{-1}$	$.19\times10^{-1}$	2.5
	$\frac{1}{10}$	.16	4.8	$.20\times10^{-5}$	2.5	$.58\times10^{-5}$	100
0.001	$\frac{1}{20}$	$.85\times10^{-1}$	1.4	$.16\times10^{-4}$	.93	$.44\times10^{-4}$	50
	$\frac{1}{40}$	$.50\times10^{-1}$	.85	$.13\times10^{-3}$	.73	$.33\times10^{-3}$	25

Table 4.4. $\|e\|_\infty$ with small ϵ.

CONCLUDING REMARKS. We have considered two weighted difference schemes with uniform meshes for the convection-diffusion problem and compared them with other schemes. One is a weighted central difference scheme for small Reynolds number and the other is a modified upwind method with artificial diffusion for large Reynolds number not less than 2. We obtain good numerical results for each case depending on Reynolds numbers. The suggested schemes work for the equation (1.1) with a lower order term cu under the condition $b^2 + 4\epsilon c > 0$. Unfortunately, these schemes are not ϵ-uniform. We will present an ϵ-uniform finite difference scheme with uniform meshes for singular convection-diffusion problems elsewhere.

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