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MAP/G/1/K QUEUE WITH MULTIPLE THRESHOLDS ON BUFFER

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Abstract. We consider $MAP/G/1$ finite capacity queue with multiple thresholds on buffer. The arrival of customers follows a Markovian arrival process (MAP). The service time of a customer depends on the queue length at service initiation of the customer. By using the embedded Markov chain method and the supplementary variable method, we obtain the queue length distribution at departure epochs and at arbitrary epochs. This gives the loss probability and the mean waiting time by Little's law. We also give a simple numerical examples to apply the overload control in packetized networks.

1. Introduction

We analyze $MAP/G/1/K$ queue with queue length dependent service times. The arrival of customers is assumed to follow a Markovian arrival process (MAP) introduced by Lucantoni et al. [1]. The MAP is a non-renewal process which includes the phase-type renewal process [2], the Markov-modulated Poisson process (MMPP) [3] and the superpositions of such processes [1] as a particular case. Recently, Asmussen and Koole [4] have shown that the MAP is weakly dense in the class of stationary point processes. Therefore, the MAP is a generalized arrival process. The service time of a customer depends on the queue length at service initiation of the customer. We assume the finite capacity of the buffer to apply the practical applications.

An analysis of such a queueing system with queue length dependent service times is motivated by performance evaluation of a traffic control which often occurs in ATM networks [6]. To obtain better performance

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for voice traffic, Sriram et al. [5,6] proposed the method which discards packets with the less significant information at output of buffer without transmission during congestion. They analyzed such a system by $M/D/1/K$ queueing system with one or two thresholds on buffer by assuming Poisson arrival process. However, the Poisson process may not be suitable to model the superposition of packetized voice traffics with a fair amount of correlation [7].

Recently, as an application for cell discarding scheme in ATM networks, Choi et al. [8] analyzed $MMPP/G/1/K$ queue with one threshold on the buffer. They assumed the arrival to be $MMPP$ for bursty voice traffic. By using the embedded Markov chain method, they analyzed the queueing system and obtained loss probability and mean waiting time. They also gave numerical examples to show the effect of cell discarding and burstiness. When the arrival is Poisson process, Choi et al. [9] also analyzed the same model by using the supplementary variable method and constructed exact solutions for stationary queue length distribution and asymptotic approximations of the solutions which yield simple formulas for performance measures such as loss rates and tail probabilities. This paper is an extension of the queueing system analyzed by Choi et al. [8,9].

In Section 2, we describe the MAP and model in details. In Section 3, we analyze the queueing system using the embedded Markov chain method and obtain the queue length distribution at departure epochs. Then, by using of the supplementary variable method, we give the queue length distribution at an arbitrary time, the loss probability and the mean waiting time. In Section 4, we give some simple numerical examples and discussion.

2. Preliminary for analysis

We describe the Markovian arrival process (MAP) introduced by Lucantoni et al. [1]. Consider a continuous-time Markov process on the state space $\{1, 2, \cdots, m, m+1\}$, where the states $1, \cdots, m$ are transient and the state $m+1$ is absorbing. Absorption, starting from any state, is certain. The epoch of absorption corresponds to an arrival of the MAP. The Markov process is instantaneously restarted in a transient state, where the selection of the new state depends on the state from which absorption occurred. When the Markov process is in a transient state $i, 1 \le i \le m$, the sojourn time in the state i is exponentially distributed with parameter λ_i. Let $p_{i,j}$ be the probability that the process enters the absorbing state from state i and is immediately restarted in state j, $1 \le i, j \le m$ and $q_{i,j}$ be the probability that the process enters another state j from state i, without being absorbed, $1 \le i, j \le m$ and $i \ne j$. Note that

$$\sum_{\substack{j=1 \\ j \ne i}}^{m} q_{i,j} + \sum_{j=1}^{m} p_{i,j} = 1, \qquad 1 \le i \le m.$$

It is convenient to represent the evolution of the system in terms of matrices C and D given below. The matrices C and D with $C_{i,j}$ and $D_{i,j}$ as (i, j)-elements are given by

$$\begin{aligned} C_{i,j} &= \lambda_i q_{i,j}, \quad i \ne j, \qquad C_{i,i} = -\lambda_i, \\ D_{i,j} &= \lambda_i p_{i,j}, \qquad 1 \le i, j \le m. \end{aligned}$$

Then, the irreducible matrix $C + D$ is the infinitesimal generator of the Markov process restricted to the states $\{1, \cdots, m\}$, called the underlying Markov process. We assume the matrix C is nonsingular. In other words C is a stable matrix (i.e., all of its eigenvalues have negative real parts; see e.g., pp. 251 of Bellman [10]). This implies that the interarrival times are finite with probability one (see Lemma 2.2.1 of Neuts [2]) and that the arrival process does not terminate. Let $J(t)$ be the state of the underlying Markov process at time t and π be its stationary probability vector. The probability vector π is given by solving the equations

$$\pi(C + D) = 0, \qquad \pi e = 1,$$

where e is a column vector with all elements equal to one.
Then, the mean arrival rate λ^* of the MAP is given by $\lambda^* = \pi De$. As a special cases of the MAP we obtain the following:

(a) Poisson process with rate λ. In this case, $C = -\lambda, D = \lambda$.
(b) PH-renewal process. The phase type (PH) renewal process is introduced in Neuts [2]. A PH-renewal process with representation (α, T) is a MAP with $C = T, D = -Te\alpha$.

(c) Markov-modulated Poisson process (MMPP). The MMPP is a doubly stochastic Poisson process whose arrival rate is given by $\lambda[J(t)] \geq 0$, where $J(t), t \geq 0$, is an m-state irreducible Markov process. The arrival rate therefore takes on only m values $\lambda_1, \cdots, \lambda_m$, and is equal to λ_j whenever the Markov process is in the state j. If the underlying Markov process has infinitesimal generator R and $\Lambda = diag(\lambda_1, \cdots, \lambda_m)$, then with $C = R - \Lambda, D = \Lambda$. The MMPP is a MAP.

(d) A superposition of MAP's. The class of MAP's is closed under superposition. That is, the superposition of n independent MAP's with representation $\{C_i, D_i\}, 1 \leq i \leq n$ is also a MAP with representation $\{C_1 \oplus \cdots \oplus C_n, D_1 \oplus \cdots \oplus D_n\}$ where " $\oplus$ " denotes the matrix Kronecker sum [10].

Let $M(t)$ be the number of customers arriving during the interval $(0, t]$. Now we define the conditional probabilities

$$p_{i,j}(n,t) = Pr\{M(t) = n, J(t) = j | M(0) = 0, J(0) = i\},$$
$$n \geq 0, \ 1 \leq i, j \leq m.$$

By the Chapman-Kolmogorov's forward equation, we have the following set of the differential-difference equations for the $m \times m$ matrix $P(n,t) \triangleq (p_{i,j}(n,t))_{1 \leq i,j \leq m}$,

$$P^{'}(n,t) = P(n,t)C + P(n-1,t)D, \qquad n \geq 0, \quad t \geq 0,$$

where $P(-1,t)$ is the matrix 0.
By simple calculation, it is easily shown that the matrix $P(n,t)$ have the probability generating function

$$\overline{P}(z,t) \triangleq \sum_{n=0}^{\infty} P(n,t) z^n$$
$$= e^{(C+zD)t}, \qquad |z| \leq 1, \ t \geq 0.$$

Customers arrive at the queue in accordance with the MAP described above. There is a single server and queue with finite capacity K (including customer in service), so that the customers arriving when the queue is full are lost. Customers in the queue are served on the first-come first-service basis. The service time of a customer depends on the queue length at service initiation epoch of the customer. Concretely, we place the threshold values $L_k(L_i \leq L_j,\ i < j,\ 1 \leq i, j \leq T)$ on the queue. If the queue length at service initiation of a customer is greater than or equal to the threshold L_{k-1} and less than the threshold L_k, the service time of the customer is $S_k(k = 1, 2, \cdots, T)$, where $L_0 = 1, L_T = K$. The service time S_k has the distribution function G_k with mean s_k and Laplace-Stieltjes transform $G_k^*(s)$.

3. Analysis

3.1 The stationary queue length distribution at departure epochs

By using the embedded Markov chain method, we first derive the queue length distribution just after departure epochs. Let $\tau_n(n \geq 1)$ be the epoch of successive departures with $\tau_0 = 0$. We also introduce the following notations:

$$N_n = \text{ the queue length at time } \tau_n+,$$
$$J_n = \text{ the state of the underlying Markov process at time } \tau_n + .$$

Then, the process $\{(N_n, J_n), n \geq 0\}$ forms a Markov chain with finite state space $\{0, 1, \cdots, K-1\} \times \{1, 2, \cdots, m\}$. In order to derive the stationary probability distribution of the Markov chain $\{(N_n, J_n), n \geq 0\}$, define the probabilities $x_{k,i}$ and the vectors as

$$x_{k,i} = \lim_{n \to \infty} Pr\{N_n = k, J_n = i\}, \quad 0 \leq k \leq K-1, \quad 1 \leq i \leq m.$$
$$x = (x_0, x_1, \cdots, x_{K-1}) \text{ with } x_k = (x_{k,1}, x_{k,2}, \cdots, x_{k,m}).$$

The one-step transition probability matrix $\overline{Q}$ of the Markov chain $\{(N_n, J_n), n \geq 0\}$ is given as follows:

$$\overline{Q} =$$

$$\left|\begin{array}{ccccccccccc}
A'_0 & A'_1 & \cdots & A'_{L_1-1} & A'_{L_1} & \cdots & A'_{L_k-1} & A'_{L_k} & \cdots & A'_{K-2} & \overline{A}'_{K-1} \\
A^1_0 & A^1_1 & \cdots & A^1_{L_1-1} & A^1_{L_1} & \cdots & A^1_{L_k-1} & A^1_{L_k} & \cdots & A^1_{K-2} & \overline{A}^1_{K-1} \\
0 & A^1_0 & \cdots & A^1_{L_1-2} & A^1_{L_1-1} & \cdots & A^1_{L_k-2} & A^1_{L_k-1} & \cdots & A^1_{K-3} & \overline{A}^1_{K-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A^1_1 & A^1_2 & \cdots & A^1_{L_k-L_1+1} & A^1_{L_k-L_1+2} & \cdots & A^1_{K-L_1} & \overline{A}^1_{K-L_1+1} \\
0 & 0 & \cdots & A^2_0 & A^2_1 & \cdots & A^2_{L_k-L_1} & A^2_{L_k-L_1+1} & \cdots & A^2_{K-L_1-1} & \overline{A}^2_{K-L_1} \\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & A^k_1 & A^k_2 & \cdots & A^k_{K-L_k} & \overline{A}^k_{K-L_k+1} \\
 & & & & & & A^{k+1}_0 & A^{k+1}_1 & \cdots & A^{k+1}_{K-L_k-1} & \overline{A}^{k+1}_{K-L_k} \\
 & & & & & & \vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & & & 0 & 0 & \cdots & A^T_1 & \overline{A}^T_2 \\
 & & & & & & 0 & 0 & \cdots & A^T_0 & \overline{A}^T_1
\end{array}\right|$$

where the blocks A^r_k are given by

$$A^r_k = \int_0^\infty P(k,x)dG_r(x), \qquad 1 \le r \le T, \quad k \ge 0,$$

: the (i,j)-element of the block A^r_k is the conditional probability that there are k customers arriving during the service time S_r and the state of the underlying Markov chain is j at the next departure epoch, given that there are at least one customer in the queue and the state of the underlying Markov chain is i just after a departure, the blocks A'_k are given by

$$A'_k = \int_0^\infty P(0,t)Ddt\ A^1_k \ = \ -C^{-1}DA^1_k, \qquad k \ge 0,$$

: the (i,j)-element of block A'_k is the conditional probability that there are k customers arriving during the service time S_1 and the state of the underlying Markov chain is j at the next departure epoch, given that the system is empty and the state of the underlying Markov chain is i just after a departure, and the blocks $\overline{A}_k^r$ are as follows:

$$\overline{A}'_{K-1} = \sum_{n=K-1}^{\infty} A'_n, \qquad \overline{A}_k^r = \sum_{n=k}^{\infty} A_n^r, \quad 1 \le r \le T, \ k \ge 0.$$

The stationary probability vector x of the Markov chain $\{(N_n, J_n), n \ge 0\}$ is given by solving the equations

$$x\,\overline{Q} = x, \qquad x\,e = 1.$$

3.2 The stationary queue length distribution at an arbitrary time

In this section we derive the formulas for the stationary queue length distribution at an arbitrary time. For an arbitrary time t, let $N(t)$ and $J(t)$ be the queue length (including customer in service) and the state of the underlying Markov process respectively. Furthermore we introduce the following notations:

$R(t) = r$ if the service time of the customer in service at time t is S_r, $\quad r = 1, \cdots, T$,

$$\xi(t) = \begin{cases} 0 & \text{if the server is idle at time } t, \\ 1 & \text{if the server is busy at time } t. \end{cases}$$

The quantities of interest are the steady state probabilities:

$$\begin{aligned} y(0,j) &= \lim_{t\to\infty} Pr\{N(t) = 0, J(t) = j, \xi(t) = 0\}, \quad 1 \le j \le m, \\ y_0 &= (y(0,1), y(0,2), \cdots, y(0,m)), \\ y^r(n,j) &= \lim_{t\to\infty} Pr\{N(t) = n, J(t) = j, R(t) = r, \xi(t) = 1\}, \\ y_n^r &= (y^r(n,1), y^r(n,2), \cdots, y^r(n,m)), \quad r = 1, \cdots, T, \\ y_n &= \sum_{r=1}^{T} y_n^r, \qquad n \ge 1. \end{aligned}$$

First, we compute the vector y_0 that the system is empty. The j-th component $y(0,j)$ of y_0 is derived by a application of the key renewal theorem [8]:

$$(1)\qquad y(0,j)=\sum_{k=1}^{m}\frac{1}{m(0,k)}\int_0^\infty p_{k,j}(0,t)dt,$$

where $m(0,k)$ is the mean recurrence time of the state $(0,k)$ in the Markov chain $\{(N_n,J_n),n\geq 0\}$ given by

$$(2)\qquad m(0,k)=\frac{1}{x_{0,k}}\sum_{n=0}^{K-1}\sum_{j=1}^{m}x_{n,j}E[\tau_l-\tau_{l-1}|\ (N_{l-1},J_{l-1})=(n,j)].$$

By the fact that the mean duration of an idle period starting in state j of the underlying Markov chain $\{J_n,n\geq 0\}$ is

$$\int_0^\infty P(0,t)dte=-C^{-1}e,$$

we easily have

$$(3)\qquad m(0,k)=Ex_{0,k}^{-1},$$

where $E=x_0(-C^{-1})e+s_1x_0e+\sum_{k=1}^{T}s_k\sum_{n=L_{k-1}}^{L_k-1}x_ne$ is the mean interdeparture time of customers.
Substituting (3) into (1), we finally obtain the vector y_0 as

$$(4)\qquad y_0=\frac{1}{E}x_0(-C^{-1}).$$

In order to obtain the stationary queue length distribution $\{y_n,n\geq 1\}$ at an arbitrary time when the server is busy, we use the supplementary variable method. Let $\hat{T}$ and $\tilde{T}$ be the remaining and the elapsed service time for the customer in service respectively. We define the joint probability distribution of the queue length and the remaining service time for the customer in service at arbitrary time τ as

$$\begin{aligned}&\alpha_r(n,j,t)dt\\&=Pr\{N(\tau)=n,J(\tau)=j,R(\tau)=r,t<\hat{T}\leq t+dt,\xi(\tau)=1\}.\end{aligned}$$

We also define the Laplace transform of $\alpha_r(n,j,t)$

$$\alpha_r^*(n,j,s) = \int_0^\infty e^{-st}\alpha_r(n,j,t)dt, \quad n \geq 1, \quad r = 1,\cdots,T,$$

and the vectors

$$\alpha_r^*(n,s) = (\alpha_r^*(n,1,s),\cdots,\alpha_r^*(n,m,s)), \quad r = 1,\cdots,T,$$
$$\alpha^*(n,s) = \sum_{r=1}^{T}\alpha_r^*(n,s), \qquad n \geq 1.$$

For calculation of $\alpha^*(n,s)$, we need to know the probability of arrivals during the elapsed service time $\tilde{T}$. Define the conditional probability $\beta_r(n,j_1,j_2,t)dt$ as

$$\begin{aligned}&\beta_r(n,j_1,j_2,t)dt\\ &= Pr\{\ n \text{ arrivals during } \tilde{T}, J(\tau) = j_2, R(\tau) = r,\\ &\qquad t < \hat{T} \leq t + dt \mid J(\bar{\tau}) = j_1\}, \quad n \geq 0, \quad r = 1,\cdots,T,\end{aligned}$$

where $\bar{\tau}$ is the starting time of the service time which includes the time τ.
Let $\beta_r^*(n,j_1,j_2,s)$ be the Laplace transform of $\beta_r(n,j_1,j_2,t)$ and $\beta_r^*(n,s)$ the matrix with $\beta_r^*(n,j_1,j_2,s)$ as (j_1,j_2)-element:

$$\begin{aligned}\beta_r^*(n,j_1,j_2,s) &= \int_0^\infty e^{-st}\beta_r(n,j_1,j_2,t)dt,\\ \beta_r^*(n,s) &= (\beta_r^*(n,j_1,j_2,s))_{1\leq j_1,j_2\leq m}, \quad n \geq 0, \quad r = 1,\cdots,T.\end{aligned}$$

Then, the vectors $\alpha_r^*(n,s)$ satisfy the following equations:

(5a)

$$\alpha_1^*(n,s) = \frac{s_1}{E}\left[x_0(-C^{-1})D\beta_1^*(n-1,s) + \sum_{k=1}^{\min(n,L_1-1)} x_k\beta_1^*(n-k,s)\right],$$
$$1 \leq n \leq K-1,$$

(5b)

$$\alpha_1^*(K,s) = \frac{s_1}{E}\left[x_0(-C^{-1})D\left\{\sum_{l=K-1}^{\infty}\beta_1^*(l,s)\right\} + \sum_{k=1}^{L_1-1} x_k\left\{\sum_{l=K-k}^{\infty}\beta_1^*(l,s)\right\}\right].$$

(6a)

$$\alpha_r^*(n,s) = \frac{s_r}{E} \sum_{k=L_{r-1}}^{\min(n,L_r-1)} x_k \beta_r^*(n-k,s), \qquad L_{r-1} \le n \le K-1,$$

(6b)

$$\alpha_r^*(K,s) = \frac{s_r}{E} \sum_{k=L_{r-1}}^{L_r-1} x_k \left[\sum_{l=K-k}^{\infty} \beta_r^*(l,s) \right],$$

$$\alpha_r^*(n,s) = 0, \text{ for all } n \text{ else.}$$

We finally obtain that

(7)

$$\begin{aligned} \alpha^*(n,s) &= \sum_{r=1}^{T} \alpha_r^*(n,s) \qquad 1 \le n \le K-1 \\ &= \frac{1}{E} \left\{ s_1 x_0 (-C^{-1}) D \beta_1^*(n-1,s) + \sum_{r=1}^{T} s_r \sum_{k=L_{r-1}}^{\min\{n,L_r-1\}} x_k \beta_r^*(n-k,s) \right\}. \end{aligned}$$

As shown in Appendix, $\beta_r^*(n,s)$ is given as follows:

$$\text{(8)} \quad \beta_r^*(n,s) = \frac{1}{s_r} \left[\sum_{l=0}^{n} A_l^r R_{n-l}(s) - G_r^*(s) R_n(s) \right], \qquad r = 1, \cdots, T,$$

where $R_n(s) = (sI+C)^{-1}(-D(sI+C)^{-1})^n$.

Finally, substituting $\beta_r^*(n,s)(r = 1, \cdots, T)$ into (7), we obtain the results.

THEOREM. *The stationary queue length probabilities $y_n = \alpha^*(n,0)$ is given by*

For $1 \le n < K$,

$$(9) \qquad y_n = \frac{1}{E}\Bigg[x_0(-C^{-1})D\sum_{l=0}^{n-1} A_l^1 C^{-1}\{D(-C^{-1})\}^{n-1-l} - \sum_{k=0}^{n} x_k C^{-1}\{D(-C^{-1})\}^{n-k} + \sum_{r=1}^{T}\sum_{k=L_{r-1}}^{\min\{n,L_r-1\}} x_k \sum_{l=0}^{n-k} A_l^r C^{-1}\{D(-C^{-1})\}^{n-k-l}\Bigg],$$

where $\sum_{k=a}^{b} x_k = 0$ if $b < a$, and

$$(10) \qquad y_K = \pi - \sum_{k=0}^{K-1} y_k.$$

Using the stationary queue length distribution $\{y_n, n \ge 0\}$, we obtain the following performance measures:

a. The loss probability:

$$P_{\text{loss}} = \frac{y_K De}{\sum_{i=0}^{K} y_i De} = \frac{y_K De}{\pi De}.$$

b. The mean queue length:

$$M = \sum_{i=0}^{K} i y_i e.$$

c. By Little's law, we obtain the mean waiting time in the system:

$$W = \frac{M}{\lambda^*(1 - P_{\text{loss}})}.$$

4. Numerical examples and discussion

In this section, we present some simple numerical examples for performance measures of our queueing system. We put two threshold values L_1 and L_2 on buffer and investigate loss probability and mean waiting time of our queueing system and the uncontrolled system. As an input process for numerical examples, we use a two-state MAP with C and D given by

$$C = \begin{bmatrix} -q_{12} - a_1 & q_{12} \\ q_{21} & -q_{21} - a_2 \end{bmatrix}, \qquad D = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}.$$

The infinitesimal generator of the underlying Markov process is the sum $(C+D)$ of the matrices C and D. Then, the mean arrival rate λ^* of the MAP is given by

$$\lambda^* = \frac{q_{21}a_1 + q_{12}a_2}{q_{12} + q_{21}}.$$

For the numerical examples, we take the buffer size $K = 12, q_{12} = 0.1, q_{21} = 0.2$, and the deterministic service times $S_1 = 3, S_2 = 2, S_3 = 1 (S_1 = S_2 = S_3 = 3$ or $L_1 = L_2 = B$ for uncontrolled system without overload control).

In figures below we display the loss probability and the mean waiting time when $L_1 = 3, L_2 = 6$ and $L_1 = 5, L_2 = 8$ in our queueing system, in which keeping $a_2/a_1 = 4$.

Fig. 1 displays the loss probability as a function of the mean arrival rate λ^*. The loss probability of the system with overload control (our queueing system) is improved considerly compared with the uncontrolled system, and also the small threshold values give low loss probability than the large threshold values.

Fig. 2 also displays the mean waiting time as a function of the mean arrival rate. We can see same results as in Fig. 1. Since very small threshold values may cause congestion of network by fast transmission, the threshold values must be taken by considering congestion of the network in applying the overload control in ATM networks. Therefore, appropriate threshold values must be given by considering Quality of Service of traffic and congestion of network.

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Appendix

Consider at an arbitrary point of time that the server is busy and the service time is S_r. This sampling point will fall in one of the service time (S_r^0) in the sequence of service time $\{S_r^i, i \geq 1\}$ which is independent and identically distributed with distribution function $G_r(\cdot)$ and density function $g_r(\cdot)$. Let s_r and $G_r^*(s)$ be the mean and the Laplace transform of the distribution function $G_r(\cdot)$. This particularly selected service time S_r^0 has density function $g_r^0(x) = xg_r(x)/s_r$, where $g_r^0(x)$ is the density of S_r^0. The remaining service time $\hat{T}$ and the elapsed service time $\tilde{T}$ have the same distribution function, that is, $\hat{T}^*(s) = \tilde{T}^*(s) = [1 - G_r^*(s)]/ss_r$, and the conditional distribution

$$E[e^{-s\hat{T}}|S_r^0] = (1 - e^{-sS_r^0})/sS_r^0.$$

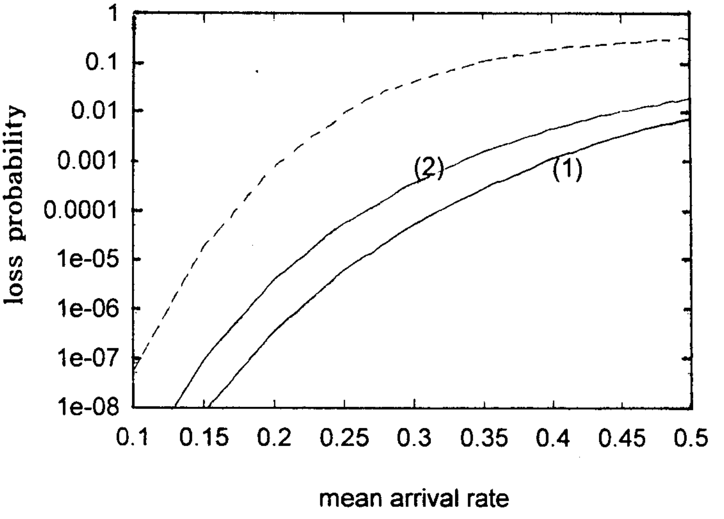

Fig. 1 Loss probability against mean arrival rate

········ : uncontrolled system

——(1) : $L_1=3, L_2=6$

——(2) : $L_1=5, L_2=8$

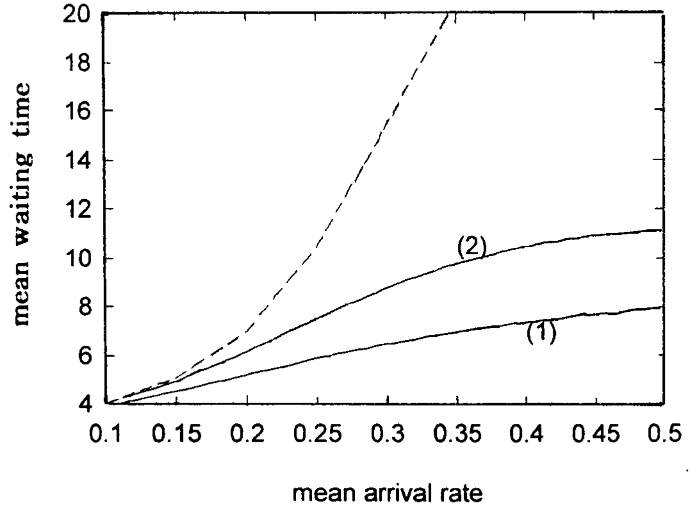

Fig. 2 Mean waiting time against mean arrival rate

········ : uncontrolled system

——(1) : $L_1=3, L_2=6$

——(2) : $L_1=5, L_2=8$

Let $R(z) = C + zD$. Then, $\beta_r^*(n,s)$ can be derived as follows:

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\beta_r^*(n,s)z^n \\
&= E\left[e^{-s\hat{T}}e^{R(z)\tilde{T}}\right] \\
&= E\left[E[e^{-s\hat{T}}e^{R(z)\tilde{T}}|S_r^0]\right] = E\left[e^{R(z)S_r^0}E\left[e^{-(sI+R(z))\tilde{T}}|S_r^0\right]\right] \\
&= E\left[(e^{R(z)S_r^0} - e^{-sS_r^0})/S_r^0\right](sI+R(z))^{-1} \\
&= \int_0^{\infty}\left[(e^{R(z)x} - e^{-sx})/x\right]\frac{xg_r(x)}{\mu}dx(sI+R(z))^{-1} \\
&= \frac{1}{s_r}\int_0^{\infty}(e^{R(z)x} - e^{-sx})g_r(x)dx(sI+R(z))^{-1} \\
&= \frac{1}{s_r}\left[\sum_{n=0}^{\infty}A_n^r z^n - G_r^*(s)I\right]\left[\sum_{n=0}^{\infty}R_n(s)z^n\right] \\
&= \frac{1}{s_r}\left[\sum_{n=0}^{\infty}\sum_{k=0}^{n}A_k^r R_{n-k}(s) - \sum_{n=0}^{\infty}G_r^*(s)R_n(s)\right]z^n,
\end{aligned}
$$

where $R_n(s) = (sI+C)^{-1}[(-D)(sI+C)^{-1}]^n$.
By coefficient comparison, we have

$$\beta_r^*(n,s) = \frac{1}{s_r}\left[\sum_{k=0}^{n}A_k^r R_{n-k}(s) - G_r^*(s)R_n(s)\right], \qquad 1 \le r \le T.$$

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