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# A CLT FOR WEAKLY DEPENDENT RANDOM FIELDS

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ABSTRACT. In this article we prove a central limit theorem for strictly stationary weakly dependent random fields with some interlaced mixing conditions. Mixing coefficients are not assumed. The result is basically the same to Peligrad ([4]), which is a CLT for weakly dependent arrays of random variables. The proof is quite similar to that of Peligrad.

## 1. Introduction

Let $d$ be a positive integer. Suppose $X = (X_k, k \in \mathbf{Z}^d)$ is a centered strictly stationary random field on a probability space $(\Omega, F, P)$. Let us denote the usual Euclidean norm of a vector $k = (k_1, \dots, k_d) \in \mathbf{Z}^d$ by $||k||$ and $|||k||| = k_1 \cdots k_d$. The distance between two disjoint nonempty subsets $S, T \subset \mathbf{Z}^d$ will be denoted by

$$\mathrm{dist}(S,T) = \min_{j \in S, k \in T} ||j - k||.$$

Let $\mathcal{A}, \mathcal{B}$ be two sub $\sigma$-algebras of $F$. Define the strong mixing coefficient by

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(AB) - P(A)P(B)| \tag{1}$$

and the maximal coefficient of correlation by

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{f \in L^2(\mathcal{A}), g \in L^2(\mathcal{B})} |\mathrm{corr}(f, g)|. \tag{2}$$

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Then the following inequality is elementary

$$\alpha(\mathcal{A},\mathcal{B}) \leq \frac{1}{4}\rho(\mathcal{A},\mathcal{B}).$$

We may extend the definitions to a centered strictly stationary random filed $X = (X_t, t \in \mathbf{R}^d)$ on $(\Omega, F, P)$. The distance between any two disjoint nonempty subsets $S, T \subset \mathbf{R}^d$ will be denoted by $\mathrm{dist}(S,T) = \inf\{||s - t|| \ : s \in S, t \in T\}$. For disjoint nonempty sets $S$ and $T$, we use the abbreviations

$$\alpha(S,T) = \alpha(\sigma(X_t, t \in S), \sigma(X_t, t \in T))$$

$$\rho(S,T) = \rho(\sigma(X_t, t \in S), \sigma(X_t, t \in T)),$$

where $\sigma(X_t, t \in S)$ denotes the $\sigma$-field generated by $\{X_t, t \in S\}$. For any real number $r > 0$, define the following dependence coefficients for the given random field $X = (X_t, t \in \mathbf{R}^d)$

$$\alpha(r) = \sup \alpha(S,T) \tag{3}$$

$$\rho(r) = \sup \rho(S,T), \tag{4}$$

where in both (3) and (4) the supremum is taken over all pairs of disjoint closed $d$-dimensional half-spaces $S$ and $T \subset \mathbf{R}^d$ with $\mathrm{dist}(S,T) \geq r$. For any real $r > 0$, define also

$$\alpha^*(r) = \sup \alpha(S,T), \qquad S, T \subset \mathbf{R}^d, \ \ \mathrm{dist}(S,T) \geq r$$

$$\rho^*(r) = \sup \rho(S,T), \qquad S, T \subset \mathbf{R}^d, \ \ \mathrm{dist}(S,T) \geq r.$$

Comparing the definitions of $\alpha(r), \rho(r), \alpha^*(r)$, and $\rho^*(r)$, the followings are obvious that $\alpha(r) \leq \alpha^*(r)$, and $\rho(r) \leq \rho^*(r)$. We state some known results.

THEOREM 1. *Suppose $d \geq 2$ and $X = (X_t, t \in \mathbf{R}^d)$ is a strictly stationary random field which is mixing and $r > 0$ is a real number. Then the following statements hold* (Bradley ([2])).

$$\begin{array}{ll} (a) & \alpha(r) \leq \rho(r) \leq 2\pi\alpha(r) \\ (b) & \alpha(r) = \frac{1}{4} \Leftrightarrow \rho(r) = 1 \\ (c) & \alpha^*(r) \leq \rho^*(r) \leq 2\pi\alpha^*(r) \\ (d) & \alpha(r) = \frac{1}{4} \Leftrightarrow \rho^*(r) = 1. \end{array}$$

It is not hard to notice that the condition $\alpha(r) \to 0$ implies that $X$ is strong mixing. Therefore by Theorem 1 the conditions $\alpha(r) \to 0$ and $\rho(r) \to 0$ are equivalent to each other for strictly stationary random fields $X = (X_t, t \in \mathbf{R}^d), d \geq 2$. Also we may say the conditions $\alpha^*(r) \to 0$ and $\rho^*(r) \to 0$ are equivalent to each other. Note that, for strictly stationary mixing random processes $\{X_t : t \in \mathbf{R}\}$ or $\{X_k : k \in \mathbf{Z}\}$, $(a)$ and $(b)$ are not true in general. Bradley ([2]) proved for the strictly stationary sequences that the condition $\alpha^*(n) \to 0$ as $n \to \infty$ contains enough information to assure the CLT without any additional rate or moment conditions higher than 2. We state the result.

THEOREM 2. *If $(X_k, k \in \mathbf{Z})$ is a strictly stationary sequence of real centered square integrable random variables such that $\sigma_n^2 = \mathrm{var}(\sum_{i=1}^n X_i) \to \infty$, and $\alpha^*(n) \to 0$ as $n \to \infty$, then*

$$\frac{\sum_{i=1}^{n} X_i}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \to \infty.$$

Consider some notations for $d$-dimensional block sum. Suppose $j = (j_1, \dots, j_d)$ and $l = (l_1, \dots, l_d)$ are elements in $\mathbf{Z}^d$ such that $j_u \leq l_u, \ u = 1, \dots, d$. Denote

$$S(X : j_1, \dots, j_d; l_1, \dots, l_d) = \sum_k X_k,$$

where the sum is taken over all $k = (k_1, \dots, k_d) \in \mathbf{Z}^d$ such that $j_u \leq k_u \leq l_u, \ u = 1, \dots, d$. For positive integers $l_1, \dots, l_d$, we use the simple notations

$$S(X : l) = S(X : l_1, \dots, l_d) = S(X : 1, \dots, 1; l_1, \dots, l_d).$$

Let $\mathbf{N}$ be the set of all positive integers. Considering the asymptotic normality of $S(X : n)$ as $n \in \mathbf{N}^d$ becomes large in the sense of the usual Euclidean norm, Bradley have proved CLT for the strictly stationary random fields with finite second moments and the corresponding unrestricted $\rho$-mixing conditions. Also no mixing rate is assumed. We state the result which is appeared in Bradley ([1]).

THEOREM 3. *Suppose $d$ is a positive integer and $X = (X_k, k \in \mathbf{Z}^d)$ is a centered real strictly stationary random field such that $0 < EX_0^2 < \infty, \rho^*(r) \to 0$ as $r \to \infty$, and the continuous spectral density $f$ of $X$ on*

*$T^d$ satisfies $f(1,\dots,1)>0$. Then as $||n||\to\infty$, it follows that $||S(X:n)||_2\to\infty$ and*

$$\frac{S(X:n)}{||S(X:n)||_2}\xrightarrow{d}\mathcal{N}(0,1).$$

In Theorem 3, $T$ is the unit circle in the complex plane and $f$ on $T^d$ is a spectral density function for $X=(X_k,k\in\mathbf{Z}^d)$ and $||\cdot||_2$ is the $L^2$ norm. Peligrad ([4]) proved CLT for strongly mixing sequences satisfying the Lindeberg condition and an additional assumption imposed to an interlaced mixing coefficient. The conditions used in the theorem in Peligrad ([4]) are weaker than the conditions used in Theorem 2 and Theorem 3 above. Consider a triangular arrays of strongly mixing random variables, $\{\xi_{ni},1\le i\le k_n\}$ where $k_n\to\infty$. We shall define the following

$$\overline{\alpha}_{nk}=\sup_{s\ge 1}\alpha(\sigma(\xi_{ni},i\le s),\sigma(\xi_{nj},j\ge s+k)) \tag{5}$$

and $\overline{\alpha}_k=\sup_n\overline{\alpha}_{nk}$. The array will be called strongly mixing if $\overline{\alpha}_k\to 0$ as $k\to\infty$. Similarly we define

$$\overline{\rho}^*_{nk}=\sup_k\rho(\sigma(\xi_{ni},i\in T),\sigma(\xi_{nj},j\in S)), \tag{6}$$

where $T,S\subset\{1,2,\dots,k_n\}$ are nonempty subsets with $\mathrm{dist}(T,S)\ge k$ and

$$\overline{\rho}^*_k=\sup_n\overline{\rho}^*_{nk}. \tag{7}$$

We state the theorem proved by Peligrad ([4]).

THEOREM 4. *Let $\{\xi_{ni};1\le i\le k_n\}$ be a triangular array of centered random variables, which is strongly mixing and have finite second moments. Assume that $\lim_{k\to\infty}\overline{\rho}^*_k<1$. Denote by $\sigma_n^2=\mathrm{var}(\sum_{i=1}^{k_n}\xi_{ni})$ and assume*

$$\sup_n\frac{1}{\sigma_n^2}\sum_{i=1}^{k_n}E\xi_{ni}^2<\infty \tag{8}$$

*and for any $\varepsilon>0$*

$$\frac{1}{\sigma_n^2}\sum_{i=1}^{k_n}E\xi_{ni}^2I(|\xi_{ni}|>\varepsilon\sigma_n)\to 0\quad\text{as}\quad n\to\infty. \tag{9}$$

*Then*

$$\frac{\sum_{i=1}^{k_n} \xi_{ni}}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as} \quad n \to \infty. \tag{10}$$

Note that the stationarity condition is not assumed in Theorem 4. When we assume the stationary condition on a sequence of random variables we can formulate a theorem as follows, from Corollary 2.3 of Peligrad ([4]),

THEOREM 5. *Suppose $\{X_k, k \in \mathbf{N}\}$ is a strongly mixing strictly stationary sequence of random variables which is centered and has finite second moments. Assume that $\lim_{n\to\infty} \rho^*(n) < 1$ and $\sigma_n^2 \to \infty$. Then*

$$\liminf \frac{\sigma_n^2}{n} > 0 \tag{11}$$

*and*

$$\frac{\sum_{k=1}^{n} X_k}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } n \to \infty. \tag{12}$$

## 2. The Main Result and Lemmas

We note that, since $\alpha(n) \leq \alpha^*(n)$, the mixing condition $\alpha^*(n) \to 0$ used in Theorem 2 is stronger than those conditions strong mixing and $\lim_{n\to\infty} \rho^*(n) < 1$ used in Theorem 5. We shall consider a strictly stationary random field $X = \{X_n, n \in \mathbf{Z}^d\}$ which are centered and have finite second moments. Also we impose the interlaced mixing conditions which is quite similar to those in Theorem 5. Then we have the following result.

THEOREM 6. *Let $X = \{X_n, n \in \mathbf{Z}^d\}$ be a strictly stationary random field which is centered and has finite second moments. Assume that*

(a) $\alpha(r) \to 0 \quad \text{as } r \to \infty$

(b) $\lim_{r\to\infty} \rho^*(r) < 1$

(c) $\sigma_n^2 = \text{var} S(X : n) \to \infty \quad \text{as } ||n|| \to \infty.$

*Then*

$$\liminf \frac{\sigma_n^2}{|||n|||} > 0$$

*and*

$$\frac{S(X:n)}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } ||n|| \to \infty.$$

The proof of Theorem 6 is quite similar to that of Theorem 5. For the proof we will state some lemmas.

LEMMA 1. *Suppose $0 < r < 1$. Suppose $X_1, \ldots, X_n$ is a family of centered square integrable random variables with the following property: For any two nonempty disjoint subsets $S, T \subset \{1, 2, \ldots, n\}$, one has that*

$$|E\left(\sum_{k\in S} X_k\right)\left(\sum_{k\in T} X_k\right)| \le r \cdot \left\|\sum_{k\in S} X_k\right\|_2 \cdot \left\|\sum_{k\in S} X_k\right\|_2 .$$

*Then*

$$\frac{(1-r)}{(1+r)}\sum_{k=1}^{n} E|X_k|^2 \le E\left|\sum_{k=1}^{n} X_k\right|^2 \le \frac{(1+r)}{(1-r)}\sum_{k=1}^{n} E|X_k|^2.$$

For the detail of the proof of Lemma 1, see Bradley ([1]). Lemma 1 provides with the bounds for the variance of partial sums in terms of a correlation coefficient of two partial sums which apart from each other. The following lemma gives us some informations about the estimate of higher moments of partial sums with mixing condition. It is the Lemma 3 in Bryc and Smolenski ([3]).

LEMMA 2. *Suppose $\{X_1, X_2, \ldots, X_n\}$ is a family of centered random variables which are in $L_q$ for $q$ a fixed real, $2 \le q \le 4$. Assume $\rho^*(1) < 1$. Then there is a positive constant $C$ depending only on $q$ and $\rho^*$ such that*

$$E\left(\left|\sum_{k=1}^{n} X_k\right|^q\right) \le C\left(\sum_{k=1}^{n} E|X_k|^q + \left(\sum_{k=1}^{n} EX_k^2\right)^{q/2}\right). \tag{13}$$

The following lemma is a variation of Lemma 2, which is Lemma 3.3 in Peligrad ([4]).

LEMMA 3. *Suppose $\{X_1, \ldots, X_n\}$ are centered random variables which are in $L_q$ for $q$ a fixed real, $2 \le q \le 4$. Assume that there is a positive* *number $p$, $1 \le p \le n$ such that $\rho^*(p) < 1$. Then there is a constant $C$ depending only on $p, q$ and $\rho^*(p)$ such that (13) holds for this $C$.*

## 3. Proof of Theorem 6

In the proof of Theorem 6 we will use the same technique used in the proof of Theorem 2.1 in Peligrad ([4]). We give some preliminary concepts of notations and sketch the proof. For the detail refer to Peligrad ([4]).

### 3.1. Normalization and truncation

For a given $n \in \mathbf{Z}^d$, define $\xi_{nk} = X_k/\sigma_n$, $k \in \mathbf{Z}^d$ such that $1 \le k_u \le n_u, u = 1, 2, \ldots, d$ and $\sigma_n^2 = \mathrm{var}(S(X : n))$. Let $S(n) = \{k = (k_1, \ldots, k_d) \in \mathbf{Z}^d | 1 \le k_u \le n_u, u = 1, 2, \ldots, d\}$. Using those notations we can rewrite the conditions in Theorem 6 as follows

$$\mathrm{var}\left(\sum_{k \in S(n)} \xi_{nk}\right) = 1$$

and

$$\sum_{k \in S(n)} E\xi_{nk}^2 = \frac{|||n|||\sigma^2}{\sigma_n^2}, \tag{14}$$

where $|||n||| = n_1 \cdots n_d$ and $\sigma^2 = EX_0^2$. Note that $||n|| \to \infty$ is equivalent to $|||n||| \to \infty$. By Lemma 1, since $\lim_{r \to \infty} \rho^*(r) < 1$, there exist positive numbers $C_1$ and $C_2$ such that

$$C_1|||n|||\sigma^2 \le E\left(\sum_{k \in S(n)} X_k\right)^2 \le C_2|||n|||\sigma^2. \tag{15}$$

Note that the term in the middle of (15) is exactly $\sigma_n^2$. Therefore we have

$$\liminf(\sigma_n^2/|||n|||) > 0. \tag{16}$$

Also, for a given $\varepsilon > 0$, we have the following. When $||n|| \to \infty$,

$$\frac{|||n|||}{\sigma_n^2} EX_0^2 I(|X_0| > \varepsilon\sigma_n) \to 0. \tag{17}$$

Since $EX_0^2 < \infty$ and $P\{|X_0| > \varepsilon\sigma_n\} \to 0$ as $||n|| \to \infty$, we can show that (17) holds by Chebyshev's Inequality. By (17) we can construct a sequence of positive numbers $\{\varepsilon_n : n \in \mathbf{Z}^d\}$ such that $\varepsilon_n = o(|||n|||^{-1})$ and satisfying

$$\frac{|||n|||}{\sigma_n^2} EX_0^2 I(|X_0| > \varepsilon_n \sigma_n) \to 0.$$

We define the following two random variables by truncation at the level $\varepsilon_n$. Define

$$\eta_{nk} = \xi_{nk} I(|\xi_{nk}| \le \varepsilon_n) - E\xi_{nk} I(|\xi_{nk}| \le \varepsilon_n)$$

and

$$\gamma_{nk} = \xi_{nk} I(|\xi_{nk}| > \varepsilon_n) - E\xi_{nk} I(|\xi_{nk}| > \varepsilon_n).$$

Since condition $(b)$ in Theorem 6 holds we can find a positive integer $p$ such that $\rho^*(p) < 1$. By Lemma 3 applied with $q = 2$ there exists a positive number $C$ such that

$$\mathrm{var}\left(\sum_{k \in S(n)} \gamma_{nk}\right) \le C \sum_{k \in S(n)} \mathrm{var}(\gamma_{nk}) \le 2C \frac{|||n|||}{\sigma_n^2} EX_0^2 I(|X_0| > \varepsilon_n \sigma_n),$$

where the last term tends to 0 as $||n|| \to \infty$ by (17). Now it is equivalent to show the CLT for a random field $\{\eta_{nk}, k \in S(n)\}$, which satisfies the following conditions

$$|\eta_{nk}| \le 2\varepsilon_n, \quad \text{where } \varepsilon_n \to 0, \tag{18}$$

$$\mathrm{var}\left(\sum_{k \in S(n)} \eta_{nk}\right) \to 1 \text{ as } n \to \infty, \tag{19}$$

and there is a finite number M independent of $n \in \mathbf{Z}^d$ such that

$$\sum_{k \in S(n)} \mathrm{var}\ \eta_{nk} \le \frac{|||n|||}{\sigma_n^2} EX_0^2 I(|X_0| < \varepsilon_n \sigma_n) < M. \tag{20}$$

### 3.2. Blocking procedure

Now we divide the variables in big blocks and small blocks. Eventually the sum of the variables in big blocks will be asymptotically a sum of independent random variables and the sum of the variables in small blocks is negligible in the sense of convergence in distribution. This technique is so called the Bernstein's method which is very useful in dealing with weakly dependent random variables. For a given $\{\varepsilon_n, n \in \mathbf{Z}^d\}$ we construct a corresponding sequence of numbers $\{q_n, n \in \mathbf{Z}^d\}$ such that the following conditions hold, as $||n|| \to \infty$,

$$q_n \to \infty, \tag{21}$$

$$q_n \varepsilon_n \to 0, \tag{22}$$

and

$$q_n \alpha([\varepsilon_n^{-1}]) \to 0. \tag{23}$$

For a given $n = (n_1, \dots, n_d) \in \mathbf{Z}^d$, define $V_n = \mathrm{var} \eta_{nk}$. Note that $V_n$ is finite and depends on only $n$ and $\varepsilon_n$. We can find the smallest positive integer $m$ which satisfies the following condition

$$m(n_2 \cdots n_d)V_n \geq q_n^{-1}. \tag{24}$$

Note that, for $m' < m$,

$$m'(n_2 \cdots n_d)V_n < q_n^{-1}.$$

Let $n_1 = l_1(m + [\varepsilon_n^{-1}]) + d_1$, where $d_1 < m + [\varepsilon_n^{-1}]$. If $d_1 \geq m$, then we can rewrite $n_1 = (l_1 + 1)m + l_1[\varepsilon_n^{-1}] + d_1'$, where $d_1' < [\varepsilon_n^{-1}]$. Therefore we may assume that $d_1 < [\varepsilon_n^{-1}]$. Also $\bar{l}_1$ stands for $l_1$ or $l_1 + 1$. For $1 \leq j \leq \bar{l}_1$, $A_j = \{k = (k_1, \dots, k_d) \in S(n) : \ (j-1)(m + [\varepsilon_n^{-1}]) < k_1 \leq jm\}$ and $B_j = \{k = (k_1, \dots, k_d) \in S(n) : \ jm + (j-1)[\varepsilon_n^{-1}] < k_1 \leq j(m + [\varepsilon_n^{-1}])\}$. After constructing the final $A_{\bar{l}_1}$ we put all the remaining variables, if any, into the last block denoted by $B_{\bar{l}_1}$. By definition of the integer $m$, we have the following inequality

$$q_n^{-1} \leq m(n_2 \cdots n_d)V_n < q_n^{-1} + (n_2 \cdots n_d)V_n. \tag{25}$$

For each $1 \leq j \leq \bar{l}_1$, denote by

$$Y_{nj} = \sum_{k \in A_j} \eta_{nk}$$

and

$$Z_{nj} = \sum_{k \in B_j} \eta_{nk}.$$

By (20) and the definition of $A_j$ we have the following

$$M > \sum_{k \in S_n} \mathrm{var}\ \eta_{nk} \geq \sum_{j=1}^{\bar{l}_1} \sum_{k \in A_j} \mathrm{var}\ \eta_{nk} \geq \bar{l}_1 n_2 \cdots n_d q_n^{-1}.$$

Therefore we have

$$\bar{l}_1(n_d \cdots n_d) \le M q_n. \tag{26}$$

We estimate the variance of $\sum_{j=1}^{\bar{l}_1} Z_{nj}$. By Lemma 3 with $q = 2$ we have, for an integer $m' < m$, the following inequality

$$\text{var} Z_{n\bar{l}_1} = ([\varepsilon_n^{-1}] + m')(n_2 \cdots n_d)V_n \le [\varepsilon_n^{-1}]n_2 \cdots n_d V_n + q_n^{-1}. \tag{27}$$

By the construction of $\varepsilon_n$ the last term in (27) is less than $[\varepsilon_n^{-1}](n_2 \cdots n_d)4\varepsilon_n^2$ $+q_n^{-1}$, which converges to 0 as $||n|| \to \infty$. By Lemma 3 with $q = 2$, applying twice, we have

$$\begin{aligned} \text{var}\left(\sum_{j=1}^{\bar{l}_1 - 1} Z_{nj}\right) &\le C_1 \sum_{j=1}^{\bar{l}_1 - 1} \text{var} Z_{nj} \\ &\le C_2 \sum_{j=1}^{\bar{l}_1 - 1} \sum_{k \in B_j} \text{var}\eta_{nk} \le C_3 \bar{l}_1 [\varepsilon_n^{-1}] n_2 \ldots n_d V_n. \end{aligned} \tag{28}$$

By (26), (22), and (18), the last term in (29) is less than $Cq_n[\varepsilon_n^{-1}](4\varepsilon_n^2)$, which converges to 0 as $||n||$ approaches infinity. Therefore $\sum_{j=1}^{\bar{l}_1} Z_{nj}$ is negligible for the convergence in distribution. By (19) we may write

$$\text{var}\left(\sum_{j=1}^{\bar{l}_1} Y_{nj}\right) \to 1 \quad \text{as } ||n|| \to \infty. \tag{29}$$

By Lemma 1 and the condition $(b)$ in Theorem 6, we have two positive numbers $K_1$ and $K_2$ such that for $n$ with large $||n||$

$$K_1 \le \sum_{j=1}^{\bar{l}_1} \text{var} Y_{nj} \le K_2. \tag{30}$$

Let $a_n = (\sum_{j=1}^{\bar{l}_1} \text{var} Y_{nj})^{1/2}$. Then we can apply a standard argument based on recurrence and the definition of the mixing coefficient $\alpha(r)$ for every $t$. Therefore we have, for some positive number $D$, the following estimation

$$(31) \left|E \exp\left(ita_n^{-1} \sum_{j=1}^{\bar{l}_1} Y_{nj}\right) - \prod_{j=1}^{\bar{l}_1} E\exp(ita_n^{-1} Y_{nj})\right| \le D\bar{l}_1 n_2 \cdots n_d \alpha([\varepsilon_n^{-1}]),$$

which converges to 0 as $||n|| \to \infty$ by (23) and (26). Consequently the problem can be reduced to investigate the asymptotic behavior of the triangular array $\{Y^*_{nj}, 1 \le j \le \bar{l}_1\}$ of independent random variables which have the identical distribution to $Y_{nj}$ with the same property of (30).

### 3.3. The proof of the CLT

In this step we claim that the array $\{a_n^{-1}Y^*_{nj} : 1 \le j \le \bar{l}_1\}$ satisfies the CLT. Since $\mathrm{var}(\sum_{j=1}^{\bar{l}_1} a_n^{-1}Y^*_{nj}) = 1$ it is enough to show that the following Lindeberg condition holds. For a given $\varepsilon > 0$,

$$\sum_{j=1}^{\bar{l}_1} EY^2_{nj} I(|Y_{nj}| > \varepsilon) \to 0 \quad \text{as} \quad ||n|| \to \infty. \tag{32}$$

Since $\sum_{j=1}^{\bar{l}_1} EY^4_{nj} \to 0$ implies (32), we shall show that the sum of fourth moments approaches zero as $||n|| \to \infty$. For a fixed $1 \le j \le \bar{l}_1$, we apply Lemma 3 with $q = 4$ to every $Y_{nj}$. Then we have a positive number $D_1$, which is independent of $n$, such that

$$EY^4_{nj} \le D_1 \left( \sum_{k \in A_j} E\eta^4_{nk} + \left( \sum_{k \in A_j} E\eta^2_{nk} \right)^2 \right). \tag{33}$$

By (25), (18) and the definition of $A_j$ we have the following inequality

$$\begin{aligned} \sum_{k \in A_j} E\eta^2_{nk} &= m(n_2 \ldots n_d)V_n < q_n^{-1} + (n_2 \cdots n_d)V_n \\ &\le q_n^{-1} + 4(n_2 \cdots n_d)\varepsilon_n^2. \end{aligned} \tag{34}$$

Moreover, by (18), we have

$$E\eta^4_{nk} \le 4\varepsilon_n^2 E\eta^2_{nk}. \tag{35}$$

Therefore, by (33), (35), (35) and (20) there exists a positive constant $K_1$ such that

$$\sum_{j=1}^{\bar{l}_1} EY^4_{nk} \le K_1\{|||n|||\varepsilon_n^2 q_n^{-1} + \varepsilon^4 q_n + (|||n|||)^{-1} + \varepsilon_n^2 + |||n|||q_n\varepsilon_n^4),$$

which converges to 0 as $||n||$ approaches infinity by the construction of $\varepsilon_n$, (21) and (22). Consequently we have (32). And we have the following conclusion

$$a_n^{-1}\sum_{j=1}^{\bar{l}_1} Y_{nj}^* \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } ||n|| \to \infty.$$

By (31), (29) and (30) we have

$$a_n^{-1}\sum_{k\in S(n)} \eta_{nk} \xrightarrow{d} \mathcal{N}(0,1) \quad \text{as } ||n|| \to \infty. \tag{36}$$

To prove Theorem 6 we only have to show that $\lim_{||n||\to\infty} a_n = 1$.

### 3.4. The convergence of $a_n$

Here we will prove that $\{(\sum_{k\in S(n)} \eta_{nk})^2\}$ is a uniformly integrable family. Then, by (19) and (36), we can show that the sequence $\{a_n\}$ converges to 1 using the convergence of the moments. Since

$$E\{|X_n|I(|X_n| > c)\} \le c^{-1}E|X_n|^2,$$

$\sup_{n\in\mathbf{Z}^d} EX_n^2 < \infty$ implies the uniform integrability of $X_n$. Therefore we need to estimate $E(\sum_{k\in S(n)} \eta_{nk})^4$. By Lemma 3, applied with $q = 4$, there is a positive constant $D_1$ independent of $n \in \mathbf{Z}^d$ such that

$$E\left(\sum_{k\in S(n)} \eta_{nk}\right)^4 \le D_1\left[\left(\sum_{k\in S(n)} E\eta_{nk}^4\right) + \left(\sum_{k\in S(n)} E\eta_{nk}^2\right)^2\right]$$

and, by (35) and (20), we can find a positive constant $D_2$ such that

$$E\left(\sum_{k\in S(n)} \eta_{nk}\right)^4 \le D_2, \quad \text{for all } n \in \mathbf{Z}^d.$$

This completes the proof.

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