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A LAW OF ITERATED LOGARITHM FOR OCCUPATION TIME OF BROWNIAN MOTION IN l_2

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ABSTRACT. We consider a random measure defined by the total occupation time of Brownian motion in l_2. If it is normalized by $\lambda^2 \log\log\lambda$ then we show that its cluster set as $\lambda \to \infty$ can be represented by I-function on σ-finite measure space in l_2.

1. Introduction

Let l_2 be as usual and $\{\beta(s), 0 \leq s < \infty\}$ be Brownian motion in l_2 starting from $x \in l_2$. For a Borel set E in l_2, we define

$$\sigma(E,\omega) = \int_0^\infty \chi_E(\beta(s,\omega))ds$$

i.e., the total time the set E is occupied by the particular path $\omega = \beta(\cdot)$. For each ω, $\sigma(\cdot,\omega)$ is a σ-finite measure on l_2. Let $\mathcal{M}$ be the space of all σ-finite measures on l_2 and let U be the space of functions $u(x) \in C^2$ on l_2 satisfying

$$0 < \inf_{x\in l_2} u(x) < \sup_{x \in l_2} u(x) < \infty$$

and Lu has compact support, where L is the infinitesimal generator of $\beta(s)$. For each σ-finite measure $\sigma \in \mathcal{M}$, we define

$$I(\sigma) = -\inf_{u\in U} \int_{l_2} \frac{Lu}{u}(x)\sigma(dx).$$

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For $\lambda \geq 3$, let

$$Z_\lambda(\cdot,\omega) = \frac{\sigma(\lambda\cdot,\omega)}{\lambda^2 \log\log\lambda} \tag{1.2}$$

so that for each ω, i.e., each Brownian motion path $\beta(\cdot), Z_\lambda(\cdot,\omega)$ is a family in the parameter λ of σ-finite measures on l_2.

Let C_ω be the cluster set for the family $\{Z_\lambda(\cdot,\omega)\}$ (as $\lambda \to \infty$), and $\mathcal{B} = \{\sigma \in \mathcal{M} : I(\sigma) \leq 1\}$. The main result of this paper is to show that

$$C_\omega = \mathcal{B} \text{ for almost all } \omega. \tag{1.3}$$

In fact this result is a generalization of the same result on Brownian motions in $R^d, d \geq 3$, which is proved by Donsker and Varadhan ([3]). It is an analogue of the case in R^d to show that $C_\omega \subset \mathcal{B}$ in l_2. However, it is highly dependent on the dimension d to show that $\mathcal{B} \subset C_\omega$ in R^d, so we have to seek another approach. We are going to use the operator which determines Brownian motions in l_2 and project Brownian motions in l_2 onto R^d by considering the truncated process.

This result is also related with a classical Strassen type of the iterated logarithm. Let $B(s)$ be Brownian motion in R^d starting from the origin with $d \geq 3$, and let $\sigma_d(\lambda,\omega)$ be the total time that a particular path $\omega = B(\cdot)$ occupies the sphere with center at the origin of radius λ. In [1] Ciesielski and Taylor showed that, for almost all Brownian motion paths,

$$\limsup_{\lambda\to 0} \frac{\sigma_d(\lambda\cdot,\omega)}{\lambda^2 \log\log(\frac{1}{\lambda})} = \frac{2}{p_d^2}, \tag{1.4}$$

where p_d is the first zero of Bessel function $J_\nu(x)$ with $\nu = \frac{1}{2}d - 2$. If $F(\sigma)$ is continuous in the vague topology at the points of $\mathcal{B}$, then (1.3) implies that for almost all Brownian motion paths ω

$$\limsup_{\lambda\to\infty} F(Z_\lambda(\cdot,\omega)) = \sup_{\{\sigma: I(\sigma)\leq 1\}} F(\sigma). \tag{1.5}$$

Hence, if we let $F(\sigma) = \sigma(S_d(0,1))$, where $S_d(0,1)$ is the unit sphere in R^d with center at 0, then we get (1.4).

2. Preliminaries

Let l_2 be as usual with the norm $\|\cdot\|_{l_2}$ and β_t be a standard l_2-valued Brownian motion defined over $(\Omega, \mathcal{F}, P)$ such that $\beta_0 = 0$. We have

$$E(\beta_t, h) = 0 \text{ and } E(\beta_t, g)(\beta_s, h) = (t \wedge s)(Tg, h),$$

for all $g, h \in l_2$, where $T : l_2 \to l_2$ is a nuclear (trace class) covariance operator. The existence of such a Brownian motion is well known ([4],[5]).

Let $\{e_i\}_{i=1}^{\infty}$ be the usual orthonormal set in l_2 and suppose that T has the orthonormal eigensystem $\{e_i, \xi_i\}$ so that

$$T(e_i) = \xi_i^2 e_i, \; i = 1, 2, \cdots,$$

where $\xi_i > 0$, $\xi_1 \geq \xi_2 \geq \xi_3 \geq \cdots$, and $\sum_{i=1}^{\infty} \xi_i^2 < \infty$. Then the following representation holds almost surely:

$$\beta_t = \sum_{i=1}^{\infty} \xi_i B_t^i e_i, \tag{2.1}$$

where $B_t^i, i = 1, 2, \cdots$ are independent, identically distributed, standard Brownian motions in one dimension. Moreover we assume that $\xi_1^2 < \sum_{i=2}^{\infty} \xi_i^2$.

We also define for each positive integer n and each $x \in l_2$, a probability measure $Q_n^x(A)$ on $\mathcal{M}$ as follows: If $A \subset \mathcal{M}$,

$$Q_n^x(A) = P_x\left\{\omega; \frac{1}{n}\sigma(\cdot, \omega) \in A\right\}, \tag{2.2}$$

where $P_x\{\cdot\}$ is the measure on the paths $\omega = \beta(\cdot)$ in l_2 starting from x. Let σ be defined as before. In this section we introduce the asymptotic behavior of the measure Q_n^x as $n \to \infty$. The proof of the following theorems are not intrinsically different from those in the case of Brownian motion in R^d (See in detail [3]).

On the space $\mathcal{M}$, we impose the vague topology, i.e., if $\sigma_0 \in \mathcal{M}$, r is a positive integer and $\phi_1, \phi_2. \cdots, \phi_r$ are functions in $C_0(l_2)$, then for each $\epsilon > 0$ a neighborhood of σ_0 is defined by

$$\begin{aligned} N(\sigma_0) &= N(\sigma_0, r, \phi_1, \cdots, \phi_r, \epsilon) \\ &= \{\sigma \in \mathcal{M} \Big| \int_{l_2} \phi_i(x)\sigma(dx) - \int_{l_2} \phi_i(x)\sigma_0(dx)\Big| < \epsilon, \; 1 \leq i \leq r\}. \end{aligned}$$

THEOREM 2.1. *For each $K \subset \mathcal{M}$ that is compact*

$$\limsup_{n\to\infty} \frac{1}{n} \log Q_n^x(K) \leq - \inf_{\sigma\in K} I(\sigma),$$

uniformly in x.

THEOREM 2.2. *Let N be a neighborhood of some $\sigma_0 \in \mathcal{M}$ with $I(\sigma_0) = l < \infty$. Then*

$$\liminf_{n\to\infty} \frac{1}{n} \log Q_n^x(N) \geq -l,$$

uniformly in x in compact sets of l_2.

LEMMA 2.3. *Given $\sigma \in \mathcal{M}$ and a neighborhood N_1 of σ, there exists another neighborhood N of σ and $\epsilon > 0$ such that if $\sigma_1 \in N, |\alpha - 1| < \epsilon$ and $|\gamma - 1| < \epsilon$, then*

$$\sigma_2(\cdot) \equiv \gamma\sigma_1(\alpha\cdot) \in N_1.$$

For a Brownian path ω in l_2, let C_ω and $\mathcal{B}$ be defined as before.

THEOREM 2.4. *For almost all ω,*

$$C_\omega \subset \mathcal{B}.$$

3. Main Result

THEOREM 3.1. *For almost all ω,*

$$\mathcal{B} \subset C_\omega. \tag{3.1}$$

PROOF. Since C_ω is closed, it is sufficient to show that for every σ such that $I(\sigma) < 1$, $\sigma \in C_\omega$ for almost all ω. If we let $I(\sigma) < 1$, we can pick $\alpha > 1$ and $\delta > 0$ such that $\alpha(I(\sigma) + \delta) < 1$. For this α and each positive integer n, let

$$\lambda_n = \exp\{n^\alpha\},\ a_n = n^2 \exp\{n^\alpha\},\ c_n = 2n^2 \exp\{n^\alpha\}$$

so that

$$c_{n-1} < \lambda_n < a_n < c_n < \lambda_{n+1},$$

for all large n. Consider a Brownian path in l_2, $\beta(t) = (\xi_1 B_1(t), \cdots, \xi_d B_d(t) \cdots)$ starting from the origin. Let τ_{a_n} be the first exit time from the ball $B(0, a_n)$ in l_2 and τ_{c_n} be the first exit time from the ball $B(0, c_n)$ in l_2. For any set $A \subset l_2$ and any positive integer n, define

$$\text{(3.2)} \qquad \begin{aligned} \alpha'_n(A,\omega) &= \int_0^{\tau_{a_n}} \chi_A(\beta(s))ds, \quad \alpha_n(A,\omega) = \frac{\alpha'_n(\lambda_n A,\omega)}{\lambda_n^2 \log\log \lambda_n} \\ \beta'_n(A,\omega) &= \int_{\tau_{c_{n-1}}}^{\tau_{a_n}} \chi_A(\beta(s))ds, \quad \beta_n(A,\omega) = \frac{\beta'_n(\lambda_n A,\omega)}{\lambda_n^2 \log\log \lambda_n} \end{aligned}$$

and recall that

$$Z_{\lambda_n}(A,\omega) = \frac{\sigma(\lambda_n A,\omega)}{\lambda_n^2 \log\log \lambda_n}.$$

Let N be a neighborhood of σ. To prove $\mathcal{B} \subset C_\omega$, it is sufficient to show that

$$P\{Z_{\lambda_n}(\cdot,\omega) \in N \text{ for infinitely many } n\} = 1.$$

<u>*Step* 1.</u> We claim that

$$\text{(3.3)} \qquad P\{Z_{\lambda_n}(\cdot,\omega) \notin N, \alpha_n(\cdot,\omega) \in N \text{ for infinitely many } n\} = 0.$$

From the Borel-Cantelli lemma it suffices to show

$$\text{(3.4.)} \qquad \sum P\{Z_{\lambda_n}(\cdot,\omega) \notin N, \alpha_n(\cdot,\omega) \in N\} < \infty.$$

We specify the vague neighborhood N of σ as the following; for some positive integer r, $\epsilon > 0$, and $\phi_i \in C_0(l_2)$, $i = 1, 2 \cdots r$,

$$\begin{aligned} N &= N(\sigma; r, \epsilon, \phi_i, \cdots \phi_r) \\ &= \left\{\sigma' \in \mathcal{M}; \left| \int \phi_i(x)\sigma'(dx) - \int \phi_i(x)\sigma(dx) \right| < \epsilon,\ i = 1 \cdots r \right\}. \end{aligned}$$

We note that
(3.5)
$$\begin{aligned}
&P\{Z_{\lambda_n}(\cdot,\omega) \notin N, \alpha_n(\cdot,\omega) \in N\}\\
&\leq P\left\{\int \phi_i(x) Z_{\lambda_n}(dx,\omega) \neq \int \phi_i(x)\alpha_n(dx,\omega) \text{ for some } 1 \leq i \leq r\right\}\\
&= P\left\{\int_0^\infty \phi_i\left(\frac{\beta(s)}{\lambda_n}\right) ds \neq \int_0^{\tau_{a_n}} \phi_i\left(\frac{\beta(s)}{\lambda_n}\right) ds \;\text{ for some } 1 \leq i \leq r\right\}\\
&= P\left\{\int_{\tau_{a_n}}^\infty \phi_i\left(\frac{\beta(s)}{\lambda_n}\right) ds \neq 0 \;\text{ for some } 1 \leq i \leq r\right\}.
\end{aligned}$$

Let $R = \inf\{\, l;\ \phi_i(x) = 0, \text{ for } \|x\|_{l_2} > l, i = 1, 2 \cdots r\}$ and let $\theta_{\lambda_n R}$ be the last exit time of $\beta(s)$ for $B(0, \lambda_n R)$ in l_2. Since for n large enough, $\lambda_n R < a_n$ and since the support $\phi_i(x)$ is contained $B(0, R)$ for all i, $i = 1, \cdots, r$, we see that the event (3.5) can occur only if $\theta_{\lambda_n R} > \tau_{a_n}$. Therefore

$$P\{Z_{\lambda_n}(\cdot,\omega) \notin N, \alpha_n(\cdot,\omega) \in N\} \leq P\{\theta_{\lambda_n R} > \tau_{a_n}\}.$$

Let η denote the first hitting time of $B(0, \lambda_n R)$. From the strong Markov property

$$P\{\theta_{\lambda_n R} > \tau_{a_n}\} = E[P_{\beta(\tau_{a_n})}\{\eta < \infty\}].$$

<u>Step 2.</u> We want to show that

$$\sum E[P_{\beta(\tau_{a_n})}\{\eta < \infty\}] < \infty. \tag{3.6}$$

The infinitesimal generator for $\beta(t)$ is $L = \frac{1}{2}\sum_{i=1}^\infty \xi_i^2 \frac{\partial^2}{\partial x_i^2}$ (see [4]). Then by Ito's formula ([4])

$$E_x[f(\beta(t,\omega))] = f(x) + E_x\left[\int_0^t \frac{1}{2}\sum_{i=1}^\infty \xi_i^2 \frac{\partial^2 f}{\partial x_i^2}(\beta(s,\omega))ds\right].$$

Note that $\|T\| = \xi_1^2$. Denote $T_0 = \sum_{i=1}^\infty \xi_i^2$. By the assumptions on $\{\xi_i\}_{i=1}^\infty$, we can choose a number Λ satisfying $1 - \frac{T_0}{2\xi_1^2} < \Lambda < 0$. If we take $f(x) = (\sum_{i=1}^\infty \xi_i^2 x_i^2)^\Lambda$, then

$$E_x\left[\int_0^t \frac{1}{2}\sum_{i=1}^\infty \xi_i^2 \frac{\partial^2 f}{\partial x_i^2}(\beta(s,\omega))ds\right] < 0,$$

for all $x \in l_2$. For some r and r_1 satisfying $0 < r < r_1$, let

$$E_r = \left\{ x \in l_2; \sum_{i=1}^{\infty} \xi_i^2 x_i^2 \le r^2 \right\}, \quad E_{r_1} = \left\{ x \in l_2; \sum_{i=1}^{\infty} \xi_i^2 x_i^2 \le r_1^2 \right\}.$$

If $y \in E_{r_1} - E_r$, define η_y be the first hitting time of $\beta(t)$ starting from y to E_r or E_{r_1}. From this we get

$$E_y[f(\beta(\eta_y, \omega))] < f(y) < C \cdot \left(\sum_{i=1}^{\infty} y_i^2 \right)^{\Lambda},$$

where $C = \xi_1^{2\Lambda}$. Thus

$$r^{2\Lambda} P(\eta_y = r) + r_1^{2\Lambda}(1 - P(\eta_y = r)) < C \cdot \left(\sum_{i=1}^{\infty} y_i^2 \right)^{\Lambda},$$

$$P(\eta_y = r) < C \cdot \frac{(\sum_{i=1}^{\infty} y_i^2)^{\Lambda}}{r^{2\Lambda}}$$

for sufficiently large r_1. Substituting $y = \beta(\tau_{a_n})$ and $r = \lambda_n R$,

$$\sum E[P_{\beta(\tau_{a_n})}\{\eta < \infty\}] \le C \cdot \sum \left(\frac{a_n}{\lambda_n R} \right)^{2\Lambda} \le C \cdot \sum n^{2\alpha\Lambda}.$$

Since we may choose α such that $2\alpha\Lambda < -1$ and $\sum n^{2\alpha\Lambda} < \infty$, we get (3.6). □

We need more lemmas to prove Theorem 3.1. Note that we specified the neighborhood $N = N(\sigma)$. Now define $N_1 \subset N$ to be the neighborhood of σ,

$$N_1 = \left\{ \sigma'; \left| \int \phi_i(x)\sigma'(dx) - \int \phi_i(x)\sigma(dx) \right| < \frac{2}{3}\epsilon, \, i = 1, 2, \cdots, r \right\}.$$

LEMMA 3.2. *We have*

$$P\{\beta_n(\cdot, \omega) \in N_1, \alpha_n(\cdot, \omega) \notin N \text{ for infinitely many } n\} = 0.$$

PROOF. Let $d \geq 3$ and P_d be an orthogonal projection of l_2 onto R^d. Define

$$\beta_d(s) = P_d\beta(s) = \sum_{i=1}^{d} \xi_i B_i(s) e_i, \quad B_d(s) = \sum_{i=1}^{d} B_i(s) e_i.$$

Let $\tau_{c_{n-1}}$ be the first exit time of $\beta(s)$ of $B(0, c_{n-1})$ in l_2 and $\tilde{\tau}_{c_{n-1}}$ be the first exit time of $\beta_d(s)$ for the sphere of radius c_{n-1} in R^d. Then $\tau_{c_{n-1}} \leq \tilde{\tau}_{c_{n-1}}$. If for some n and some $\omega, \beta_n(\cdot,\omega) \in N_1$ but $\alpha_n(\cdot,\omega) \notin N$, then for at least one $i, i = 1, \cdots, r$,

$$\left| \int \phi_i(x)\beta_n(dx,\omega) - \int \phi_i(x)\alpha_n(dx,\omega) \right| \geq \frac{1}{3}\epsilon,$$

which means by change of variable,

$$\left| \int_{\tau_{c_{n-1}}}^{\tau_{a_n}} \phi_i\left(\frac{\beta(s)}{\lambda_n}\right) ds - \int_0^{\tau_{a_n}} \phi_i\left(\frac{\beta(s)}{\lambda_n}\right) ds \right| \geq \frac{1}{3}\epsilon\lambda_n^2 \log\log\lambda_n. \tag{3.7}$$

Let $M = \sup_{1\leq i\leq r, x\in l_2} |\phi_i(x)|$. Then (3.7) implies that $M \cdot \tau_{c_{n-1}} \geq \frac{1}{3}\epsilon\lambda_n^2 \log\log\lambda_n$. Hence, for each positive integer n,

$$\begin{aligned} P\{\beta_n(\cdot,\omega) \in N_1, \, \alpha_n(\cdot,\omega) \notin N\} &\leq P\left\{\tau_{c_{n-1}} \geq \frac{\epsilon}{3M}\lambda_n^2 \log\log\lambda_n\right\} \\ &\leq P\left\{\tilde{\tau}_{c_{n-1}} \geq \frac{\epsilon}{3M}\lambda_n^2 \log\log\lambda_n\right\}. \end{aligned} \tag{3.8}$$

Let

$$E_{c_{n-1}} = \left\{ x \in R^d : \sum_{i=1}^{d} \frac{x_i^2}{\xi_i^2} \leq r_0^2, \text{ for some } r_0 \text{ satisfying } r_0\xi_d = c_{n-1} \right\}.$$

Since $\xi_1 \geq \xi_2 \geq \cdots \xi_d$, we have $B(0, c_{n-1}) \subset E_{c_{n-1}}$. Again, let $\tilde{\tau}_{E_{c_{n-1}}}$ be the first exit time of $\beta_d(s)$ from $E_{c_{n-1}}$ and $\hat{\tau}_{r_0}$ be the first exit time of $B_d(s)$ from $B(0,r_0)$ respectively in R^d. As the proof of Step 2 in Theorem 3.1, we get $\hat{\tau}_{r_0} \geq \tilde{\tau}_{E_{c_{n-1}}} \geq \tilde{\tau}_{c_{n-1}}$ and

$$\begin{aligned} P\left\{\tilde{\tau}_{c_{n-1}} \geq \frac{\epsilon}{3M}\lambda_n^2 \log\log \lambda_n\right\} &\leq P\left\{\tilde{\tau}_{E_{c_{n-1}}} \geq \frac{\epsilon}{3M}\lambda_n^2 \log\log \lambda_n\right\} \\ &\leq P\left\{\hat{\tau}_{r_0} \geq \frac{\epsilon}{3M}\lambda_n^2 \log\log \lambda_n\right\} \\ &\leq k \exp\left\{\frac{-\epsilon k \lambda_n^2 \log\log \lambda_n}{3Mc_{n-1}^2}\right\} \\ &\leq C \cdot \frac{c_{n-1}^2}{\lambda_n^2}, \end{aligned}$$

where C, k are some constants and the third inequality comes from the result in [1]. Since $\sum \frac{c_{n-1}^2}{\lambda_n^2} < \infty$, we obtain the result. □

LEMMA 3.3. *Let $E_n = \{\omega; \beta_n(\cdot,\omega) \in N_1\}$. Then, for $n > m$, there exists a constant C, independent of n and m such that*

$$P\{E_n \cap E_m\} \leq C \cdot P\{E_n\}P\{E_m\}.$$

PROOF. See [3]. □

Let $N_2 \subset N_1 \subset N$ be specified by

$$N_2(\sigma) = \left\{\sigma' : \left|\int \phi(x)\sigma(dx) - \int \phi(x)\sigma'(dx)\right| \leq \frac{\epsilon}{3}, i = 1, \cdots, r\right\}.$$

LEMMA 3.4.

$$P\{\beta_n(\cdot,\omega) \in N_1, \textit{ for infinitely many } n\} = 1. \tag{3.9.}$$

PROOF. Note that

$$\begin{aligned} &P\{\beta_n(\cdot,\omega) \in N_1\} \\ &\geq P\{\alpha_n(\cdot,\omega) \in N_2\} - P\{\beta_n(\cdot,\omega) \notin N_1, \alpha_n(\cdot,\omega) \in N_2\} \end{aligned} \tag{3.10}$$

$$
(3.11) \qquad \begin{aligned} &P\{\alpha_n(\cdot,\omega) \in N_2\} \\ &\geq P\{Z_{\lambda_n}(\cdot,\omega) \in N_2\} - P\{Z_{\lambda_n}(\cdot,\omega) \notin N_2,\, \alpha_n(\cdot,\omega) \in N_2\}. \end{aligned}
$$

Firstly, from the scaling property of $\beta(s)$

$$
(3.12) \qquad \begin{aligned} P\{Z_{\lambda_n}(\cdot,\omega) \in N_2\} &= P\left\{\frac{\sigma(\cdot,\omega)}{\lambda_n^2 \log\log \lambda_n} \in N_2\right\} \\ &\geq \exp\{(\lambda_n^2 \log\log \lambda_n)(I(f)+\delta)\}, \end{aligned}
$$

by Theorem 2.2. Since $\lambda_n = \exp(n^\alpha)$, we get

$$
P\{Z_{\lambda_n}(\cdot,\omega) \in N_2\} \geq \exp\{-(\lambda_n^2 \log\log \lambda_n)(I(f)+\delta)\} = \frac{1}{n^{\alpha(I(f)+\delta)}}.
$$

By the way we pick α and δ, $\alpha(I(f)+\delta) < 1$ and hence we conclude that

$$
(3.13) \qquad \sum P\{Z_{\lambda_n}(\cdot,\omega) \in N_2\} = \infty.
$$

By the claim of Step 1 and (3.13),

$$
\sum P\{\alpha_n(\cdot,\omega) \in N_2\} = \infty,
$$

and by the same argument used in Lemma 3.2

$$
\sum P\{\beta_n(\cdot,\omega) \notin N_1, \alpha_n(\cdot,\omega) \in N_2\} < \infty.
$$

Hence we get

$$
\sum P\{\beta_n(\cdot,\omega) \in N_1\} = \infty.
$$

(3.13) and Lemma 3.3 satisfy the hypotheses of a refined form of Borel-Cantelli lemma ([1]) stated below. Finally, we get (3.9) by that lemma.□

LEMMA 3.4. *Let $E_n, n = 1, 2, \cdots$ be a sequence of events such that $P\{E_n\} \to 0$ as $n \to \infty$, while $\sum_{n=1}^{\infty} P\{E_n\}$ diverges and $P\{\limsup E_n\} = 0$ or 1. If there exists an absolute constant C such that, for $n \neq r$, $P\{E_n \cap E_r\} \leq C \cdot P\{E_n\}P\{E_r\}$ then infinitely many of the events E_n occur with probability 1.*

CONTINUED PROOF OF THEOREM 3.1. Combining Lemma 3.2 and Lemma 3.4, we conclude that

$$P\{\alpha_n(\cdot,\omega) \in N \text{ for infinitely many } n\} = 1 \tag{3.14}$$

and (3.14) together with the claim of Step 1 implies that

$$P\{Z_{\lambda_n}(\cdot,\omega) \in N \text{ for infinitely many } n\} = 1.$$

□

References

[1] Z. Ciesielski and S. J. Taylor, *First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path*, Trans. Ann. Math. Soc. **103** (1962), 434-450.

[2] P. Chow and J. Menaldi, *Exponential estimates in exit probability for some diffusion process in Hilbert space*, Stochastics and Stochastic reports **29** (1990), 377-393.

[3] M. D. Donsker and S. R. Varadhan, *A law of iterated logarithm for total occupation times of transient Brownian motion*, Comm. on pure and Appl. Math. **33** (1980), 365-393.

[4] H. Kuo, *Stochastic integrals in abstract wiener space*, Pacific J. Math. **41** (1972), 469-483.

[5] J. Kuelbs and R. Lepage, *The law of the iterated logarithm for Brownian motion in a Banach space*, Trans. Ann. Math. Soc. **185** (1973), 253-264.

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