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THE NON-EXISTENCE OF HOPF BIFURCATION IN A DOUBLE-LAYERED BOUNDARY PROBLEM SATISFYING THE DIRICHLET BOUNDARY CONDITION

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ABSTRACT. A free boundary problem is derived from a singular limit system of a reaction diffusion equation whose reaction terms are bistable type. In this paper, we shall consider a free boundary problem with two layers satisfying the zero flux boundary condition and shall show that the Hopf bifurcation can not occur as a parameter varies.

1. Introduction

A reaction diffusion system satisfying bistable condition is reduced to a free boundary problem as a layer parameter tends to zero. For the multiple free boundary problem, the well posedness of solutions was shown in [2], however, an occurrence of a Hopf bifurcation in this cases has not been shown yet. For a double free boundary problem satisfying the Neumann boundary condition, the authors in [3] proved the Hopf bifurcation. We want to examine a Hopf bifurcation for double-layered problem satisfying the Dirichlet boundary condition. We will see quite different behaviors from this problem. In this paper, we shall show that there does not exist a Hopf bifurcation for the double-layered problem satisfying the Dirichlet boundary condition and thus it suggests that the Hopf bifurcation does not occur in this multiple-layered boundary problem.

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We shall consider a free boundary problem with two layers s and m as an abstract evolution equation

$$\text{(F)} \qquad \begin{cases} \dfrac{d}{dt}(v,s,m) + \widetilde{A}(v,s,m) = F(v,s,m) \\ (v,s,m)(0) = (v_0(\cdot), s_0, m_0)\,. \end{cases}$$

The operator $\widetilde{A}$ is a 3×3 matrix that the (1,1)-entry is a differential operator $A = -D\frac{d^2}{dx^2} + c^2$ and all the other entries are zero. The operator $\widetilde{A}$ is defined by

$$\begin{cases} \widetilde{A} : D(\widetilde{A}) \subset_{\text{dense}} \widetilde{X} \longrightarrow \widetilde{X},\ \widetilde{X} := L_2((0,1)) \times \mathbf{R} \times \mathbf{R} \\ D(\widetilde{A}) := D(A) \times \mathbf{R} \times \mathbf{R}, D(A) := \{v \in H^{2,2}(0,1) \,:\, v(0) = 0 = v(1)\}\,. \end{cases}$$

The nonlinear operator F is given by

$$F(v,s,m) = \begin{pmatrix} F_1(v(\cdot,t), s(t), m(t)) \\ F_2(v(\cdot,t), s(t), m(t)) \\ F_3(v(\cdot,t), s(t), m(t)) \end{pmatrix} := \begin{pmatrix} H(\cdot - s(t)) - H(\cdot - m(t)) \\ \frac{1}{\tau} C(v(s(t),t)) \\ -\frac{1}{\tau} C(v(m(t),t)) \end{pmatrix}$$

where the velocity of the free boundary $C(\cdot)$, is a continuously differentiable function defined on $\mathbf{I} := (-a, 1-a)$ and given by (see in [1], [8])

$$C(r) = \frac{2(r+a) - 1}{\sqrt{(1-r-a)(r+a)}}.$$

The well-posedness of (F) was shown in [2] and [10], and we shall adopt a few things in order to examine a Hopf bifurcation. The authors obtained a regularized problem of (F) using Green's function for A and a following transformation

$$u(t)(x) := v(x,t) - g(x, s(t), m(t))\,,$$

where a real function g is defined in $[0,1]^3$ and is given by

$$g(x,s,m) := \int_s^m G(x,y)\,dy = A^{-1}(H(\cdot - s) - H(\cdot - m))(x)\,.$$

Moreover, we define γ and $\eta : [0,1]^2 \longrightarrow \mathbf{R}$

$$\gamma(s,m) := g(s,s,m) \quad \text{and} \quad \eta(s,m) := g(m,s,m)\,.$$

Then the problem (F) is transformed to the following problem

$$(\mathrm{R}) \qquad \begin{cases} \dfrac{d}{dt}(u,s,m) + \widetilde{A}(u,s,m) = \dfrac{1}{\tau}\, f(u,s,m) \\ (u,s,m)(0) = (u(0), s(0), m(0)) = (u_0, s_0, m_0) \end{cases}$$

where the nonlinear term f is a mapping from W to $L_2(0,1)\times\mathbf{R}\times\mathbf{R}$ with

$$W := \{(u,s,m) \in C^1([0,1]) \times (0,1) \times (0,1) \,:\, u(s) + \gamma(s,m) \in \mathbf{I} \\ u(m) + \eta(s,m) \in \mathbf{I}\} \subset_{\text{open}} C^1([0,1]) \times \mathbf{R} \times \mathbf{R}$$

and is given by

$$f(u,s,m) = \begin{pmatrix} C(u(s)+\gamma(s,m))\, G(x,s) + C(u(m)+\eta(s,m))\, G(x,m) \\ C(u(s)+\gamma(s,m)) \\ -C(u(m)+\eta(s,m)) \end{pmatrix}.$$

In [2], the authors proved the well-posedness of (R) using domains of fractional powers $\alpha \in [0,1]$ of A and the imbedding theorem [4]. They obtained that $f : W \cap \widetilde{X}^\alpha \to \widetilde{X}$ is a continuously differentiable function where $\widetilde{X}^\alpha := X^\alpha \times \mathbf{R}, X^\alpha := D(A^\alpha) \subset C^1([0,1])$. In the next section, we shall examine a Hopf bifurcation of (R) in this space.

2. Non-existence of a Hopf bifurcation

We shall investigate the behaviors of eigenvalues for the problem (R). In order to do this, we need the linearized eigenvalue problem for (R) which is linearized at the stationary solutions.

The stationary problem for (R) is given by

$$\begin{cases} Au^* &= \frac{1}{\tau} C(u^*(s^*) + \gamma(s^*,m^*)) \cdot G(\cdot, s^*) \\ & \quad + \frac{1}{\tau} C(u^*(m^*) + \eta(s^*,m^*)) \cdot G(\cdot, m^*) \\ 0 &= \frac{1}{\tau} C(u^*(s^*) + \gamma(s^*,m^*)) \\ 0 &= -\frac{1}{\tau} C(u^*(m^*) + \eta(s^*,m^*)) \end{cases}$$

for $(u^*, s^*, m^*) \in D(\widetilde{A}) \cap W$. For nonzero τ, the above system is equivalent to the pair of equations

$$(2.1) \qquad u^* = 0, \quad C(\gamma(s^*,m^*)) = 0 \text{ and } C(\eta(s^*,m^*)) = 0.$$

We thus obtain

PROPOSITION 2.1. *If* $0 < 1 - 2a < \frac{2}{c^2 \sinh c} \sinh^2(\frac{c}{4}) \sinh(\frac{c}{2})$, *then* (R) *has a unique stationary solution* $(0, s^*, m^*)$ *for all* $\tau \neq 0$ *with* $m^* = 1 - s^*$, $s^* \in (0, 1/2) \setminus \{1/4\}$. *The linearization of* f *at* $(0, s^*, m^*)$ *is*

$$
\begin{aligned}
&Df(0, s^*, m^*)(\hat{u}, \hat{s}, \hat{m}) \\
&= \Big(\hat{u}(s^*) + \gamma_s(s^*, m^*)\hat{s} + \gamma_m(s^*, m^*)m^*\Big) \cdot \Big(G(s^*, m^*), 1, 0\Big) \\
&\quad + \Big(\hat{u}(m^*) + \eta_s(s^*, m^*)\hat{s} + \eta_m(s^*, m^*)m^*\Big) \cdot \Big(G(s^*, m^*), 0, -1\Big).
\end{aligned}
$$

The pair $(0, s^*, m^*)$ *corresponds to a unique steady state* (v^*, s^*, m^*) *of* (F) *for* $\tau \neq 0$ *with* $v^*(x) = g(x, s^*, m^*)$.

PROOF. We rewrite $\gamma(s, m)$ and $\eta(s, m)$ as

$$\gamma(s, m) = \eta(s, m) + \frac{1}{c^2 \sinh c} \sinh(c(s + m - 1)) \cdot \sinh \frac{c(s - m)}{2}.$$

Note that $C(r) = 0$ if and only if $\gamma(s, m) = 1/2 - a$ and $\eta(s, m) = 1/2 - a$ which implies $\sinh(c(s + m - 1)) = 0$. Thus we have $C(r) = 0$ if and only if $s + m = 1$. Therefore we only need to show the existence of s^* which satisfies $\gamma(s^*, 1 - s^*) = 1/2 - a$ for $s^* \in (0, 1/2)$. Now we define

$$
\begin{aligned}
\Gamma(s) &:= \gamma(s, 1 - s) \\
&= \frac{2}{c^2 \sinh c} \sinh \frac{c}{2} \cdot \sinh \frac{c(1 - 2s)}{2} \cdot \sinh cs.
\end{aligned}
$$

Then $\Gamma(s)$ is satisfied that $\Gamma(0) = 0 = \Gamma(1/2)$ and $\Gamma'(s) = \dfrac{\sinh(1/2 - 2s)}{c \cdot \cosh(c/2)}$. Therefore, $\Gamma(s) - (1 - 2a) = 0$ has solutions in $(0, 1/2) \setminus \{1/4\}$. It follows that $m^* = 1 - s^*$ with $m^* \in (1/2, 1) \setminus \{3/4\}$. The formula for $Df(0, s^*, m^*)$ follows from Lemma 2.5 in [2] and the relation $C'(1/2 - a) = 4$. Using Theorem 2.7 in [2], we obtain the corresponding steady state (v^*, s^*, m^*) for (F). □

Since $C'(1/2 - a) = 4$, we define a new parameter $\mu = 4/\tau$. We now introduce the definition of a Hopf bifurcation.

DEFINITION 2.2. Under the assumptions of Proposition 2.1, define (for $3/4 < \alpha \leq 1$) the operator $B \in L(\widetilde{X}^\alpha, \widetilde{X})$ as

$$B := \frac{1}{4} Df(0, s^*, m^*).$$

We then define $(0, s^*, m^*, \mu^*)$ to be a Hopf point for (R) if there exists an $\epsilon_0 > 0$ and a C^1-curve

$$(-\epsilon_0 + \mu^*, \mu^* + \epsilon_0) \mapsto (\lambda(\mu), \phi(\mu)) \in \mathbf{C} \times \widetilde{X}_{\mathbf{C}}$$

($X_{\mathbf{C}}$ denotes the complexification of the real space X) of eigendata for $-\widetilde{A} + \mu B$ such that

(i) $(-\widetilde{A} + \mu B)(\phi(\mu)) = \lambda(\mu)\phi(\mu), \quad (-\widetilde{A} + \mu B)(\overline{\phi(\mu)}) = \overline{\lambda(\mu)}\,\overline{\phi(\mu)}$;
(ii) $\lambda(\mu^*) = i\beta$ with $\beta > 0$;
(iii) $\mathrm{Re}\,(\lambda) \neq 0$ for all $\lambda \in \sigma(-\widetilde{A} + \mu^* B) \setminus \{\pm i\beta\}$;
(iv) $\mathrm{Re}\,\lambda'(\mu^*) \neq 0$ (transversality).

The linearized eigenvalue problem of (R) is given by

$$-\widetilde{A}(u, s, m) + \mu B(u, s, m) = \lambda(u, s, m)$$

which is equivalent to

(2.2)
$$\begin{cases} (A + \lambda)u &= \mu \cdot \big(\gamma_s(s^*, m^*)s + \gamma_m(s^*, m^*)m + u(s^*)\big) \cdot G(\cdot, s^*) \\ &\quad + \mu \cdot \big(\eta_s(s^*, m^*)s + \eta_m(s^*, m^*)m + u(m^*)\big) \cdot G(\cdot, m^*) \\ \lambda s &= \mu \cdot \big(\gamma_s(s^*, m^*)s + \gamma_m(s^*, m^*)m + u(s^*)\big) \\ \lambda m &= -\mu \cdot \big(\eta_s(s^*, m^*)s + \eta_m(s^*, m^*)m + u(m^*)\big)\,. \end{cases}$$

Here we note that

$$\gamma_s(s^*, m^*) = -G(s^*, s^*) + \int_{s^*}^{m^*} G_x(s^*, y)\, dy = -\eta_m(s^*, m^*)$$

and

$$\gamma_m(s^*, m^*) = G(s^*; m^*) = -\eta_s(s^*, m^*).$$

Furthermore, $\gamma_s(s^*, m^*) < 0$ for $1/4 < s < 1/2$ and $\gamma_s(s^*, m^*) > 0$ for $0 < s < 1/4$. Also $\displaystyle\int_{s^*}^{m^*} G_x(s^*, y)\, dy = (v^*)'(s^*) > 0$ for $0 < s < 1/2$.

We now show the non-existence of pure imaginary eigenvalues and the Hopf critical point in the following theorem.

THEOREM 2.3. *There is no a purely imaginary eigenvalue* $\lambda = i\beta$, $\beta > 0$, *of* (2.2) *and a critical point* μ^* *in order for* $(0, s^*, m^*, \mu^*)$ *to be a Hopf point.*

PROOF. Let $v(x,t) = u(x,t) + g(x, s(t), m(t))$ to (2.2). Then we have the following linearized eigenvalue problem of (F) which is represented by

$$(A+\lambda)v = -\delta_{s^*}\cdot s + \delta_{m^*}\cdot m\,, \tag{2.3}$$

$$\lambda\, s = \mu\cdot\Big((v^*)'(s^*)\, s + v(s^*)\Big) \tag{2.4}$$

and

$$\lambda\, m = -\mu\cdot\Big((v^*)'(m^*)\, m + v(m^*)\Big) \tag{2.5}$$

where δ_{s^*} ia the Dirac delta function.

The solution of (2.3) is given by $v(x) = -G_\lambda(x, s^*)\, s + G_\lambda(x, m^*)\, m$ where G_λ is Green's function of $A+\lambda$ satisfying the Dirichlet boundary condition. Multiply (2.4) by s and (2.5) by m, and multiply (2.4) by m and (2.5) by s and then add together. Then we have

$$\lambda\,(s+m)^2 = \mu\Big((v^*)'(s^*) - G_\lambda(s^*, s^*) + G_\lambda(s^*, m^*)\Big)(s+m)^2\,. \tag{2.6}$$

We let $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda = \beta > 0$ in (2.6), and obtain the real part

$$(v^*)'(s^*) - \operatorname{Re}\Big(G_{i\beta}(s^*, s^*) - G_{i\beta}(s^*, m^*)\Big) = 0\,. \tag{2.7}$$

The imaginary part is

$$\beta + \mu\,\operatorname{Im}\Big(G_{i\beta}(s^*, s^*) - G_{i\beta}(s^*, m^*)\Big) = 0\,. \tag{2.8}$$

We have to check that the equation (2.7) has a solution for β. So define

$$T(\beta) := (v^*)'(s^*) - \operatorname{Re}\left(G_{i\beta}(s^*, s^*) - G_{i\beta}(s^*, m^*)\right).$$

Then $T(\beta)$ is a strictly increasing continuous function of β^2. Furthermore,

$$\begin{aligned}\lim_{\beta\to\infty} T(\beta) &= (v^*)'(s^*)\\ &= \frac{1}{c\sinh c}\Big(\cosh cs^*(\cosh c(1-s^*) - 1)\Big) > 0\end{aligned}$$

and

$$\begin{aligned}T(0) &= (v^*)'(s^*) - \operatorname{Re} G(s^*, s^*) + \operatorname{Re} G(s^*, m^*)\\ &= \frac{2}{c\sinh c}\sinh^2(c\,(s^* - 1/2))\,.\end{aligned}$$

In order to have a solution of β, $T(0)$ must be negative, but $T(0)$ is always nonnegative. Thus, a pair of pure imaginary complex conjugate eigenvalues cannot exist. Hence a critical point of μ does not exist. $\square$

So we have shown that there does not exist a Hopf bifurcation in a double-layered problem of (R) satisfying the Dirichlet boundary condition, but there does exist this phenomena in (R) satisfying the Neumann boundary condition.

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