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A NOTE ON FELLER'S THEOREM

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ABSTRACT. In this note we have generalization of Feller's theorem to real separable Banach spaces, from which we obtain easily Chow-Robbins "fair" games problem in the Banach spaces.

1. Introduction

Let $(B, \| \ \|)$ be a real separable Banach space. The law of large numbers for Banach-valued random variables have been studied by many authors.

In this paper, we apply Chung's $SLLN$ in a Banach space [2] to obtain Feller's $SLLN$ [4] in a Banach space. From this result, Chow-Robbins "fair" games problem can be easily obtained.

2. Main results

Throughout this section, let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed B-valued random variables, and put $S_n = \sum_{i=1}^{n} X_i$. Let ϕ be a positive, even and continuous function on R such that as $|x|$ increases $\frac{\phi(x)}{x} \uparrow$ and $\frac{\phi(x)}{x^2} \downarrow$.

The following lemma plays an essential role in our main theorem.

LEMMA 1. (Choi and Sung [2]) *Let $\{X_n, n \geq 1\}$ be a sequence of independent B-valued random variables and $\{a_n, n \geq 1\}$ constants such that $0 < a_n \uparrow \infty$. Assume $\sum_{n=1}^{\infty} E\phi(\|X_n\|)/\phi(a_n) < \infty$. Then the following*

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are equivalent:

$$\begin{aligned} &\text{(i)} \qquad E\|S_n\|/a_n \to 0,\\ &\text{(ii)} \qquad S_n/a_n \to 0 \text{ a.s.},\\ &\text{(iii)} \qquad S_n/a_n \to 0 \text{ in probability.} \end{aligned}$$

THEOREM 2. *Let $E\|X_1\| = \infty$ and let $\{b_n, n \geq 1\}$ be a sequence of positive numbers such that $\{b_n/n\}$ is nondecreasing. Then*

$$\text{(i)} \qquad \sum_{n=1}^{\infty} P(\|X_1\| > b_n) = \infty \text{ implies } \limsup_{n\to\infty} \|S_n\|/b_n = \infty \text{ a.s. and}$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} P(\|X_1\| > b_n) < \infty \text{ implies } \limsup_{n\to\infty} \|S_n\|/b_n = 0 \text{ a.s.}$$

PROOF. Let $F(x) = P(\|X_1\| \leq x)$, and assume that $\sum_{n=1}^{\infty} P(\|X_1\| > b_n) = \infty$. By a well-known lemma on series of tail probabilities ([8], p. 131), we have that $\sum_{n=1}^{\infty} P(\|X_1\| > \lambda b_n) = \infty$ for all $\lambda > 0$. By Borel-Cantelli lemma, we have

$$P(\limsup_{n\to\infty} \|X_n\|/b_n \geq \lambda) \geq P(\|X_n\|/b_n > \lambda \text{ i.o }) = 1,$$

and since λ is arbitrary,

$$\limsup_{n\to\infty} \frac{\|X_n\|}{b_n} = \infty \text{ a.s.} \tag{2.1}$$

Thus we note via the triangle inequality that for $n \geq 2$

$$\frac{\|X_n\|}{b_n} = \frac{\|S_n - S_{n-1}\|}{b_n} \leq \frac{\|S_n\|}{b_n} + \frac{\|S_{n-1}\|}{b_n}$$

which, in view of (2.1), proves (i).

Assume that $\sum_{n=1}^{\infty} P(\|X_1\| > b_n) < \infty$. Let $X_k' = X_k I(\|X_k\| \leq b_k), X_k'' = X_k I(\|X_k\| > b_k), S_k' = \sum_{i=1}^{k} X_i'$ and $S_k'' = \sum_{i=1}^{k} X_i''$. Then by Borel-Cantelli lemma, $S_n''/b_n \to 0$ a.s. Now we complete the proof by showing that $S_n'/b_n \to 0$ a.s. But it suffices to show that $\sum_{n=1}^{\infty} E\|X_n'\|^2/b_n^2 < \infty$ and $E\|S_n'\|/b_n \to 0$ by Lemma 1 with $\phi(x) = x^2$. If we set $b_0 = 0$, then we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{E\|X_n'\|^2}{b_n^2} &= \sum_{n=1}^{\infty}\frac{1}{b_n^2}\int_{|x|\le b_n} x^2 dF(x)\\ &= \sum_{n=1}^{\infty}\frac{1}{b_n^2}\sum_{k=1}^{n}\int_{b_{k-1}\le|x|<b_k} x^2 dF(x)\\ &\le \sum_{k=1}^{\infty}\int_{b_{k-1}\le|x|<b_k} dF(x) b_k^2 \sum_{n=k}^{\infty}\frac{1}{b_n^2}.\end{aligned}$$

Since the sequence $\{b_n/n, n\ge 1\}$ is nondecreasing,

$$\sum_{n=k}^{\infty}\frac{1}{b_n^2}\le\frac{k^2}{b_k^2}\sum_{n=k}^{\infty}\frac{1}{n^2}\le\frac{2k}{b_k^2},$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty}E\|X_n'\|^2/b_n^2 &\le \sum_{k=1}^{\infty}2kP(b_{k-1}\le\|X_1\|<b_k)\\ &\le 2\sum_{k=0}^{\infty}P(\|X_1\|>b_k)<\infty.\end{aligned}$$

We now estimate the quantity $\sum_{k=1}^{n}E\|X_k'\|/b_n$, as $n\to\infty$. Clearly for any $N<n$, it is bounded by

$$\frac{n}{b_n}\left(b_N+\int_{b_N\le|x|<b_n}|x|dF(x)\right). \tag{2.2}$$

Since $E\|X_1\|=\infty$ and $\sum_{n=1}^{\infty}P(\|X_1\|>b_n)<\infty$, b_n/n cannot be bounded. Hence for fixed N the term $(n/b_n)b_N$ in (2.2) tends to 0 as $n\to\infty$ and the second term of (2.2) is bounded by

$$\frac{n}{b_n}\sum_{j=N+1}^{n}b_j\int_{b_{j-1}\le|x|<b_j}dF(x)\le\sum_{j=N+1}^{n}j\int_{b_{j-1}\le|x|<b_j}dF(x),$$

since $nb_j/b_n\le j$ for $j\le n$. If we replace the n in the right hand side above by ∞, it tends to 0 as $N\to\infty$ since

$$\sum_{k=1}^{\infty}k\int_{b_{k-1}\le\|x\|<b_k}dF(x)\le\sum_{n=1}^{\infty}P(\|X_1\|\ge b_n)<\infty.$$

This completes the proof of Theorem 2. □

Using Theorem 2 and following the line of proof of Theorem 2 [3], we obtain the following Chow-Robbins "fair" games problem in general Banach spaces.

THEOREM 3. *If* $E\|X_1\| = \infty$, *then for any sequence of* $\{b_n, n \geq 1\}$ *either* $\liminf_{n\to\infty} \|\frac{S_n}{b_n}\| = 0$ *a.s. or* $\limsup_{n\to\infty} \|\frac{S_n}{b_n}\| = \infty$ *a.s., and consequently,* $P(\lim_{n\to\infty} \|\frac{S_n}{b_n}\| = 1) = 0$.

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