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HOMOGENEOUS FUNCTION AND ITS APPLICATION IN A FINSLER SPACE

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ABSTRACT. We deal with a differential equation which is constructed from homogeneous function, and its geometrical meaning in a Finsler space. Moreover, we prove that a locally Minkowski space satisfying a differential equation $F_{\beta\beta\beta} = 0$ is flat-parallel.

1. Introduction

Homogeneity of a function on a Finsler geometry plays an important role. In fact, a Finsler metric $L(x, y)$ is called an (α, β)-metric if L is a positive homogeneous function of degree 1 in α and β, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. The (α, β)-metric has been sometimes treated in theoretical physics ([1],[2]), and studied by some authors ([3],[4],[5]).

The purpose of the present paper is to give an (α, β)-metric in a Finsler space satisfying a differential equation, and show its geometrical meaning.

2. Homogeneous function

In order to prove the theorems of this section, we shall show two lemmas as follows. First, from the Euler's Theorem we get

LEMMA 2.1. *If the function $H(\alpha)$ is positively homogeneous of degree n in α, then we have*

$$H(\alpha) = c\alpha^n, \ c = constant.$$

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Next we consider the function $F(\alpha,\beta)$ of two variables, and denote by the subscripts α,β of F the partial derivatives of F with respect to α,β respectively, that is,

$$F_\alpha = \frac{\partial F}{\partial \alpha},\ F_\beta = \frac{\partial F}{\partial \beta},\ F_{\alpha\beta} = \frac{\partial^2 F}{\partial\alpha\partial\beta},\ etc.$$

LEMMA 2.2. *Let the function $F(\alpha,\beta)$ be a positively homogeneous of degree n in α and β. If $F_{\underbrace{\beta\beta\dots\beta}_{n+1}} = 0$, then we have*

$$F(\alpha,\beta) = \sum_{k=0}^{n} c_k \alpha^{n-k}\beta^k,$$

where $c_0, c_1, \dots$ are constants.

PROOF. Let us find the solution of $F_{\underbrace{\beta\beta\dots\beta}_{n+1}} = 0$. Integrating this in β repeatedly and paying attention to the homogeneity of F, from Lemma 2.1 we get

$$F_{\underbrace{\beta\beta\dots\beta}_{n}} = b_0,\quad F_{\underbrace{\beta\beta\dots\beta}_{n-1}} = b_0\beta + b_1\alpha,$$

$$F_{\underbrace{\beta\beta\dots\beta}_{n-2}} = b_0'\beta^2 + b_1\alpha\beta + b_2\alpha^2,$$

$$F_{\underbrace{\beta\beta\dots\beta}_{n-3}} = b_0''\beta^3 + b_1'\alpha\beta^2 + b_2\alpha^2\beta + b_3\alpha^3,$$

$$\vdots$$

$$F = c_0\alpha^n + c_1\alpha^{n-1}\beta + \cdots + c_n\beta^k = \sum_{k=0}^{n} c_k\alpha^{n-k}\beta^k,$$

where b's and c's are constants. This completes the proof. □

Let the function $F(\alpha,\beta)$ be a positively homogeneous of degree 2 in α, and β. From the homogeneity of F, we obtain

$$F_{\alpha\alpha\alpha}\alpha + F_{\alpha\alpha\beta}\beta = 0,\ F_{\alpha\beta\alpha}\alpha + F_{\alpha\beta\beta}\beta = 0,\ F_{\beta\beta\alpha}\alpha + F_{\beta\beta\beta}\beta = 0.$$

If $F_{\beta\beta\beta} = 0$, from the above we have $F_{\beta\beta\alpha} = F_{\alpha\beta\alpha} = F_{\alpha\alpha\alpha} = 0$.

Next, we consider a positively homogeneous function $F(\alpha, \beta)$ of degree 3 in α and β. Then we have

$$F_{\alpha\alpha\alpha\alpha}\alpha + F_{\alpha\alpha\alpha\beta}\beta = 0, \ F_{\alpha\alpha\beta\alpha}\alpha + F_{\alpha\alpha\beta\beta}\beta = 0,$$
$$F_{\alpha\beta\beta\alpha}\alpha + F_{\alpha\beta\beta\beta}\beta = 0, \ F_{\beta\beta\beta\alpha}\alpha + F_{\beta\beta\beta\beta}\beta = 0.$$

In this case, if $F_{\beta\beta\beta\beta} = 0$, we easily get $F_{\beta\beta\beta\alpha} = F_{\alpha\beta\beta\alpha} = F_{\alpha\alpha\beta\alpha} = F_{\alpha\alpha\alpha\alpha} = 0$. Next, for a positively homogeneous function $F(\alpha, \beta)$ of degree 4 in α and β, if $F_{\beta\beta\beta\beta\beta} = 0$, then we have $F_{\beta\beta\beta\beta\alpha} = F_{\alpha\beta\beta\beta\alpha} = F_{\alpha\alpha\beta\beta\alpha} = F_{\alpha\alpha\alpha\beta\alpha} = F_{\alpha\alpha\alpha\alpha\alpha} = 0$. In the similar way, paying attention to the homogeneity of F, consequently we have

THEOREM 2.1. *Let $F(\alpha, \beta)$ be a positively homogeneous function of degree n in α and β. Then the condition $F_{\underbrace{\beta\beta\ldots\beta}_{n+1}} = 0$ is equivalent to*

$$F_{i_1 i_2 \ldots i_{n+1}} = 0, \ i_1, i_2, \ldots, i_{n+1} \in \{\alpha, \beta\}. \tag{2.1}$$

REMARK. This theorem means that a general solution of the differential equation $F_{i_1 i_2 \ldots i_{n+1}} = 0$, $i_1, i_2, \ldots, i_{n+1} \in \{\alpha, \beta\}$, does not depend on the choice of the subscript variables α and β.

Since an (α, β)-metric $L(\alpha, \beta)$ in a Finsler space is a positively homogeneous of degree 1 in α and β, it is possible to give an (α, β)-metric by putting $F = L^n$. From Lemma 2.2 and Theorem 2.1 we have

THEOREM 2.2. *Let $F(\alpha, \beta)$ be a positively homogeneous function of degree n in α and β. Then the solution of the differential equation $F_{i_1 i_2 \ldots i_{n+1}} = 0$, $i_1, i_2, \ldots, i_{n+1} \in \{\alpha, \beta\}$, is an (α, β)-metric as follows:*

$$L(\alpha, \beta) = \left(\sum_{k=0}^{n} c_k \alpha^{n-k} \beta^k \right)^{1/n}, \ F = L^n. \tag{2.2}$$

REMARK. The metric (2.2) is exactly regarded as a generalization of the Randers metric. In 1984 Shibata ([7]) dealt with an interesting (α, β)-metric

$$L(\alpha, \beta) = (\alpha^s + \cdots + c_k \alpha^{s-k} \beta^k + \cdots + \beta^s)^r,$$

where $rs = 1$ and c's are constants. This metric is only a special case of the equation (2.2) if $c_0 = c_n = 1$.

3. A Finsler space satisfying $F_{\beta\beta\beta} = 0$

We consider a Finsler space $F^n = (M^n, L(\alpha, \beta))$ with an (α, β)-metric, where M^n is an n-dimensional differential manifold equipped with a fundamental function L. The function $L(\alpha, \beta)$ is a positively homogeneous of degree 1 in α and β, where $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential 1-form. If we put $F = L^2/2$ and $\dot{\partial}_i = \partial/\partial y^i$, then the fundamental tensor $g_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j F$ and the Cartan's C-tensor C_{ijk} are given by ([1])

$$g_{ij} = (F_\alpha/\alpha)k_{ij} + (F_{\alpha\alpha}/\alpha^2)y_i y_j + (F_{\alpha\beta}/\alpha)(y_i b_j + y_j b_i) + F_{\beta\beta} b_i b_j,$$

$$(3.1)\quad 2C_{ijk} = (F_{\alpha\beta}/\alpha)(K_{ij}p_k + K_{jk}p_i + K_{ki}p_j) + F_{\beta\beta\beta}\, p_i p_j p_k.$$

where we put $K_{ij} = a_{ij} - y_i y_j$, $y_i = a_{ij}y^j$, and $p_i = b_i - (\beta/\alpha^2)y_i$.

It is well known that any 2-dimensional Finsler space is strongly non-Riemannian and has the Berwald frame ([1]). But a 3-dimensional Finsler space is strongly non-Riemannian, if and only if vector $C_i = C_i{}^r{}_r$ does not vanish. Any Finsler space of dimension $n > 3$ with (α, β)-metric is not strongly non-Riemannian. Since the set of Riemannian space is characterized by the equation $C_{ijk} = 0$, it is proper that we should pay attention to the behaviour of this tensor C_{ijk} in a Finsler space with (α, β)-metric. Then, what is a suitable and simple form of C_{ijk}?

Matsumoto early dealt with a C-reducible ([1], [5]), that is,

$$C_{ijk} = h_{ij}A_k + h_{jk}A_i + h_{ki}A_j,$$

where $h_{ij} = g_{ij} - l_i l_j$, $l_i = \dot{\partial}_i L$ and A_i is some vector. It is well known that a C-reducible Finsler space is induced to a Randers space and a Kropina space.

On the other hand, we can consider another simple form of C_{ijk}. If $F_{\beta\beta\beta} = 0$, from (3.1) we obtain

$$C_{ijk} = K_{ij}B_k + K_{jk}B_i + K_{ki}B_j,$$

for some vector B_i. Putting $F = L^2/2$, and using Lemma 1.2, the solution of the differential equation $F_{\beta\beta\beta} = 0$ is $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c's are constants. Therefore we have

LEMMA 3.1. *In the Finsler space $F^n = (M^n, L(\alpha,\beta))$ the followings are equivalent to each other:*

$$\text{(3.2)}\qquad \begin{aligned} &a)\ C_{ijk} = K_{ij}B_k + K_{jk}B_i + K_{ki}B_j,\\ &b)\ F_{\beta\beta\beta} = 0,\\ &c)\ L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2,\ c_1, c_2, c_3 \neq 0, 1. \end{aligned}$$

Let $\gamma_j{}^i{}_k(x)$ be Christoffel symbols of the Riemannian metric α and $G_j{}^i{}_k(x,y)$ be connection coefficients of the Berwald connection $B\Gamma$ of an (α,β)-metric $L(\alpha,\beta)$. Then the previous paper ([1],[3]) gives the equation to find the difference $B_j{}^i{}_k = G_j{}^i{}_k - \gamma_j{}^i{}_k$:

$$\text{(3.3)}\qquad L_\alpha B_j{}^k{}_i y^j y_k = \alpha L_\beta (b_{j;i} - B_j{}^k{}_i b_k) y^j,$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$ and (;) denotes the covariant differentiation with respect to the Riemannian connection $\gamma_j{}^i{}_k(x)$.

We consider a *locally Minkowski* space $F^n = (M^n, L)$, that is, M^n admits a covering by coordinate neighborhoods in each of which the fundamental function L is a function of y^i alone. We denote by $\overset{r}{R}_{hijk}$ a Riemannian curvature tensor with respect to the $\gamma_j{}^i{}_k$.

DEFINITION 3.1 ([5]). A locally Minkowski space with (α,β)-metric is called *flat-parallel*, if α is locally flat($\overset{r}{R}_{hijk} = 0$) and b_i is parallel with respect to $\alpha(b_{i;j} = 0)$.

THEOREM A ([3]). *A $F^n = (M^n, (\alpha,\beta))$ is a locally Minkowski if and only if $B_j{}^k{}_i$ are functions of x alone and $\overset{r}{R}_h{}^i{}_{jk}$ of the Riemannian α is written as:*

$$\text{(3.4)}\qquad \overset{r}{R}_h{}^i{}_{jk} = -\mathcal{U}(jk)\{B_h{}^i{}_{j;k} + B_h{}^r{}_j B_r{}^i{}_k\},$$

where $\mathcal{U}(jk)$ denotes the terms obtained from the preceding terms by interchanging indices j and k.

On the other hand, we shall find some (α,β)-metrics which define flat-parallel Minkowski spaces. Putting $P_{i00} = B_j{}^k{}_i y^j y_k$ and $Q_{i0} = (b_{j;i} - B_j{}^k{}_i b_k) y^j$, the equation (3.3) is written as

$$\text{(3.5)}\qquad L_\alpha P_{i00} = \alpha L_\beta Q_{i0},$$

where the index 0 denotes as usual the transvection by y^i. It is remarked that for a locally Minkowski space these P_{i00} and Q_{i0} are polynomials in y^i of degree 2 and 1 respectively. If (3.5) gives $P_{i00} = Q_{i0} = 0$ necessarily, then we have $B_j{}^k{}_i = 0$ and $b_{j;i} = 0$, and (3.4) shows $\overset{r}{R}_h{}^i{}_{jk} = 0$. Consequently this $L(\alpha, \beta)$ defines a flat-parallel Minkowski space.

We shall apply this procedure to the metric (3.2) c).

THEOREM 3.1. *A locally Minkowski space with the metric* (3.2) *c) is flat-parallel.*

PROOF. From (3.2) c), we have $L_\alpha = (c_1\alpha + c_2\beta)/L,\ L_\beta = (c_2\alpha + c_3\beta)/L$. Substituting this into (3.5), we get

$$(3.6) \qquad (c_1 P_{i00} - c_3\beta Q_{i0})\alpha + (c_2\beta P_{i00} - c_2\alpha^2 Q_{i0}) = 0.$$

Since α is irrational in y^i, (3.6) leads us to

$$\begin{cases} c_1 P_{i00} - c_3\beta Q_{i0} = 0, \\ c_2\beta P_{i00} - c_2\alpha^2 Q_{i0} = 0. \end{cases}$$

From $det \begin{pmatrix} c_1 & -c_3\beta \\ c_2\beta & -c_2\alpha^2 \end{pmatrix} \neq 0$, it follows that $P_{i00} = Q_{i0} = 0$. This completes the proof. □

For arc-length s, a geodesic in F^n is given by the differential equations

$$d^2x^i/ds^2 + 2G^i(x, dx/ds) = 0,$$

where $2G^i = g^{ir}(y^j \dot\partial_r \partial_j F - \partial_r F)$, $\partial_j = \partial/\partial x^j$ and $F = L^2/2$.

Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces on the same underlying manifold M^n. If any geodesic of F^n is a geodesic of $\bar{F}^n$ and vice versa, then F^n is called projective to $\bar{F}^n$ and change $\sigma : L \to \bar{L}$ of metric is called projective. It is well-known that σ is projective if and only if there exists a positively homogeneous function $P(x, y)$ of degree 1 in y^i satisfying $\bar{G}^i = G^i + Py^i$. Throughout the following we indicate by putting bar the corresponding quantities of $\bar{F}^n$. Assume that a change $\sigma : F^n = (M^n, L(\alpha, \beta)) \to \bar{F}^n = (M^n, \bar{L}(\alpha, \beta))$ of (α, β)-metric.

On the other hand, we shall introduce a β-change ([5]) as follows:

DEFINITION 3.2. Let $L(\alpha, \beta)$ be an (α, β)-metric. The change $\varphi : \alpha \to L(\alpha, \beta)$ of metric is called a β-change.

If we denote by R^n the associated Riemannian space with a Finsler space F^n with (α, β)-metric, then the β-change is the change from R^n to F^n. There is a theorem between projective change and β-change as follows:

THEOREM B ([5]). *A β-change is projective, if and only if we have*

$$L_\beta \Psi_{ij} + L_{\beta\beta} \Omega_{ij} = 0, \tag{3.7}$$

where $\Psi_{ij} = (b_{i;j} - b_{j;i})/2$, $\Omega_{ij} = (p_i \beta_{;j} - p_j \beta_{;i})/2$ *and* $\beta_{;j} = b_{i;j} y^j$.

Now we consider a change $\pi : \alpha \to L = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$. Then, from Theorem B we can obtain the condition for a change π to be projective.

THEOREM 3.2. *A change* $\pi : \alpha \to L = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ *is projective if and only if we have* $b_{i;j} = 0$.

PROOF. From (3,2)c), we have $L_\beta = (c_2\alpha + c_3\beta)/L$, $L_{\beta\beta} = \Delta_1 \alpha^2/L^3$, where $\Delta_1 = c_1c_3 - c_2^2 \neq 0$ is supposed because (3.2) c) is not a Randers metric. Substituting this into (3.7), we get

$$c_2\alpha(c_1\alpha^2 + 3c_3\beta^2)\Psi_{ij} + (\Delta_2\alpha^2\beta + c_3^2\beta^3)\Psi_{ij} + \Delta_1\alpha^2\Omega_{ij} = 0, \tag{3.8}$$

where $\Delta_2 = c_1c_3 + 2c_2^2$. Since α is an irrational polynomial of y^i, (3.8) leads us to $(c_1\alpha^2 + 3c_3\beta^2)\Psi_{ij} = 0$. From $c_1\alpha^2 + 3c_3\beta^2 \neq 0$, we have $\Psi_{ij} = 0$. Substituting this into (3.8), by virtue of $\Delta_1 \neq 0$ we have $\Omega_{ij} = 0$. Further, transvecting this by y^i and using $p_i y^i = 0$ and $\Psi_{ij} = 0$, we obtain $b_{i;j} = 0$. Conversely, if $b_{i;j} = 0$, it satisfies (3.8). This completes the proof. □

On the other hand, Park and Choi (cf. [6], Theorem 3.2) dealt with the condition that a Finsler space with a metric (3.2) c) to be a projectively flat. Combining this Theorem and Theorem 3.2, we can give more geometrical meaning like a Matsumoto's Theorem (cf. [4], Theorem 2). Thus we have

COROLLARY 3.1. *A Finsler space with an (α, β)-metric (3.2) c) is projectively flat if and only if a change $\pi : \alpha \to L = \sqrt{c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2}$ is projective and the associated Riemannian space with the metric α is projectively flat.*

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